Journal of the Washington Academy of Sciences

This Journal, the official organ of the Washington Academy of Sciences, publishes: (1) Short original papers, written or communicated by members of the Academy; (2) proceedings and programs of meetings of the Academy and affiliated societies; (3) notes of events connected with the scientific life of Washington. The Journal is issued monthly, on the fifteenth of each month. Volumes correspond to calendar years.

Manuscripts may be sent to any member of the Board of Editors. It is urgently requested that contributors consult the latest numbers of the Journal and conform their manuscripts to the usage found there as regards arrangement of title, subheads, synonymies, footnotes, tables, bibliography, legends for illustrations, and other matter. Manuscripts should be typewritten, double-spaced, on good paper. Footnotes should be numbered serially in pencil and submitted on a separate sheet. The editors do not assume responsibility for the ideas expressed by the author, nor can they undertake to correct other than obvious minor errors.

Illustrations in excess of the equivalent (in cost) of one full-page halftone are to be paid for by the author.

Proof.—In order to facilitate prompt publication one proof will generally be sent to authors in or near Washington. It is urged that manuscript be submitted in final form; the editors will exercise due care in seeing that copy is followed.

Unusual cost of foreign, mathematical, and tabular material, as well as alterations made in the proof by the author, may be charged to the author.

Author's Reprints.—Reprints will be furnished in accordance with the following schedule of prices (approximate):

Copies	4 pp.	8 pp.	12 pp.	16 pp.	20 pp.	Covers
100	$2.48	$4.96	$ 7.44	$ 9.92	$12.40	$2.09
150	3.72	7.44	11.16	14.88	18.60	3.14
200	4.96	9.92	14.88	19.84	24.80	4.18
300	7.44	14.88	22.32	29.76	37.20	6.27
400	9.92	19.84	29.76	39.68	49.60	8.36

Subscriptions or requests for the purchase of back numbers or volumes of the Journal or the Proceedings should be sent to Harald A. Rehder, Custodian and Subscription Manager of Publications, U. S. National Museum, Washington 25, D. C.

Subscription Rates for the Journal.—Per year $7.50

Price of back numbers and volumes:	*Per Vol.*	*Per Number*
Vol. 1 to vol. 10, incl.—not available.*	—	—
Vol. 11 to vol. 15, incl. (21 numbers per vol.)	$10.00	$0.70
Vol. 16 to vol. 22, incl. (21 numbers per vol.)	8.00	0.60
Vol. 23 to current vol. (12 numbers per vol.)	7.50	0.90

* Limited number of complete sets of the Journal (vol. 1 to vol. 39, incl.) available for sale to libraries at $333.50.

Proceedings, vols. 1–13 (1899–1911) complete	$25.00
Single volumes, unbound	2.00
Single numbers	.25

Missing Numbers will be replaced without charge provided that claim is made to the Treasurer within 30 days after date of following issue.

Remittances should be made payable to "Washington Academy of Sciences" and addressed to the Treasurer, H. S. Rappleye, U. S. Coast and Geodetic Survey, Washington 25, D. C.

Exchanges.—The Academy does not exchange its publications for those of other societies.

JOURNAL
OF THE
WASHINGTON ACADEMY OF SCIENCES

VOLUME 40 March 15, 1950 No. 3

MATHEMATICS.—*Comments on the classical theory of integral equations.*[1] D. C. LEWIS, The Johns Hopkins University.

§1. INTRODUCTION

The purpose of this paper is to show how a large part of the theory of the symmetric kernel can be adapted to the unsymmetric case merely by introducing the concept of a characteristic pair. By this we mean a pair of functions, $f(x)$ and $g(x)$, not identically vanishing, which with a suitable value of λ satisfy the following pair of equations:

$$
(1.1)\qquad
\begin{aligned}
g(x) &= \lambda \int_a^b f(y)K(y, x)\, d\alpha(y) \\
f(x) &= \lambda \int_a^b K(x, y)g(y)\, d\alpha(y).
\end{aligned}
$$

Here $K(x, y)$ is the generally nonsymmetric kernel which we assume to be continuous, and hence bounded, on the closed square $a \leq x \leq b$, $a \leq y \leq b$. The function $\alpha(x)$ is monotonic increasing (and nowhere constant) on the interval $a \leq x \leq b$.

The number λ corresponding to such a nontrivial solution of (1.1) is called a characteristic number of the kernel. It is clear that if λ is thus a characteristic number accompanying the pair $[f(x), g(x)]$, then λ^2 is a characteristic value in the ordinary sense of the symmetric iterated kernel with characteristic function $f(x)$ or $g(x)$, according to whether the symmetric iterated kernel is

$$\int_a^b K(x, \xi)K(y, \xi)\, d\alpha(\xi) \text{ or }$$

$$\int_a^b K(\xi, x)K(\xi, y)\, d\alpha(\xi).$$

Erhard Schmidt has taken advantage of this fact in adapting his theory of the symmetric kernel to the unsymmetric case.[2] Our method, on the contrary, makes no use of the iterated kernel. We attack the problem directly by a method devised by Kellogg[2] for the symmetric case. We show that this method can be easily modified so as to fit also the unsymmetric case. We prove a maximum property of our characteristic numbers and function pairs and prove expansion theorems in a manner entirely analogous to the classical theory for the symmetric case.[2]

