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THE DRAG AND SHAPE OF AIR BUBBLES MOVING IN LIQUIDS

by

Benjamin Rosenberg

September 1950

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THE DRAG AND SHAPE OF AIR BUBBLES MOVING IN LIQUIDS

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Benjamin Rosenberg

ABSTRACT

A survey is made of the available information on the motion of air bubbles in liquids. These results show considerable scatter and uncertainty. Therefore, some tests were repeated in order to determine the rate of rise of air bubbles in water.

Dimensional analysis is used to obtain the parameters which can best be used to describe the results. These are the drag coefficient, the Reynolds number, and a third parameter which involves the properties of the liquid only. The dependence of the drag, shape, and path of the bubble on the Reynolds number is determined. Since the tests were confined to one liquid, the importance of the third parameter could not be ascertained. The results indicate that this parameter may have considerable influence upon the motion and shape of a bubble for a large range of Reynolds numbers.

INTRODUCTION

The tests described in this report were initiated because of the need for these data in conjunction with the work under project NS713-201 and NE051-237.

The behavior of bubbles in pressure gradients is of some interest in hydrodynamic research, having been considered in connection with a variety of problems.^{1,2,3} Since a gravity field is the simplest pressure field in which to study bubble shape and motion, the experiments consisted of the determination of the velocity of rise of air bubbles in water along with observations of the shapes and paths of the rising bubbles as a function of their size. In addition, some theoretical discussion is given concerning the parameters which determine the bubble characteristics. Pertinent data from other experiments are included and discussed.

The results show that the velocity of rise, bubble shape, path, and general kinematic behavior depend upon the bubble size, the pressure gradient, the density and viscosity of the medium and, to a lesser extent not fully indicated by available data, upon the surface tension.

¹References are listed on page 22.

HISTORICAL BACKGROUND

Considerable theoretical and experimental work has been done on the motion of bubbles in liquids. However, no single report presents a comprehensive discussion of the subject. Most of the theoretical work has been confined to a dimensional analysis with suggested solutions for the rate of rise for bubbles in the Stokes region of flow. The experimental work has been devoted to measurements of the rate of rise of bubbles in various liquids with some observations as to the bubble shape and path but there is considerable question concerning the reliability of some of the techniques involved.

DIMENSIONAL ANALYSIS

In nearly all papers on the motion of bubbles in liquids, the authors have recognized the following factors as pertinent:

- U the velocity of rise of the bubble,
- g the acceleration of gravity,
- ρ the density of the liquid,
- l the length parameter indicative of the bubble size,
- μ the coefficient of viscosity of the liquid, and
- γ the surface tension for the bubble-liquid interface.

Although this list does not complete the set of quantities needed to specify the system, the effect of other factors such as absolute pressure and the viscosity and density of the gas constituting the bubble are believed to be negligible. A convenient length parameter is the radius of the bubble. Since large bubbles are not spherical, they are described in terms of the equivalent spherical radius r_e which is the radius of a sphere having the same volume as the bubble.

Applying the methods of dimensional analysis,⁴ it is possible to group these factors into three dimensionless parameters in terms of which the relation among the six physical quantities can be described. The usual choice of combinations results in a statement of the following form:

$$f_1\left(\frac{gr_e}{U^2}, \frac{2Ur_e}{\mu/\rho}, \frac{\rho r_e U^2}{\gamma}\right) = 0 \quad [1]$$

For bubbles rising at their terminal velocity in liquids, the first ratio will now be shown to be related to the drag coefficient

$$C_D \equiv \frac{\text{Drag force}}{\frac{1}{2}\rho U^2 A} \quad [2]$$

where C_D is the drag coefficient, and A is the projected area. For convenience, A can be redefined to be the projected area of a sphere of equal volume. In addition, the drag force for a bubble at its terminal velocity is equal to the buoyant force $V\rho g$ where V is the volume of the bubble. Hence, in terms of the equivalent radius, the expression for the drag coefficient reduces to

