G. signatæ Stol. sat affinis differt corpore in parte granuloso (in *G. signata* omnino granulosa) et scuto abdominali flavo quadrivittato (in *G. signata* tantum flavo-marginato).

IX.—*On the Differential Equation of a Trajectory.*—*By* Asutosh Mukhopadhyay, M. A., F. R. A. S., F. R. S. E. *Communicated by* The Hon'ble Mahendralal Sarkar, M. D., C. I. E.

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§ 1. The problem of determining the oblique trajectory of a system of confocal ellipses, appears to have been first solved by the Italian Mathematician Mainardi, in a memoir in the *Annali di Scienze Mathematische e Fisiche*, t. I, page 251, which has been reproduced by Boole (*Differential Equations*, 4th edition, pp. 248-251). Representing half the distance between the foci by h, and the tangent of the angle of intersection by n, we obtain for the equation of the trajectory,

$$-2n\tan^{-1}\sqrt{\left(\frac{h^2}{x\mathrm{M}}-1\right)}+\log\frac{1-\sqrt{1-\frac{\mathrm{M}}{x}}}{1+\sqrt{1-\frac{\mathrm{M}}{x}}}=\mathrm{C}\ \ldots\ldots\ldots\ldots\ (1)$$

where C is the constant of integration, and M is given by the quadratic

$$(x^2+y^2+h^2)\mathrm{M}=x\,(\mathrm{M}^2+h^2)\ \ldots\ldots\ldots\ldots\ (2)$$

Now, this form of the equation is so complicated that it would be a hopeless task to have to trace the curve from it; indeed, it is so unsymmetrical and inelegant that Professor Forsyth in his splendid work on Differential Equations (page 131) does not at all give the answer. In the present note, the curve is represented by a pair of remarkably simple equations which admit of an interesting geometrical interpretation.

§ 2. Assume then

$$x\mathrm{M}=h^2\cos^2\phi$$
$$\mathrm{C}=2n\lambda$$

where λ is a new constant. Substituting in (1), we have

$$\log\frac{1-\sqrt{1-\frac{\mathrm{M}}{x}}}{1+\sqrt{1-\frac{\mathrm{M}}{x}}}=2n\lambda+2n\phi.$$

$$\therefore \quad \frac{1-\sqrt{1-\dfrac{M}{x}}}{1+\sqrt{1-\dfrac{M}{x}}} = e^{2n(\lambda+\phi)}$$

$$\therefore \quad \frac{1}{\sqrt{1-\dfrac{M}{x}}} = \frac{1+e^{2n(\lambda+\phi)}}{1-e^{2n(\lambda+\phi)}}$$

$$\therefore \quad 1-\frac{M}{x} = \left\{\frac{1-e^{2n(\lambda+\phi)}}{1+e^{2n(\lambda+\phi)}}\right\}^2$$

$$\therefore \quad \frac{M}{x} = 1-\left\{\frac{1-e^{2n(\lambda+\phi)}}{1+e^{2n(\lambda+\phi)}}\right\}^2 = \frac{4e^{2n(\lambda+\phi)}}{\left\{1+e^{2n(\lambda+\phi)}\right\}^2}.$$

But, $$M = \frac{h^2}{x}\cos^2\phi$$

$\therefore$ Substituting in the above and extracting the square root, we get

$$\frac{h}{x}\cos\phi = \frac{2e^{n(\lambda+\phi)}}{1+e^{2n(\lambda+\phi)}}$$

$$\therefore \quad x = h\cos\phi\,\frac{1+e^{2n(\lambda+\phi)}}{2e^{n(\lambda+\phi)}}$$

$$= h\cos\phi\,\tfrac{1}{2}\left\{e^{n(\lambda+\phi)}+e^{-n(\lambda+\phi)}\right\}$$

$$= h\cos\phi.\ \cosh n(\lambda+\phi).$$

Again, substituting the value of M in (2), we have

$$(x^2+y^2+h^2)\,\frac{h^2\cos^2\phi}{x} = x\left(\frac{h^4\cos^4\phi}{x^2}+h^2\right)$$

$$\therefore \quad x^2+y^2+h^2 = h^2\cos^2\phi+x^2\sec^2\phi$$

$$\therefore \quad y^2+h^2\sin^2\phi = x^2\tan^2\phi$$

$$\therefore \quad \frac{x^2}{h^2\cos^2\phi}-\frac{y^2}{h^2\sin^2\phi} = 1,$$

and, since we have shewn that

$$x = h\cos\phi.\ \cosh n(\lambda+\phi),$$

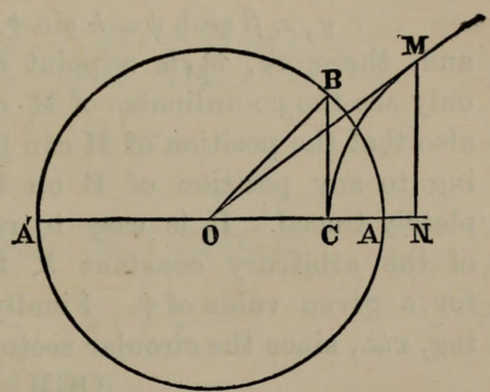
we see at once that

$$y = h\sin\phi\ .\ \sinh n(\lambda+\phi).$$

Therefore, the co-ordinates of any point on the trajectory may be expressed in a very neat and symmetrical form, in terms of a parameter ϕ *viz.*, we have

