RELATIONS BETWEEN THE ECCENTRICITIES AND INCLINATIONS OF THE ORBITS OF THE PLANETS JUPITER AND SATURN.

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It is well known that the eight major planets revolve around the Sun in nearly the same plane and in elliptical orbits which do not depart greatly from circularity.

In the cases of the Earth, Mars, Jupiter, and Saturn, the equators of these planets are not greatly inclined to the common plane of the orbits; the equators of Mercury, Venus, Uranus, and Neptune have not been seen, but from the motion of the satellites of Uranus and Neptune it is inferred that their equators have large inclinations.

The precise calculations of astronomers, which unfortunately can only extend to a few hundreds of years, agree with observation in showing that the eccentricities and inclinations of the various orbits are changing slowly. Lagrange, Laplace, and Poisson have proved that the mean-distances of the planets are essentially invariable, and the two former have, by very rough and inexact methods, shown that the sums of simple functions of the eccentricities and inclinations of all the planets will always remain small, but on account of the preponderating masses of the four outer planets the equations prove nothing as to the stability of the orbits of the four inner planets, of which group the Earth is one. It cannot to-day be proved that the orbit of such a planet as the Earth is stable. The large uniformity of flora and fauna for great ages gives rise to a no doubt well-founded belief that the Earth's orbit is stable, although there may be no mathematical proof yet available.

The case is different with the planets Jupiter and Saturn; the perturbations which these planets undergo, due to the presence of the other planets, are so insignificant compared to their own action on each other that they can be considered apart. In fact, the preponderating masses of the Sun, Jupiter, and Saturn in the solar system make these three bodies a nearly ideal case of the "problem of three bodies," and in consequence of this the rigorous integrals given by Jacobi in his celebrated paper on the " Elimination of the Nodes " are closely approximated to.*

If we adopt the notation for Jupiter—

ι = inclination of orbit to the invariable plane,
e = eccentricity of orbit and $S\phi = e$,
m = mass,
a = ratio of Jupiter's mean distance to Saturn's,

and use accented symbols for the same quantities when applied to Saturn, and take the constant of attraction as unity, we have (see Charlier, "Mechanik des Himmels" formulæ of 14,* p. 274 of vol. i.)—

$$\beta\sqrt{a}C\phi S\iota - \beta_1 C\phi_1 S\iota_1 = 0 \qquad \text{(A)}$$
$$\beta\sqrt{a}C\phi C\iota + \beta_1 C\phi_1 C\iota_1 = c_1,$$

in which—

$$\beta^2 = \frac{m^2}{1+m}, \qquad \beta_1^2 = \frac{m_1^2(1+m)}{1+m+m_1},$$

and C and S are written for cosine and sine, and c_1 is a constant. Taking—

$$1/m = 1047{\cdot}3, \qquad 1/m_1 = 3501{\cdot}6,$$
$$L\sqrt{a} = 9{\cdot}8682770\text{–}10,$$

and dividing the equation (A) by $\beta\sqrt{a}$, we get—

$$C\phi S\iota - qC\phi_1 S\iota_1 = 0 \qquad Lq = 9{\cdot}6076936\text{–}10$$
$$C\phi C\iota + qC\phi_1 C\iota_1 = c \qquad q = 0{\cdot}4052225,$$

in which both q and c are absolute constants.

Dr. G. W. Hill found by a comparison between the places given by his theory of Jupiter and Saturn and those actually observed that the reciprocals of the masses were 1047·38 and 3502·2 respectively; these values would change the value of Lq to 9·6076314–10, so that one must not lay too much stress on the value of this constant.

If we square and add the two equations, we get—

$$C^2\phi + 2qC\phi C\phi_1 C(\iota + \iota_1) + q^2 C^2\phi_1 = c^2.$$

* The not too great inequality of the three masses is also essential, and exists in the case of these three bodies—the integrals would furnish no useful information in the case of Jupiter, Sun, and Jupiter's 8th satellite, because of the smallness of the mass of JVIII, compared with Jupiter and the Sun.

We take the following figures from Hill's tables :—

$$\begin{array}{ll} \mathrm{L}e = 8{\cdot}6835317{-}10 & \mathrm{LC}\phi = 9{\cdot}9994938{-}10 \\ \mathrm{L}e_1 = 8{\cdot}7486589{-}10 & \mathrm{LC}\phi_1 = 9{\cdot}9993165{-}10 \\ \iota + \iota_1 = 1^\circ\ 15'\ 20{\cdot}9'' & \mathrm{LC}(\iota+\iota_1) = 9{\cdot}9998957{-}10 \end{array}$$

and thus have for the value of the constant c—

$$\mathrm{L}c = 0{\cdot}1471664 \qquad c = 1{\cdot}4033512$$
$$c^2 = 1{\cdot}9693950$$

We can now make a rough discussion of the last equation, viz.—

$$\mathrm{C}^2\phi + 0{\cdot}81\ \mathrm{C}\phi \mathrm{C}\phi_1 \mathrm{C}(\iota+\iota_1) + 0{\cdot}16\ \mathrm{C}^2\phi_1 = 1{\cdot}969.$$

As no cosine can exceed unity, and $\mathrm{C}\phi$, $\mathrm{C}\phi_1$, and $\mathrm{C}(\iota+\iota_1)$ are essentially positive, it is at once evident that at all times the C quantities must each be less than but yet exceedingly close to unity. If each is taken as unity the equation fails to balance, but only just fails, for we then have—

$$1 + 0{\cdot}810445 + 0{\cdot}164205 = 1{\cdot}974650,$$
whilst c^2 is actually $= 1{\cdot}969395$.