$$C_D = \frac{8}{3} g \frac{r_e}{U^2} \quad [3]$$

The above expression for C_D is, except for a numerical factor, the first dimensionless parameter of Equation [1]. More generally, since the essential effect of gravity is to produce a uniform pressure gradient ρg , the drag coefficient for a bubble at terminal velocity may be written $8/3 r_e \nabla p / \rho U^2$ where ∇p is any pressure gradient in which the bubble may be placed. The second and third ratios are the Reynolds and Weber numbers respectively. Therefore the relation sought is of the form

$$f_2(C_D, Re, W) = 0 \quad [4]$$

No attempt was made to determine theoretically the form of f_2 except in the Stokes region of flow, that is, for very small spherical bubbles, where inertial terms may be neglected.^{5,6} These have not proven satisfactory beyond a Reynolds number of 1. Actually, up to a Reynolds number of 70, bubbles in water behave exactly like rigid spheres.

Since a theoretical solution to the hydrodynamic equation of motion for a bubble at its terminal velocity in a liquid does not appear feasible, methods for the empirical solution of Equation [4] are indicated. However, the form of these parameters is inconvenient for presenting the data as ordinarily obtained. The experimental procedure is to vary the size of the bubble and measure the resulting velocity of rise. If, instead of the Weber number, we chose a dimensionless ratio not involving the bubble size or speed, presentation and interpretation of the results are facilitated. A suitable parameter⁷ is $g\mu^4/\rho\gamma^3$. This new parameter will be denoted by the symbol M . As explained before, the function of the gravitational field is to produce a pressure gradient. Therefore M can be rewritten as $\mu^4 \nabla p / \rho^2 \gamma^3$.

The experimental data may then be presented as

$$f_3(C_D, Re, M) = 0 \quad [5]$$

An empirical solution of Equation [5] can be obtained by determining the rate of rise as a function of bubble size in different liquids.

PREVIOUS EXPERIMENTAL WORK

Much experimental work has been done on the rise of bubbles in liquids. However, when an attempt was made to compile results from various sources, it was found that the data were in poor agreement. The most successful work appears to have been done by Allen⁸ who determined the rate of rise of very small bubbles in water and aniline.

The scatter and unreliability of the previous results may be due to several causes:

1. Temperature Fluctuations. The parameter M contains the viscosity of the medium as one of its factors. Since the viscosity varies considerably with temperature and it is desirable to maintain M constant for any series of tests, temperature control is important.

2. Inaccurate Methods of Measurement. The two important measurements are the rate of rise and the size of the bubbles. Rate of rise was often determined by timing the ascent of the bubble over a predetermined distance by means of a stop watch. Most frequently the bubble size was measured by collecting the bubble in an inverted graduated cylinder and reading the volume displaced, the usual uncertainty of measurement being about 0.1 cc. These techniques were not considered very satisfactory for the accuracy desired, particularly for non-spherical bubbles. Allen obtained results for spherical bubbles only. He caught the individual bubble and then measured the radius by means of a microscope equipped with a micrometer eyepiece. The rate of rise was determined by use of a stop watch. For the small bubbles investigated, the terminal velocity is so low and the tube used was sufficiently long that a precise measurement of the rate of rise could be made.

3. Wall Effect. The influence of the walls of the tube is such as to reduce the terminal velocity of the bubbles. For bubbles in the viscous region of flow a correction to the terminal velocity often used was that developed by Ladenberg⁹

$$U_{\infty} = \left(1 + 2.4 \frac{r}{R'}\right) U \quad [6]$$

where U is the terminal velocity of the bubble in the tube,

U_{∞} is the terminal velocity of the bubble in an infinite medium,

r is the radius of the bubble, and

R' is the radius of the tube.

Further experiments on rigid spheres at the Iowa Institute of Hydraulic Research¹⁰ indicate that a better correction is

$$U_{\infty} = \left[1 + \frac{9}{4} \frac{r}{R_1} + \left(\frac{9}{4} \frac{r}{R_1} \right)^2 \right] U \quad [7]$$

The nature of the correction for bubbles outside the Stokes region of flow is uncertain but should be of the same order of magnitude. Almost all previous tests were done in fairly confined tanks, which could result in errors of 4 or 5 percent in the terminal velocity.

