ON THE ZEROS OF A CERTAIN FUNCTION INVOLVING BESSEL FUNCTIONS.

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With two Text-figures.

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In a paper published some time ago (Griffith, 1957) I stated, without proof, some facts concerning the distribution of the zeros of

$$w(z,\, C,\, \nu)\equiv w(z)\equiv zH^{(1)}_{\nu+1}(z)-CH^{(1)}_{\nu}(z), \quad \nu\geqq 0, \qquad \ldots\ldots \quad (1)^*$$

where C is a *real* constant. The results quoted were sufficient for the needs of the paper. I submit in what follows an analysis of the zeros of $w(z)$ for $-\frac{1}{2}\pi\leqq\arg z\leqq\frac{3}{2}\pi$, which will include a statement of, and a proof of, the previous assertions. It will be obvious that our conclusions can be modified trivially to give information concerning the zeros of $zK_{\nu+1}(z)-CK_{\nu}(z)$ in the region $-\pi<\arg z<\pi$.

Suppose that $z_0=re^{i\alpha}$, $-\frac{1}{2}\pi\leqq\alpha\leqq\frac{1}{2}\pi$ is a zero of $w(z)$. Then writing $\bar{z}=re^{-i\alpha}$, we see that

$$z_0H^{(1)}_{\nu+1}(z_0)-CH^{(1)}_{\nu}(z_0)=0$$

and

$$\bar{z}_0H^{(2)}_{\nu+1}(\bar{z}_0)-CH^{(2)}_{\nu}(\bar{z}_0)=0$$

Then by Erdélyi (1953, p. 80, (43)) we obtain

$$-\bar{z}_0e^{i\pi(\nu+1)}H^{(1)}_{\nu+1}(\bar{z}_0e^{i\pi})+Ce^{i\pi\nu}H^{(1)}_{\nu}(\bar{z}_0e^{i\pi})=0$$

that is

$$(\bar{z}_0e^{i\pi})H^{(1)}_{\nu+1}(\bar{z}_0e^{i\pi})-CH^{(1)}_{\nu}(\bar{z}e^{i\pi})=0$$

Since the order of these equations may be reversed, we observe that the *zeros of* w(z) *are symmetrically placed with regard to the imaginary axis.*

We now show that *if a multiple zero of* w(z) *occurs, it must lie on one of the axes.*

Banerjee has proved that $H^{(1)}_{\nu+1}(z)$ and $H^{(1)}_{\nu}(z)$ have no common zero (quoted in Erdélyi, 1953, p. 62). Thus it immediately follows that no zero of $H^{(1)}_{\nu}(z)$ will coincide with a zero of $w(z)$.

Now $w(z)$ may be written in either of the forms

$$w(z)=zH^{(1)}_{\nu+1}(z)-CH^{(1)}_{\nu}(z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2a)$$

or

$$w(z)=-zH^{(1)}_{\nu-1}(z)+(2\nu-C)H^{(1)}_{\nu}(z) \qquad \ldots\ldots\ldots\ldots \quad (2b)$$

(Watson, 1953, p. 74).

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* The Bessel Functions $J_\nu(z)$, $Y_\nu(z)$, $H^{(1)}_\nu(z)$, $H^{(2)}_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ used in this paper are those defined by Watson (1953, pp. 40, 64, 73, 77 and 78). We will write $w(z)$ whenever it is not necessary to specify C and ν.

Now, if $w(z)$ has a multiple zero, z_0, then z_0 is a zero of both $z^\nu w(z)$ and $d[z^\nu w(z)]/dz$. Thus

$$z_0^\nu[-z_0 H_{\nu-1}^{(1)}(z_0)+(2\nu-C)H_\nu^{(1)}(z_0)]=0$$

and

$$z_0^\nu[z_0 H_\nu^{(1)}(z_0)-CH_{\nu-1}^{(1)}(z_0)]=0$$

(Watson, 1953, p. 74).

Then eliminating $\mathrm{H}_{\nu-1}^{(1)}(z_0)$ from these two equations, we have

$$z_0^\nu[z_0^2+C(C-2\nu)]H_\nu^{(1)}(z_0)=0.$$

If we delete the branch point from our consideration and recall that $H_\nu^{(1)}(z_0)\neq 0$, we see that $z_0^2+C(C-2\nu)=0$.