We use the following abbreviation throughout the paper:

$$(u, v) = \int_a^b u(x)v(x)\, d\alpha(x).$$

We recall also in this introductory section that the class of functions $f(x)$, which may be represented in the form

$$f(x) = \int_a^b K(x, y)\Phi(y)\, d\alpha(y) \text{ or }$$

$$\int_a^b \Phi(y)K(y, x)\, d\alpha(y),$$

in which $(\Phi, \Phi) = 1$, is equicontinuous. This follows from the uniform continuity of $K(x, y)$ in the closed square if we apply Schwarz's inequality to the right member of the equality

$$f(x) - f(x') = \int_a^b [K(x, y) - K(x', y)]\Phi(y)\, d\alpha(y).$$

The fact that this same class of functions is

[1] Received December 8, 1949.

[2] Cf. Literature Cited at end of paper.

bounded is even more obvious. We make these remarks to justify our subsequent use of Ascoli's theorem in the application of Kellogg's method. They also serve for an immediate proof of the fact that the functions of any characteristic pair must be continuous, a fact we tacitly use in some of our subsequent proofs.

§2. ELEMENTARY PROPERTIES OF CHARACTERISTIC NUMBERS AND PAIRS

First, it is clear from (1.1) that zero cannot be a characteristic number, inasmuch as the characteristic pair $[f, g]$ must not vanish identically.

Although we hereby assume throughout the paper that K and α are real-valued functions of real variables, we temporarily consider the possibility that f, g, and λ may be complex-valued. We notice from (1.1) that, if f and g are a characteristic pair corresponding to a characteristic number λ, then f and $-g$ are a characteristic pair corresponding to the characteristic number $-\lambda$, and conversely. Hence there is no essential loss of generality, if we assume in the sequel, as we shall do, that λ always lies in the right half plane including, possibly, the axis of pure imaginaries.

Let $[f_1, g_1]$ *and* $[f_2, g_2]$ *be two characteristic pairs corresponding respectively to the characteristic numbers* λ_1 *and* λ_2. *Then, in any case,*

$$(2.1) \qquad |(f_1, f_2)| = |(g_1, g_2)|,$$

and, if $|\lambda_1| \neq |\lambda_2|$, *we have zero for the common value of* (f_1, f_2) *and* (g_1, g_2).

Indeed, from

$$f_2(x) = \lambda_2 \int_a^b K(x, y) g_2(y)\, d\alpha(y),$$

we have

$$\begin{aligned}(f_1, f_2) &= \lambda_2 \int_a^b \int_a^b f_1(x) K(x, y) g_2(y)\, d\alpha(y)\, d\alpha(x) \\ &= \lambda_2 \int_a^b \left[\int_a^b f_1(x) K(x, y)\, d\alpha(x) \right] g_2(y)\, d\alpha(y).\end{aligned}$$

But, since the integral within the bracket is known to be equal to $\lambda_1^{-1} g_1(y)$, we have

$$(2.2) \qquad (f_1, f_2) = (\lambda_2/\lambda_1)(g_1, g_2).$$

Interchanging the subscripts 1 and 2 and remembering that $(u, v) = (v, u)$, we also obtain $(f_1, f_2) = (\lambda_1/\lambda_2)(g_1, g_2)$. Hence, either $\lambda_2/\lambda_1 = \lambda_1/\lambda_2$, or else $(g_1, g_2) = 0$, and the italicized statement is proved.

We also can prove very easily that *no characteristic number can be complex with non-vanishing imaginary part.* For, suppose λ_1 were a complex characteristic number with characteristic pair $[f_1, g_1]$. Then, if the complex quantities conjugate to λ_1, f_1, g_1 be denoted respectively by λ_2, f_2, g_2, we see from (1.1) that λ_2 is a characteristic number with characteristic pair $[f_2, g_2]$. Moreover (2.2) holds, and, since both (f_1, f_2) and (g_1, g_2) are obviously positive, (2.1) shows that $\lambda_1 = \lambda_2$. Hence λ_1 is real as we wished to show.

The result of all this is that, from now on, we may assume the characteristic numbers λ to be real and positive and the characteristic pairs to be real.

There may be many linearly independent characteristic pairs $[f_i, g_i]$, $i = 1, 2, \cdots$, corresponding to a single characteristic number λ. It is readily seen from (1.1) that any linear combination of these f_i taken together with the same linear combination of the g_i will furnish a characteristic pair for the characteristic number λ. Furthermore the f's corresponding to the *same* characteristic number can be *orthonormalized* so that $(f_i, f_j) = \delta_{ij}$ and then from (2.2) it is seen that the corresponding g's are also orthonormalized. It is already clear that f's corresponding to *different* characteristic numbers are orthogonal to each other, and the same is true of the g's.