4. Turbulence. Data on medium and large bubbles may show scatter because of the turbulence created by the passage of the previous bubbles.¹¹ This would affect the motion of the bubbles.

Because of the uncertainty of previous results, a series of tests was initiated at the David Taylor Model Basin in order to obtain data within the desired range of bubble sizes. Allen's data were considered reliable enough not to require duplication. However the correction for wall effect as given by Equation [7] was applied to Allen's measurements and then included in the Taylor Model Basin results. Although eventually a number of liquids may be tested, the tests described here were confined to water. This permitted the obtaining of data under very favorable conditions.

TEST APPARATUS AND PROCEDURE

The tests were performed in the transparent-wall tank, which is large enough to preclude wall effect and to insure a minimum of turbulence effects. The water temperature variation during the course of a day is negligible and is fairly small from day to day. Most of the tests were run at a water temperature of 67° F.

A sketch of the installation is shown in Figure 1. The smaller bubbles were generated by means of nozzles of various sizes attached to a long brass tube which in turn was connected to a compressed air line through a needle valve. The valve was used to regulate the supply of air so that the bubbles were formed and released at about 10-second intervals. The larger bubbles were formed by collecting a number of small bubbles in a large cup. The cup was then rotated in order to release a single bubble. Although this method was found to be best, it was still difficult to obtain a single large bubble by this dumping process. If the speed of rotation was too great, a large number of bubbles were released. If the speed was too slow, one large bubble with several satellites clinging to it or following it was obtained. This was also undesirable since the presence of the satellites

Figure 1 - Installation Used for Bubble Tests
in the Transparent-Wall Tank

affects the drag of the bubble and produces an error in measuring its size. With careful practice in manipulating the dumping mechanism, all but the largest bubbles could be generated without satellites.

Motion pictures of the rising bubbles were made on 35 mm film. Back lighting was used; a diffusing screen between the light source and the bubbles provided a bright background against which the bubbles were photographed. Film speeds of 24 to 60 frames per second were used. The camera was equipped with a timing light actuated by the 60-cycle current source. This impressed marks on the film from which the exact film speed could later be determined. In photographing the larger bubbles, the camera was located at such a distance as to make the height of the film frame represent about 12 inches in the plane of the bubbles. For the smaller bubbles, the camera was moved closer so that the field included only five inches. For each position of the camera an accurately ruled scale was placed in the plane of the bubbles and a few frames taken to provide a distance scale in measuring bubble positions from the developed film.

Readings for the displacement, shape and size of the bubble were obtained by means of the Bausch and Lomb comparator using a magnification of ten. The comparator can be used to measure displacements to within 0.01 cm. Successive frames showing the bubble can be placed on the screen and the position of the bubble determined for each. In order to determine the path and shape of the bubble as it rose, the images of the bubble on the screen were traced on a single sheet of paper, along with the reference scale. Such a trace is shown in Figure 2.

The size of the bubbles was determined gravimetrically. This method is more accurate than volumetric methods. A simple but rugged chemical balance reliable to about 2 mg was used. An inverted funnel, closed at the upper end, was hung from one edge of the balance and immersed in the water. The change in weight due to the displacement of the water by the air was obtained. For the small bubbles, a sufficient number were collected so that the change in weight was 0.1 gm or greater. The larger bubbles were weighed individually. In order to verify uniformity in the size of the bubbles during a run, several film records were obtained and compared. If the size of the bubble as determined with the comparator was different for different bubbles, the run was

Figure 2 - Path and Shape of Bubble as Traced by Means of a Bausch and Lomb Comparator

The bubble radius is 1.15 cm. The scale is in centimeters.

The time interval between successive pictures is 0.0885 sec.

rejected. This occurred very infrequently and was due mainly to failure to allow the rate of feed of compressed air to become constant before beginning the test. The change in weight due to the bubble gives the volume of the water displaced, therefore the volume of the bubble or a counted number of bubbles is known. The volume so obtained must be adjusted because the pressure at which the bubble size is measured is different from that at which the rate of rise is determined. The effect of surface tension on the pressure inside the bubble is negligible.

$$V_1 = \left(\frac{P_0 - P'}{P_1 - P'} \right) V_0 \quad [8]$$

where V_1 is the volume of the bubble when it is in the field of the camera,
 V_0 is the volume of the bubble as determined by weighing,
 P_0 is the absolute pressure at the funnel,
 P_1 is the pressure at the camera level, and
 P' is the vapor pressure of water at the test temperature.