Now C and ν are real, and we see that this proves our assertion.

To obtain many of our results we write

$$w(z)=zH_\nu^{(1)}(z)\zeta(z),$$

where

$$\zeta(z)=\frac{H_{\nu+1}^{(1)}(z)}{H_\nu^{(1)}(z)}-\frac{C}{z}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

and examine the change in arg $\zeta(z)$ as z passes around certain contours.

Thus, account must be taken of the zeros of $H_\nu^{(1)}(z)$. Combining information supplied in Erdélyi (1953, p. 62) and Watson (1953, p. 511), we obtain

A. (*a*) $H_\nu^{(1)}(z)$ has no zeros if $0\leqq \arg z<\pi$.

(*b*) If $\nu-\frac{1}{2}$ is an even integer $2k$, then $H_\nu^{(1)}(z)$ has k zeros in each of the regions $-\frac{1}{2}\pi<\arg z<0$ and $\pi<\arg z<\frac{3}{2}\pi$.

(*c*) If $\nu-\frac{1}{2}$ is an odd integer $2k-1$, then $H_\nu^{(1)}(z)$ has $k-1$ zeros in each of the regions $-\frac{1}{2}\pi<\arg z<0$ and $\pi<\arg z<\frac{3}{2}\pi$ and a single zero on the negative imaginary axis.

(*d*) If $\nu-\frac{1}{2}$ is not an integer and $2k$ is the nearest even integer, then $H_\nu^{(1)}(z)$ has k zeros in each of regions $-\frac{1}{2}\pi<\arg z<0$ and $\pi<\arg z<\frac{3}{2}\pi$.

Analysis of the case $C=2\nu$ is somewhat trivial. Here we find that

$$w(z)=-zH_{\nu-1}^{(1)}(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Thus if $C=2\nu$, *all the zeros of* $\mathrm{H}_{\nu-1}^{(1)}(\mathrm{z})$ *are zeros of* $\mathrm{w}(\mathrm{z})$. Further, by examining the behaviour of $w(z)$ in the neighbourhood of the origin, it will be found that $\mathrm{w}(0, 2\nu, \nu)=0$ *only for* $0<\nu<2$.

It is only in this special case that the origin is a zero of $w(z, C, \nu)$, since if $C\neq 2\nu$ we find that

$$w(z)\sim\begin{cases} i\pi^{-1}[C\log z-2][1+o(1)], & \nu=0 \\ i\pi^{-1}[\Gamma(\nu)2^\nu z^{-\nu}][2\nu-C][1+o(1)], & \nu\neq 0 \end{cases}$$

as $|z|\to 0$.

We tabulate first some of the formulæ to be used later. To obtain these we use Erdélyi (1953), p. 4 (4), (5) ; p. 5 (15) ; p. 8 (32) ; p. 80, (35), (39)(, (42) ; and p. 85 (1).

$$\frac{H_{\nu+1}^{(1)}(z)}{H_\nu^{(1)}(z)}\sim i[1+O(|z|^{-1})] \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

as $|z| \to \infty$;

$$\zeta(z) \sim \begin{cases} \left[\dfrac{2}{z \log z} - \dfrac{C}{z}\right][1+o(1)], & \nu=0 \\ \dfrac{2\nu - C}{z}[1+o(1)], & \nu \neq 0 \end{cases} \qquad \ldots\ldots\ldots\ldots \quad (6)$$

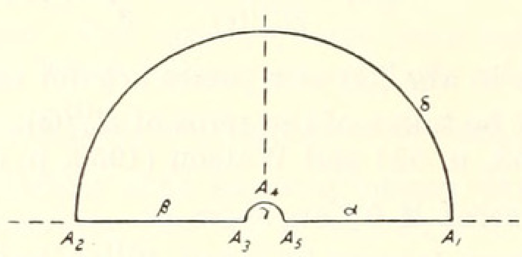
as $|z| \to 0$;

when $z=x>0$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{J_{\nu+1}(x)J_\nu(x)+Y_{\nu+1}(x)Y_\nu(x)-2i(\pi x)^{-1}}{[J_\nu(x)]^2+[Y_\nu(x)]^2}\,; \quad \ldots\ldots \quad (7)$$

when $z=re^{i\pi}$, $r>0$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{-J_{\nu+1}(r)J_\nu(r)-Y_{\nu+1}(r)Y_\nu(r)-2i(\pi r)^{-1}}{[J_\nu(r)]^2+[Y_\nu(r)]^2} \quad \ldots\ldots \quad (8)$$