Let us now consider a finite number m of characteristic pairs $[f_i, g_i]$ with corresponding characteristic numbers λ_i, $i = 1, 2, \cdots$, which are not necessarily distinct. According to what has been said, we may assume that $(f_i, f_j) = (g_i, g_j) = \delta_{ij}$. Then Bessel's inequality applied to the kernel shows that

$$\begin{aligned}&\int_a^b [K(x, y)]^2\, d\alpha(y) \\ &\geq \sum_{i=1}^m \left[\int_a^b K(x, y) g_i(y)\, d\alpha(y) \right]^2 = \sum_{i=1}^m \frac{f_i(x)^2}{\lambda_i^2}.\end{aligned}$$

Whence, a further integration with respect to x yields

$$(2.3)\quad \int_a^b \int_a^b [K(x, y)]^2\, d\alpha(y)\, d\alpha(x) \geq \sum_{i=1}^{m} \frac{1}{\lambda_i^2}.$$

Since the left member of this inequality is independent of m, it is clear that the number of characteristic numbers less than any fixed number must be finite. We can therefore assume that our notation is chosen so that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. If there is an infinite number of characteristic pairs, the inequality (2.3) then shows that the λ's must tend to infinity rapidly enough to secure the convergence of

$$\cdot \sum_{i=1}^{\infty} 1/\lambda_i^2 \leq \int_a^b \int_a^b [K(x, y)]^2\, d\alpha(x)\, d\alpha(y).$$

§3. EXISTENCE OF CHARACTERISTIC PAIRS

We first prove that every kernel that does not vanish identically has at least one characteristic pair.

We prove the following two theorems, of which the first is a special case of the second with $m = 0$:

THEOREM I. *Every kernel that is not identically zero has at least one characteristic pair and corresponding characteristic number.*

THEOREM II. *Let* $[f_1, g_1], [f_2, g_2], \cdots, (f_m, g_m]$ *be m orthonormalized characteristic pairs of the kernel $K(x, y)$, the corresponding characteristic numbers being $\lambda_1, \lambda_2, \cdots, \lambda_m$. Then either*

$$K(x, y) = \sum_{k=1}^{m} \frac{f_k(x) g_k(y)}{\lambda_k},$$

or else there exists at least one further characteristic pair $[f, g]$ with corresponding characteristic number λ, such that

$$(f_k, f) = (g_k, g) = 0, \quad k = 1, 2, \cdots, m.$$

Since Theorem I is a special case of Theorem II, we confine our attention to the latter. For this purpose, we define

$$K_m(x, y) = K(x, y) - \sum_{k=1}^{m} \frac{f_k(x) g_k(y)}{\lambda_k}.$$

Not every function $u_0(y)$ is orthogonal to $K_m(x, y)$ for all x, unless $K_m \equiv 0$. For, suppose $K_m(x_1, y_1) \neq 0$. Then the function $u_0(y) = K_m(x_1, y)$ is not orthogonal to $K_m(x_1, y)$, since the continuity of $K(x, y)$, and hence of $K_m(x, y)$, implies that

$$\int_a^b [K_m(x_1, y)]^2\, d\alpha(y) > 0.$$

Moreover $u_0(y)$, as thus defined, also has the property of being orthogonal to g_i for $i = 1, 2, \cdots, m$. This is easily proved on the basis of the known formulas,

$$(g_k, g_i) = \delta_{ki} \quad \text{and } \lambda_i^{-1} f_i(x_1) = \int_a^b K(x_1, y) g_i(y)\, d\alpha(y).$$

Now then, suppose $u_0(y)$ be *any* function in L_2 (not necessarily $K_m(x_1, y)$) which is orthogonal to $g_1, \cdots, g_m$ but which is not orthogonal to $K_m(x, y)$ for all x. Let

$$n_0 = (u_0, u_0)^{\frac{1}{2}} > 0, \qquad \bar{u}_0(y) = n_0^{-1} u_0(y).$$

Since $\bar{u}_0(y)$ is also not orthogonal to $K_m(x, y)$, although it is orthogonal to $g_1, \cdots, g_m$, the function

$$(3.1)\quad u_1(x) = \int_a^b K_m(x, y) \bar{u}_0(y)\, d\alpha(y) = \int_a^b K(x, y) \bar{u}_0(y)\, d\alpha(y),$$

does not vanish identically. Let

$$n_1 = (u_1, u_2)^{\frac{1}{2}} > 0, \qquad \bar{u}_1(y) = n_1^{-1} u_1(y).$$

We now give a formal inductive definition of a sequence, $u_i(y)$, $i = 0, 1, 2, 3, \cdots$ by means of the following equations:

$$(3.2)\quad u_{2s}(x) = \int_a^b \bar{u}_{2s-1}(y) K(y, x)\, d\alpha(y), \qquad s = 1, 2, \cdots$$

$$(3.3)\quad n_{2s} = (u_{2s}, u_{2s})^{\frac{1}{2}}, \qquad \bar{u}_{2s}(y) = n_{2s}^{-1} u_{2s}(y)$$

$$(3.4)\quad u_{2s+1}(x) = \int_a^b K(x, y) \bar{u}_{2s}(y)\, d\alpha(p), \qquad s = 0, 1, 2, \cdots$$