Since the water in the tank had been exposed to the atmosphere for a considerable length of time, it could be assumed saturated with air. Consequently, no consideration need be given to the possibility of decrease in the size of the bubble due to its dissolving in the water as it rose.

Most of the bubbles below an r_e of 0.06 cm were released through fine hypodermic needles and could not be generated with a uniform size. The sizes of these bubbles were determined with the comparator. This method was compared with the gravimetric method for several bubbles and found to give agreement within a few percent. Since the bubbles used for comparison purposes were somewhat larger than those released from the hypodermic needles, larger errors in determining the size of the smallest bubbles could be expected.

The rate of rise of the bubbles was determined by measuring the displacement of the bubbles in successive frames of the film. Figure 3 shows the displacement as a function of the time for several typical runs. The resultant curves are straight lines indicating that the velocity of the bubble is constant during the interval it traverses the camera field.

DISCUSSION OF RESULTS

Since bubbles of different size will assume different shapes, the equivalent radius r_e , defined as the radius of a sphere having the same volume as that of the bubble, was used as the length parameter. Figure 4 shows the terminal velocity of air bubbles in water at 67° F as a function of the

Figure 3 - Typical Curves of Displacement as a Function of Time

equivalent radius. Allen's data for bubbles greater than 0.01 cm are included. The correction for wall effect has been applied to these data.

As previously pointed out, a more generally applicable presentation is that showing the drag coefficient as a function of Reynolds number with the third parameter M in Equation [5] kept constant. The corresponding curve for rigid spheres¹² is also included in Figure 5.

Since it is difficult to visualize the bubble size from the value of the Reynolds number, the relation between bubble size and Reynolds number for bubbles rising at their terminal velocity in water is given in Figure 6.

Examination of Figure 5 shows that up to a Reynolds number of 70, the bubble behaves like a rigid sphere. For a Reynolds number range from 70 to 400, the bubble, although still spherical, has a drag coefficient considerably less than rigid spheres. A possible explanation for the decreased drag coefficient is the development of slip at the boundary of the fluid sphere.¹¹ Beyond a Reynolds number of 400, the hydrodynamic and surface-tension forces

Figure 4 - The Terminal Velocity of Air Bubbles in Water as a Function of Bubble Size

Figure 5 - The Drag Coefficient as a Function of Reynolds Number for Air Bubbles Rising at their Terminal Velocity in Water

Figure 6 - The Variation of Reynolds Number with Size for Bubbles Rising at their Terminal Velocity in Water at 67° F

are both important in determining the shape and consequently the drag coefficient of the bubble. As the bubble size increases, the shape of the bubble becomes flatter with a consequent rise in the value of the drag coefficient. For a Reynolds number greater than 5000, surface tension plays a relatively minor role in determining the shape of the bubble; hydrodynamic forces acting on the bubble result in the spherical cap shape.

A description of the shape and motion of the bubble as a function of bubble size is given in Table 1 for air bubbles rising at their terminal velocity in water at 67° F.

TABLE 1

Description of the Motion and Shape of Air Bubbles
in Water as a Function of the Bubble Size

r_e centimeters	Re	Description
< 0.04	< 70	Spherical bubbles traveling in rectilinear paths; same C_D as for rigid spheres.
0.04 - 0.062	70 - 400	Spherical bubbles traveling in rectilinear paths; C_D less than for solid spheres.
0.062 - 0.077	400 - 500	Oblate spheroid, rectilinear motion.
0.077 - 0.24	500 - 1100	Oblate spheroid, helical motion.
0.24 - 0.35	1100 - 1600	Oblate spheroid, essentially, but shape becomes increasingly irregular with increasing Re; motion is almost rectilinear.
0.35 - 0.88	1600 - 5000	Transition - oblate spheroid to spherical caps, shape is very irregular and fluctuates continuously, motion is almost rectilinear.
> 0.88	> 5000	Spherical caps, rectilinear motion; C_D almost constant at about 2.6.