Text-fig. 1.

when $x=te^{\frac{1}{2}\pi i}$, $t>0$

$$w(z) = (\tfrac{1}{2}\pi i)^{-1}e^{-\frac{1}{2}\nu\pi i}[tK_{\nu+1}(t)-CK_\nu(t)]\,; \quad \ldots\ldots\ldots\ldots \quad (9)$$

when $z=ue^{-\frac{1}{2}\pi i}$, $u>0$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{\pi \cos \nu\pi \,.\, u^{-1} - iP}{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where

$$P = \{\pi^2 I_{\nu+1}(u)I_\nu(u) - K_{\nu+1}(u)K_\nu(u)\} + \pi \sin \nu\pi\{I_{\nu+1}(u)K_\nu(u) - I_\nu(u)K_{\nu+1}(u)\}$$

and

$$Q = [\pi I_\nu(u) + \sin \nu\pi K_\nu(u)]^2 + \cos^2 \nu\pi[K_\nu(u)]^2.$$

Since we have completed the case $C=2\nu$, we will assume in what follows that $C \neq 2\nu$. Since C/z is real on the real axis, equations (7) and (8) show that $\zeta(z)$ does not vanish on the real axis. Thus w(z) *does not have a zero on the real axis.* Similarly, since C/z is imaginary on the negative imaginary axis, equation (10) shows that if $\cos \nu\pi \neq 0$ (i.e. $\nu - \frac{1}{2}$ does not equal an integer), $w(z)$ does not have a zero on the negative real axis.

We now determine the number of zeros above the real axis by examining the increase of arg $\zeta(z)$ as z passes around the contour in Figure 1.

It will be assumed that the large semicircle δ (with centre the origin) is sufficiently large for the estimate (5) to be valid and the small semicircle γ to be small enough for the estimate (6) to hold.

It is then easy to see that the values of arg $\zeta(z)$ are given by the following table.

	$C<2\nu$	$C>2\nu$
A_1	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$
A_2	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$
A_3	$-\pi$	0
A_4	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi$
A_5	0	π
A_1	$-\frac{1}{2}\pi$	$\frac{3}{2}\pi$

The increase in arg $\zeta(z)$ is zero if $C<2\nu$ and is 2π if $C>2\nu$. Thus, referring back to A(a) above and recalling the symmetry of the zeros, we conclude that

If $C<2\nu$, $w(z)$ *has no zero above the real axis.*

If $C>2\nu$, $w(z)$ *has one and only one zero above the real axis. This is a simple zero, which lies on the positive imaginary axis.*

In view of equation (9), we see that we have proved incidently a rather obvious result which we will need later, *viz.*: $tK_{\nu+1}(t)-CK_\nu(t)$ *has one and only one real positive zero if* $C>2\nu$ *and no real positive zero if* $C<2\nu$.

Now the recurrence formulæ (Watson, 1953, p. 79) show that

$$tK_{\nu+1}(t)-CK_\nu(t)=tK_{\nu-1}(t)-(C-2\nu)K_\nu(t).$$

If we sketch the graphs of $tK_{\nu-1}(t)$ and $(C-2\nu)K_\nu(t)$ it is immediately obvious that as $C-2\nu$ increases from 0 to ∞, the zero moves from the origin to ∞. The asymptotic formulæ for the Bessel functions show that for large C the zero approximates to $C-\nu-\frac{1}{2}$.

We now proceed to determine the distribution of the zeros of $w(z)$, which lie below the real axis.

We first assume that $\nu-\frac{1}{2}$ is not an integer. Thus $\cos\nu\pi\neq 0$, and so $w(z)$ will have no zeros on the negative imaginary axis.