Thus the eccentricities and inclinations of the orbits of these two planets can never deviate greatly from their present small values.

Both ι and ι_1 can be determined in terms of e and e_1, and *vice versâ.* We have, in fact—

$$\mathrm{S}\iota/2 = \sqrt{\frac{(q\mathrm{C}\phi_1 + \mathrm{C}\phi - c)\,(q\mathrm{C}\phi_1 - \mathrm{C}\phi + c)}{4c\mathrm{C}\phi}}$$

$$\mathrm{S}\iota_1/2 = \sqrt{\frac{(q\mathrm{C}\phi_1 + \mathrm{C}\phi - c)\,(\mathrm{C}\phi - q\mathrm{C}\phi_1 + c)}{4cq\mathrm{C}\phi_1}}$$

$$\mathrm{C}\phi = c\frac{\mathrm{S}\iota_1}{\mathrm{S}(\iota+\iota_1)}$$

$$\mathrm{C}\phi_1 = \frac{c}{q}\,\frac{\mathrm{S}\iota}{\mathrm{S}(\iota+\iota_1)}$$

The first pair of these equations are ill-suited for finding the inclinations, because $q\mathrm{C}\phi_1 + \mathrm{C}\phi$ is very nearly equal to c. It is better to find ι and ι_1 by trial from the first equation of (A) ; we thus find—

$$\left.\begin{array}{l} \iota = 21'\ 16{\cdot}6'' \\ \iota_1 = 54'\ 4{\cdot}3'' \end{array}\right\}\ 1850.$$

If we adopt the variations in e and e_1 and ι and ι_1 found by other investigators,* we can draw some further inference from the equation (A). Hill ("Collected Works," iv., p. 127) has discussed the maximum and minimum values of e and e_1, and he points out that the maximum eccentricity of one orbit corresponds with the minimum eccentricity of the other, and *vice versâ.* From Hill's figures we get—

$$\begin{array}{ll} LC\phi\ (\text{min.}) = 9{\cdot}9991924{-}10 & LC\phi_1\ (\text{max.}) = 9{\cdot}9999705{-}10 \\ LC\phi\ (\text{max.}) = 9{\cdot}9998597{-}10 & LC\phi_1\ (\text{min.}) = 9{\cdot}9984324{-}10 \end{array}$$

If we write the first equation of (A) thus—

$$S\iota = q\frac{C\phi_1}{C\phi}S\iota_1,$$

and substitute the above values of $C\phi$ and $C\phi_1$, we get—

$$\begin{array}{ll} S\iota = 0{\cdot}4044\ S\iota_1. & \text{Min. of variable factor.} \\ \quad = 0{\cdot}4051\ S\iota_1. & \text{Present value (1850).} \\ \quad = 0{\cdot}4054\ S\iota_1. & \text{Max. value.} \end{array}$$

These equations show that ι and ι_1 increase or decrease together, and that with considerable precision we have at all times—

$$\Delta\iota = 0{\cdot}405\Delta\iota_1.$$

It must be remembered that $\Delta\iota$ and $\Delta\iota_1$ are the variations of the inclinations to the invariable plane of the system.

Stockwell ("Smithsonian Contributions to Knowledge," xviii., 1870) gives the following figures:—

$$\begin{array}{lllll} \text{Maximum value of } \iota + \iota_1 = 1^\circ & 29' & 35'' & \\ \text{Minimum value of } \iota + \iota_1 = 1 & 1 & 39 & \end{array}$$

$$\begin{array}{lllll} \text{to which we add the present value} & = 1 & 15 & 20{\cdot}9 & (1850) \\ \text{Value 2,000 years hence (Le Verrier)} & = 1 & 15 & 1 & (3850) \end{array}$$

These values show that with great accuracy we can write the second equation of (A) in the approximate form—

$$C\phi + qC\phi_1 = c \qquad \left\{ \begin{array}{l} q = 0{\cdot}405 \\ c = 1{\cdot}403 \end{array} \right.$$

or making e and e_1 variable—

$$e\Delta e + qe_1\Delta e_1 = 0.$$

* The argument is complete without these, and logically they should not be used; but their use simplifies the reasoning.

This simple formula shows, as Hill has already pointed out from other considerations, that as e increases or diminishes e_1 decreases or increases, and that when e is at a maximum e_1 is at a minimum, and *vice versa*. It further shows that the variations of the eccentricities preserve the ratio $-0{\cdot}405 \times \frac{e_1}{e}$. Hill's Tables of Jupiter and Saturn do not give the variations of the eccentricities directly, but from them I have found the centennial variations to be—

$$\begin{array}{ll} \text{Jupiter} & +34{\cdot}06'' \\ \text{Saturn} & -71{\cdot}63 \end{array}$$

These figures give a ratio of $-0{\cdot}409\frac{e_1}{e}$, which is a close accordance, especially when it is remembered that Hill's variations include the action of Uranus and the other planets.

It is remarkable that the ratio of the mean motion of Jupiter to that of Saturn ($=0{\cdot}4028$) is so close to the value of q ($=0{\cdot}4052$) and that the present ratio of the great inequality in the mean longitude of Saturn to the corresponding inequality of Jupiter (by Hill's tables $=0{\cdot}4113$) is again not widely different from q. These approximations may be fortuitous.

Summary.

The following relations, which are practically rigorous, connect the variations of the eccentricities and inclinations to the invariable plane of the planets Jupiter and Saturn :—

$$\begin{array}{rl} \text{Jupiter.} & \text{Saturn.} \\ \Delta e = & -0{\cdot}405\frac{e_1}{e}\Delta e_1. \\ \Delta\iota = & +0{\cdot}405\Delta\iota_1. \end{array}$$