Figures 7 and 8 show the shapes and paths of representative bubbles. Except for the spherical bubbles, the shape fluctuates as the bubble rises. A bubble moving in a helical path always maintains its axis of symmetry parallel to the direction of motion. The paths for bubbles of the same size may vary considerably due to differences in conditions at generation and to disturbances such as turbulence and thermal currents. It was found that the helical paths generally decreased in pitch and diameter as the bubble size increased. As the bubble radius increased from 0.075 to 0.22 cm the pitch decreased from 7 cm to 4 cm while the diameter of the helix decreased from 1 cm to 0.4 cm.

The faired curve of Figure 5 showing drag coefficient as a function of Reynolds number was compared with the data obtained by previous experimenters. This comparison is shown in Figure 9 and indicates the considerable scatter in the results. Still, for Re greater than 400, the points cluster fairly well around the curve obtained from TMB tests.

As pointed out before, the term A used in Equation [2] to define the drag coefficient of a body is generally the projected area. This formulation facilitates comparison of the drag developed by different shapes. In order to compare the drag measurements for bubbles with those obtained for solid bodies of roughly similar shape, the data are presented in the more usual form in Figure 10. The values of the parameters shown were computed from the experimental data according to the definitions.

$$C_D' = \frac{8Vg}{\pi a^2 U^2} \quad [9]$$

$$Re' = \frac{\rho a U}{\mu} \quad [10]$$

in which a is the over-all transverse diameter of the flattened bubble as measured from the photographs. Although the value employed for each bubble is an average of measurements from several frames of the film, there is still considerable scatter in the computed values of the drag coefficient determined in this way. No values were computed in the range of Re' from 4000 to 10,000 where, because of the irregular and extreme fluctuations in shape accompanying the rise of the bubbles, no reliable estimate of the transverse dimension could be made. Not all of the bubble data are included. Instead, typical bubbles of various sizes were selected. The data for spherical-cap bubbles indicate an average of 0.86 for the drag coefficient C_D' . The value of C_D' for a solid of roughly similar shape has been determined experimentally¹⁷ and is 0.65 at a Reynolds number of 21,000.

A description in geometrical terms of the bubble shape in the various flow regions may be of interest. Since a bubble fluctuates in shape as it rises, all bubble dimensions which were obtained from the photographs represent average values. For the bubbles which can be approximated by oblate ellipsoids, a completely descriptive shape parameter is the ratio of the major to minor axes, a/b . This is shown as a function of bubble size in Figure 11 using data obtained for typical bubbles. As the equivalent radius increases, the bubble becomes flatter until a maximum ratio of about 2.7 is reached at an

Equivalent Radius, cm	0.055	0.073	0.159
Reynolds Number	237	448	850
Scale	1.29/1	1.37/1	0.528/1
Time Interval, sec	0.0185	0.0407	0.0588

Figure 7 - Photographs Showing the Shape and Paths of Typical Bubbles for Reynolds Numbers less than 1100

equivalent radius of 0.35 cm. Beyond this size, the ratio remains the same but the shape of the bubble becomes increasingly irregular. Finally, at a value of r_e greater than 0.55 cm, the shape varies so greatly, as the bubble rises, that no specific values of a/b can be assigned. The transition to spherical caps is completed at an equivalent radius of 1.0 cm. The shape of a spherical cap can be adequately described in terms of the radius of curvature R of the nose, the transverse dimension a of the bubble, the height b of the bubble, and the angle subtended ϕ . The angle ϕ and the ratios R/r_e and a/b for a number of representative bubbles are shown as a function of bubble size in Figure 12. Although the scatter is considerable, the data show that all three parameters are independent of the bubble size, with a/b about 4.02, R/r_e about 2.5, and ϕ approximately 50 degrees.