Keeping Figure 1 in mind, the description of Figure 2 is obvious (See page 194).

As z passes around the contour in Figure 2, the values of arg $\zeta(z)$ are given by the following table:

	$C<2\nu$		$C>2\nu$
	$\cos\nu\pi>0$	$\cos\nu\pi<0$	
A_6	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$
A_1 ..	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$
A_5	0	0	$-\pi$
A_7	$\frac{1}{2}\pi$	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$
A_6	$-\frac{1}{2}\pi$	$\frac{3}{2}\pi$	$-\frac{1}{2}\pi$

Thus arg $\zeta(z)$ is unchanged except when $C<2\nu$ and $\cos\nu\pi<0$; in which case the increase is 2π. So if $C<2\nu$ and $\cos\nu\pi<0$, the number of zeros of $w(z)$ in $-\frac{1}{2}\pi<\arg z<0$ is one more than the number of zeros of $H_\nu^{(1)}(z)$ in that region. Otherwise the number of zeros of $w(z)$ and $H_\nu^{(1)}(z)$ in $-\frac{1}{2}\pi<\arg z<0$ is the same.

Then referring back to A(d) we obtain

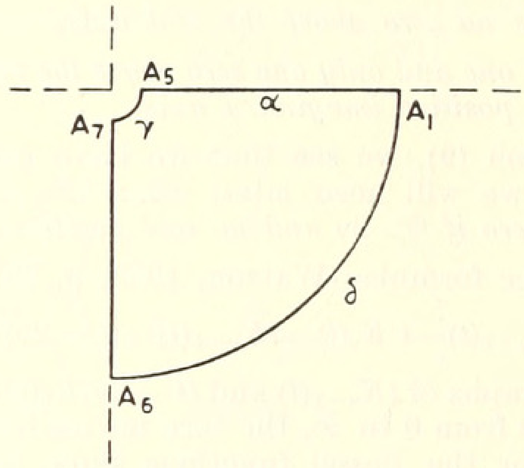
If $C>2\nu$, *then* $w(z)$ *has* k *zeros in* $-\frac{1}{2}\pi<arg\, z<0$ (*and in* $\pi<arg\, z<\frac{3}{2}\pi$) *provided* $2k-\frac{1}{2}<\nu<2k+\frac{1}{2}$ *or* $2k+\frac{1}{2}<\nu<2k+1\frac{1}{2}$.

If $C<2\nu$, *then* $w(z)$ *has* k *zeros in* $-\frac{1}{2}\pi<arg\, z<0$ (*and in* $\pi<arg\, z<\frac{3}{2}\pi$) *provided* $2k-1\frac{1}{2}<\nu<2k-\frac{1}{2}$ *or* $2k-\frac{1}{2}<\nu<2k+\frac{1}{2}$.

We now assume that $\nu-\frac{1}{2}=n$ (n, an integer). Thus $\cos \nu\pi=0$.

We cannot use a method similar to that used above, since there may be zeros on the negative imaginary axis.

Using Erdélyi (1953), p. 78 (90) to determine the explicit expansion for $w(z, C, n+\frac{1}{2})$, we observe that it may be expressed as the product of a factor which has no finite zero and a polynomial of degree $n+1$.



Text-fig. 2.

Thus $w(z)$ must have $n+1$ zeros. If $C>2\nu$, one only of these must lie above the real axis, and if $C<2\nu$, then all must lie below the real axis. If we determine the number of zeros which lie on the axis, the remainder will be symmetrically placed on either side.

We write $z=ue^{i\pi}$, $u>0$ and use Erdélyi (1953), p. 5 (15) and p. 80 (45) and Watson (1953), p. 79 to put $w(z)$ in the following forms:
if $\nu-\frac{1}{2}=2k$ (k an integer)

$$w(z)=2\pi^{-1}e^{-(k+\frac{1}{4})\pi i}p(u) \quad \ldots\ldots\ldots\ldots\ldots \quad (11a)$$

with

$$p(u)=[uK_{\nu+1}(u)-CK_\nu(u)]-\pi[uI_{\nu+1}(u)+CI_\nu(u)] \quad \ldots\ldots \quad (11b)$$