$$\left.\begin{aligned} x &= h \cos\phi.\ \cosh n(\lambda+\phi). \\ y &= h \sin\phi.\ \sinh n(\lambda+\phi). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (A).$$

§ 3. The equations (A) admit of a very simple geometrical interpretation.



Let A′A be the line joining the foci of the system of confocal ellipses, so that OA $= h$. On A′A as diameter, describe a circle having its céntre at O. Draw any radius OB, making the angle AOB $= \phi$; draw BC perpendicular to OA; then, we have OC $= h\cos\phi$, BC $= h \sin\phi$. Construct a hyperbola CM, having its centre at O, and its transverse and congugate axes equal to OC, BC respectively; then, of course, BC is a tangent and OB an asymptote to this hyperbola. Take a point M on the hyperbola, so that the area of the hyperbolic sector OCM may be $n(\lambda+\phi)$ times the area of the triangle OCB: then, I assert that M is a point on the trajectory, *viz.*, the co-ordinates of M are

$$\begin{aligned} x &= h\cos\phi.\ \cosh n(\lambda+\phi). \\ y &= h\sin\phi.\ \sinh n(\lambda+\phi). \end{aligned}$$

To see how this is, drop MN perpendicular on OA. Then, writing for the moment OC $= h\cos\phi = \alpha$, BC $= h\sin\phi = \beta$, the hyperbola is

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

and, any point (x_1, y_1) as M on this hyperbola is obviously satisfied by $x_1 = \alpha\cosh\psi$, $y_1 = \beta\sinh\psi$. Now, the area of the portion CMN is given

$$\text{by CMN} = \int_\alpha^{x_1} y\,dx = \frac{\beta}{\alpha}\int_\alpha^{x_1}\sqrt{x^2-\alpha^2}\,dx$$

$$= \frac{\beta}{2\alpha}\left\{x\sqrt{x^2-\alpha^2} - \alpha^2\log\left(x+\sqrt{x^2-\alpha^2}\right)\right\}_{x=\alpha}^{x=x_1}$$

$$= \frac{\beta}{2\alpha}\left\{\alpha^2\cosh\psi\sinh\psi - \alpha^2\log\frac{\alpha\cosh\psi+\alpha\sinh\psi}{\alpha}\right\}$$

$$[\because x_1 = \alpha\cosh\psi]$$

$$= \frac{\beta}{2\alpha}\left\{\alpha^2\cosh\psi\sinh\psi - \alpha^2\psi\right\}$$

$$= \tfrac{1}{2}\,\alpha\beta \cosh\psi \sinh\psi - \tfrac{1}{2}\,\alpha\beta\psi$$
$$= \tfrac{1}{2}\,\text{ON. NM} - \tfrac{1}{2}\,\alpha\beta\psi = \text{ONM} - \tfrac{1}{2}\alpha\beta\psi$$
$$\therefore\ \text{OCM} = \tfrac{1}{2}\alpha\beta\psi = \text{OCB.}\ \psi.$$

But, *ex hypothesi*, $\text{OCM} = \text{OCB.}\ n(\lambda+\phi)$

$$\therefore\ \psi = n(\lambda+\phi),$$

which shews that the co-ordinates of M are given by

$$x_1 = \alpha\cosh\psi = h\cos\phi.\ \cosh n(\lambda+\phi)$$
$$y_1 = \beta\sinh\psi = h\sin\phi.\ \sinh n(\lambda+\phi),$$

and, therefore, M is a point on the trajectory. We thus see that not only are the co-ordinates of M expressible in a very simple form, but also that the position of M can be determined geometrically, corresponding to any position of B on the circle; hence, the curve can be completely traced. It is easy to remark that whatever may be the value of the arbitrary constant λ, the point M lies on the hyperbola CM, for a given value of ϕ. Finally, a geometrical relation is worth noticing, *viz.*, since the circular sector $\text{AOB} = \tfrac{1}{2}h^2\phi$, we have from

$$\text{OCM} = n(\lambda+\phi)\ \text{OBC},$$

the equation

$$\text{OCM} = n\lambda\ \text{OBC} + 2n\,\frac{\text{OAB. OBC}}{h^2}$$

or,

$$\text{OCM} = n\lambda\ \text{OBC} + 2n\,\frac{\text{OAB}}{\text{OA}}.\ \frac{\text{OBC}}{\text{OB}}.$$