Equivalent Radius, cm	0.322	0.808	1.76
Reynolds Number	1460	4570	13,790
Scale	0.522/1	0.382/1	0.375/1
Time Interval, sec	0.0588	0.0633	0.0643

Figure 8 - Photographs Showing the Shape and Paths of Typical Bubbles for Reynolds Numbers Greater than 1100

SPHERICAL CAPS

Taylor and Davies¹⁷ made extensive investigations of bubbles having the shape of spherical caps. They suggest that the drag coefficient and shape of spherical-cap bubbles are independent of the Reynolds number. Their empirical relation between the rate of rise and bubble size provides a value of 2.6 for the drag coefficient which is in complete agreement with the TMB experimental results shown in Figure 5. The results of a number of measurements of selected bubble dimensions presented in Figure 12 likewise agree with their conclusion regarding the geometric similarity of spherical caps of various sizes.

By using the experimental values of the subtended angle and the ratio a/b , a somewhat idealized picture of a representative bubble can be constructed as is shown in Figure 12. The value for R/r_e cannot be utilized in

Figure 9 - The Drag Coefficient as a Function of Reynolds Number for Air Bubbles in Water

Figure 10 - The Drag Coefficient as a Function of Reynolds Number for Air Bubbles Rising at their Terminal Velocity in Water

The drag coefficient is based on projected area and the Reynolds number on the transverse dimension.

Figure 11 - Effect of Bubble Size on the Shape of Ellipsoidal Air Bubbles Rising in Water

Figure 12 - Effect of Bubble Size on the Shape of Spherical Cap Air Bubbles Rising in Water

the construction because the base of the bubble is not plane but is concave downward as is indicated by the dotted line. This shape was observed by studying the bubble from below through the transparent wall of the tank.

Taylor and Davies considered also the shape of the bubble as related to the pressure distribution over the surface. By assuming that the pressure distribution for a spherical cap with a radius of curvature R is the same as that of a sphere with the same radius, they show that a velocity of rise $2/3\sqrt{gR}$ would result in the hydrostatic and dynamic pressure gradients nearly canceling in the region around the nose of the cap. This is the required condition along the surface of the bubble if surface tension is neglected. Their

experiments with spherical-cap bubbles in nitrobenzene showed excellent agreement with their theory. This was also done at the Taylor Model Basin and the results are shown in Figure 13. It can be seen that the observed rates of rise are equal to about $0.645 \sqrt{gR}$ rather than $2/3 \sqrt{gR}$. This indicates that the dynamic-pressure gradient along the bubble surface is somewhat greater than the one computed for a sphere of the same nose radius.

Figure 13 - The Terminal Velocity of Spherical Caps as a Function of the Radius of Curvature of the Nose of the Bubble

SIGNIFICANCE OF THE PARAMETER M

Since the TMB experiments were all conducted in water at one temperature, only the bubble size being varied independently, no information concerning the effect of the parameter M can be derived from these data. This parameter was, in fact, chosen so as to be independent of the variables in the experiments. It may be varied independently of the Reynolds number by changing either the pressure gradient or the properties of the liquid. In connection with the study of bubble motion in water, the effect of varying pressure gradients is directly of interest. Unfortunately, no data are available concerning the behavior of bubbles in water at pressure gradients other than that produced by gravity. Some measurements have been made, however, to determine the

velocity of rise of bubbles in liquids other than water.^{8, 15} Data from these experiments are combined and presented in Figure 14. Because these are incomplete and show apparent inconsistencies, the conclusions which can be drawn are rather meager and tentative. A few generalizations appear justified both on the basis of the data and from the expected influence of the variables incorporated in the parameter M.

Figure 14 - Drag Coefficient as a Function of Reynolds Number for Different Values of the Parameter M as Obtained from Data on the Terminal Velocity of Bubbles in Various Liquids

For very low Reynolds numbers, the bubble behavior is almost independent of the value of the parameter M , the bubbles acting as rigid spheres. A puzzling aspect of the problem is provided by the fact that there is a critical value of the Reynolds number beyond which the drag coefficient of the bubble departs from that of rigid spheres, yet the bubbles are still spherical. The available data are not sufficient to indicate definitely whether the Reynolds number at which this break takes place is a function only of M or whether the phenomenon depends upon some property of the fluids not considered in the foregoing analysis. A tentative observation is that the greater the value of M , the lower is the critical Reynolds number.