$$=[uK_{\nu-1}(u)-(C-2\nu)K_\nu(u)]-\pi[uI_{\nu-1}(u)+(C-2\nu)I_\nu(u)]\ldots \quad (11c)$$

and if $\nu-\frac{1}{2}=2k-1$ (k an integer)

$$w(z)=-2\pi^{-1}e^{-(k+\frac{1}{4})\pi i}q(u) \quad \ldots\ldots\ldots\ldots \quad (12a)$$

with

$$q(u)=[uK_{\nu+1}(u)-CK_\nu(u)]+\pi[uI_{\nu+1}(u)+CI_\nu(u)] \quad \ldots\ldots\ldots\ldots \quad (12b)$$

$$=[uK_{\nu-1}(u)-(C-2\nu)K_\nu(u)]+\pi[uI_{\nu-1}(u)+(C-2\nu)I_\nu(u)]. \quad (12c)$$

Thus to find the zeros of $w(z)$ on the negative imaginary axis of z, we need only consider the zeros of $p(u)$ and $q(u)$ for positive u.

It will be seen that it is necessary to treat the cases $\nu-\frac{1}{2}$ and $\nu=1\frac{1}{2}$ separately.

With $\nu - \frac{1}{2}$, we have

$$w(z, C, \tfrac{1}{2}) = -(\tfrac{1}{2}\pi z)^{-\frac{1}{2}} e^{iz}[z - i(C-1)]$$

with its only zero at $i(C-1)$.

When $\nu = 1\frac{1}{2}$,

$$w(z, C, 1\tfrac{1}{2}) = i(\tfrac{1}{2}\pi z^3)^{-\frac{1}{2}} e^{iz}[z^2 + iz(3-C) - (3-C)]$$

and the explicit formula for the zeros may be written as

$$z_0 = -i\tfrac{1}{2}(3-C) \pm \tfrac{1}{2}(3-C)^{\frac{1}{2}}(1+C)^{\frac{1}{2}}.$$

Except that there is a zero at the origin when $C = 3$ $(= 2\nu)$, this indicates a typical result of the case $\nu = 2k - \frac{1}{2}$. We have :

when $C < -1$, there are two negative imaginary zeros ;

when $C = -1$, there is a double imaginary zero (at $z_0 = -2i$) ;

when $-1 < C < 3$, there are no zeros in the imaginary axis ;

when $C > 3$, there are two imaginary zeros, one positive and one negative.

We now assume that $0 \leqq C < 2\nu$, $\nu \geqq 2\frac{1}{2}$. Then $uI_{\nu+1}(u) + CI_\nu(u)$ is obviously strictly monotonic, increasing from 0 to ∞, for increasing from 0 to ∞. Our previous work shows that $uK_{\nu+1}(u) - CK_\nu(u)$ will have no zeros and never become negative. Thus $q(u)$ will have no zeros.

Using Watson (1953, p. 70), we find that

$$uK_{\nu+1}(u) - CK_\nu(u) = uK_{\nu-1}(u) + (2\nu - C)K_\nu(u)$$

and that

$$\frac{d}{du}[uK_{\nu-1}(u)] = -[uK_\nu(u) - \nu K_{\nu-1}(u)]$$

which has no zeros for $\nu < 2(\nu - 1)$. Thus $uK_{\nu+1}(u) - CK_\nu(u)$ is strictly monotonic decreasing to zero.

Thus $p(u)$ has one and only one zero.

We now assume that $C > 2\nu$, $\nu \geqq 2\frac{1}{2}$, and consider

$$u^\nu[uK_{\nu+1}(u) - CK_\nu(u)] \equiv s(u) \equiv r(v)$$

as a function of $v = u^2$. Then

$$\frac{dr}{dv} = -2u^{\nu-1}[uK_\nu(u) - CK_{\nu-1}(u)]$$

and

$$\frac{d^2r}{dv^2} = 4u^{\nu-2}[uK_{\nu-1}(u) - CK_{\nu-2}(u)].$$

So, obviously, r, dr/dv and d^2r/dv^2 each have one and only one (simple) zero. Then, keeping the asymptotic expressions for the Bessel functions in view, we observe that the graph of $y = r(v)$ starts at a point on the negative y-axis, increases steadily, and after cutting the v-axis passes through a maximum. It then decreases to an inflexional point, at which it changes from being concave downwards to being convex downwards and then finally approaches the v-axis from above.