According to the available data, the relation of the drag coefficient to the Reynolds number for ellipsoidal bubbles is greatly affected by the value of M . This is to be expected since in this range the bubble shape is neither completely controlled by surface tension as is true for spherical bubbles, nor almost completely free of it, as appears to be the case for spherical caps.

It is possible to surmise the role of the parameter M in determining the Reynolds number required for transition from ellipsoidal bubbles to spherical caps. A little manipulation of M modifies it to the form

$$\frac{1}{Re^2} \left(\frac{r_e^3}{r_e \gamma} \right) \left(\frac{\mu r_e U}{r_e \gamma} \right)^2$$

The terms in parentheses represent the ratios between pressure-gradient and surface-tension forces and viscous and surface-tension forces, respectively. For a given Reynolds number, M may be increased by increasing either the pressure gradient forces or viscous forces. If the pressure gradient is increased, the buoyant force is increased. Since in the transition region, the drag coefficient has attained a constant value, the terminal velocity of the bubble will increase. Therefore, in order to maintain the Reynolds number constant the viscosity of the medium must be increased. Hence, increasing the pressure-gradient forces results in an increase in the viscous forces. As a result, surface-tension forces become relatively smaller as M is increased at a constant Reynolds number. In addition, we know from experiment that increasing the Reynolds number at constant M results in transition from ellipsoidal bubbles to spherical caps, indicating relatively smaller surface-tension forces with increasing Reynolds numbers. Since maintenance of the ellipsoidal shape is due primarily to surface-tension forces, increasing M would result in transition to spherical caps occurring at a lower Reynolds number. The data support this conclusion fairly well.

In most hydrodynamic problems we are concerned solely with the variation of drag coefficient with Reynolds number for a single liquid, water. The form of M indicates that for a specific liquid at a given temperature, changing the pressure gradient is equivalent to varying M . Therefore in using different liquids we are obtaining the same sort of information as if we were to vary the pressure gradient, provided our assumption concerning the relevant physical variables is correct.

SUMMARY AND CONCLUSIONS

The motion of air bubbles in a liquid can best be characterized by the use of three dimensionless parameters, the Reynolds number, the drag coefficient and a third parameter M which for a specific liquid is proportional to the pressure gradient.

Tests on air bubbles at their terminal velocity in water indicate that for Reynolds numbers of less than 70, bubbles behave like rigid spheres. At greater values of the Reynolds number, the drag coefficients of the bubbles are considerably less than for rigid spheres even though the bubbles are still spherical in shape. This may be due to the development of slip at the boundaries. For Reynolds numbers from 400 to 5000 the hydrodynamic and surface-tension forces are both important in determining the shape and consequently the drag coefficient of the bubbles. Beyond this range, hydrodynamic forces almost exclusively determine the shape of the bubble, resulting in a spherical cap. The test results are shown by means of the curve of Figure 5 giving the drag coefficient as a function of the Reynolds number and Table 1 describing the motion and shape of the bubbles.

In the tests at the Taylor Model Basin, the slight change in volume resulting from the change in pressure as the bubble rose in the field of the camera resulted in no appreciable change in the terminal velocity. However, the bubble fluctuated in shape as it rose. Therefore it was difficult to assign any one set of dimensions to a particular bubble. An average was obtained. Using these average dimensions, it was shown that spherical caps are geometrically similar. Also, their drag coefficients are independent of bubble size, having a value of about 2.6.

The rate of rise of a spherical cap is a relatively simple function of the radius of curvature of the nose, the relation determined experimentally being given by:

$$U = 0.645 \sqrt{gR}$$

The effect of the parameter M on the relation between the drag coefficient and Reynolds number is uncertain since there are only very incomplete data available. The results indicate that M influences the relation between the other parameters in the region in which the bubbles no longer act like rigid spheres and have not yet attained the shape of spherical caps. The value of M affects the Reynolds-number range for which this zone exists and also the minimum value of C_D . As M increases, transitions in shape occur at lower Reynolds numbers and the minimum value for C_D increases, approaching closer to rigid spheres.

PERSONNEL

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