Since $u^\nu[uI_{\nu+1}(u) + CI_\nu(u)]$ (as a function of v) is monotonic increasing from zero, it easily follows that $q(u)$ has one and only one zero, but that $p(u)$ may have no zero, a double zero or two zeros, but no more than two zeros.

Now sketching the graphs of $u[K_{\nu-1}(u) - \pi I_{\nu-1}(u)]$ and $(C - 2\nu)[K_\nu(u) + \pi I_\nu(u)]$, it will be seen that as $C - 2\nu$ increases from zero, one of the zeros of $p(u)$ will increase from zero, while the other will decrease from the zero of $K_{\nu-1}(u) - \pi I_{\nu-1}(u)$ to a common value $u_1(\nu)$* with corresponding $C = C_1(\nu)$.

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* We have written $u_1(\nu)$ and $C_1(\nu)$ to emphasize the fact that these values are dependent on ν.

For $C > C_1(\nu)$, there will be no zero on the negative imaginary axis.

Now at (u_1, C_1) we observe that $u^\nu p(u)$ and its derivative will have a common zero. Thus

$$[u_1 K_{\nu-1}(u_1) - (C_1 - 2\nu) K_\nu(u_1)] - \pi[u_1 I_{\nu-1}(u_1) + (C_1 - 2\nu) I_\nu(u_1)] = 0$$

and

$$[-u_1 K_\nu(u_1) + C_1 K_{\nu-1}(u_1)] - \pi[u_1 I_\nu(u_1) + C_1 I_{\nu-1}(u_1)] = 0.$$

Eliminating $K_\nu(u_1)$ and $I_\nu(u_1)$ from these equations, we find that

$$[u_1^2 - C_1(C_1 - 2\nu)][K_{\nu-1}(u_1) - \pi I_{\nu-1}(u_1)] = 0,$$

in which the second factor is not zero. Thus (u_1, C_1) can be found from the point of intersection of

$$u^2 - C(C - 2\nu) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

and

$$C = \frac{u[K_{\nu+1}(u) - \pi I_{\nu+1}(u)]}{K_\nu(u) + \pi I_\nu(u)} \quad \ldots\ldots\ldots\ldots\ldots \quad (14)$$

We now assume that $C < 0$. We may use a method similar to that just given. We will then find that $p(u)$ has one and only one zero, while $q(u)$ may have no zero, a double zero or two zeros.

As C increases from $-\infty$, then one zero of $q(u)$ will decrease from $+\infty$, while the other will increase from the single zero of $K_\nu(u) - \pi I_\nu(u)$ until they coincide at $u_2(\nu)$ with $C = C_2(\nu)$. As C increases further, there will be no zero until C passes 2ν. There will then be one zero (as shown above).

The values of $u_2(\nu)$ and $C_2(\nu)$ can be found from the point of intersection of (13) and

$$C = \frac{u[K_{\nu+1}(u) + \pi I_{\nu+1}(u)]}{K_\nu(u) - \pi I_\nu(u)} \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

Collecting the results, we now summarize.

Omitting the cases $\nu = \frac{1}{2}$ and $\nu = 1\frac{1}{2}$, which have been discussed above, the distribution of the zeros of $w(z)$ when $\nu - \frac{1}{2} = n$ (n an integer) is given by the following table.

		Negative Imaginary Axis.	Positive Imaginary Axis.	Regions. $-\frac{1}{2}\pi < \arg z < 0$ $\pi < \arg z < \frac{3}{2}\pi$
$\nu = 2k + \frac{1}{2}$	$C < 2\nu$ ..	1	0	k
	$2\nu < C \leqq C_1(\nu)$ ..	2	1	$k-1$
	$C > C_1(\nu)$ ..	0	1	k
$\nu = 2k - \frac{1}{2}$	$C \leqq C_2(\nu) < 0$ ..	2	0	$k-1$
	$C_2(\nu) < C < 2\nu$ ..	0	0	k
	$C > 2\nu$ ..	1	1	$k-1$

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