RESISTANCE OF MATERIALS

SLOCUM

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UNIVERSITY OF TORONTO
DEPARTMENT OF CIVIL ENGINEERING
Municipal and Structural

RESISTANCE OF MATERIALS

FOR BEGINNERS IN ENGINEERING

BY

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Municipal and Structural

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PREFACE

The chief feature which distinguishes this volume from other American textbooks on the same subject is that the Principle of Moments is used consistently throughout in place of the usual calculus processes. By basing the work on this principle it has been found practicable to give a simple and obvious treatment of many topics for which the calculus is usually thought to be indispensable, such as the calculation of moments of inertia, the deflection of beams, the buckling of columns, and the strength of thick cylinders. Experience has shown conclusively that the average engineering graduate, and even the practicing engineer, is deficient in the ability to apply the Principle of Moments readily, but when thus used as the central and coördinating principle, it must necessarily make an indelible impression on the mind of the student and go far toward remedying this deficiency.

The mechanics of materials is of such fundamental importance in all branches of technology that it is important to begin its study as early in the course as possible. Heretofore it has been necessary to defer it -- awaiting the completion of the calculus -- until junior year, when the curriculum is already crowded with technical subjects requiring its application. This text makes it possible for the course to parallel or even to precede the calculus. In addition, it makes the subject available for trade or architectural schools where no calculus is taught.

Although simple and obvious, the treatment is adequate, and its simplicity in no way limits its range or generality. The text is supplemented by a variety of engineering applications, giving practical information as well as a mastery of the principles involved.

S. E. SLOCUM

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TABLE I
1. AVERAGE VALUES OF PHYSICAL CONSTANTS

Material	Ultimate Tensile Strength lb./in. ²	Ultimate Com- pressive Strength lb./in. ²	Ultimate Shearing Strength lb./in. ²	Ultimate Flexural Strength (Modulus of Rupture) lb./in. ²	Elastic Limit lb./in. ²	Unit Elon- gation at Elastic Limit inch	Young's Modulus of Elasticity lb./in. ²	Modulus of Shear (Modulus of Rigidity) lb./in. ²	Weight lb./ft. ³	Coefficient of Linear Expansion 1° F.
Hard steel	100,000	120,000	80,000	110,000	60,000	0.0012	30,000,000	12,000,000	490	.0000074
Structural steel . . .	60,000	60,000	50,000	60,000	35,000	0.0012	30,000,000	12,000,000	490	.0000061
Wrought iron	50,000	50,000	40,000	50,000	25,000	0.0010	25,000,000	10,000,000	480	.0000068
Cast-iron tension . . .	20,000	. . . .	20,000	35,000	6,000	0.0004	15,000,000	6,000,000	450	.0000063
“ compression	. . . .	90,000	. . . .	. . . .	20,000	. . . .	. . . .	. . . .	. . . .	. . . .
Brass, drawn	43,000	11,000	. . . .	. . . .	. . . .	. . . .	14,500,000	5,000,000	530	. . . .
“ cast	24,000	. . . .	. . . .	. . . .	. . . .	. . . .	9,500,000	. . . .	520	. . . .
Copper, drawn	32,000	. . . .	. . . .	. . . .	. . . .	. . . .	15,000,000	6,000,000	550	.0000064
“ cast	22,000	. . . .	. . . .	. . . .	. . . .	. . . .	12,000,000	. . . .	. . . .	. . . .
Timber, with grain . .	10,000	8,000	600	9,000	3,000	0.0020	1,500,000	. . . .	40	.0000028
“ across grain	. . . .	. . . .	3,000	. . . .	. . . .	. . . .	. . . .	400,000	. . . .	. . . .
Concrete	300	3,000	1,000	700	1,000	. . . .	2,000,000	. . . .	150	.0000055
Stone	. . . .	6,000	1,500	2,000	2,000	. . . .	6,000,000	1,800,000	160	.0000050
Brick	. . . .	3,000	1,000	800	1,000	. . . .	2,000,000	. . . .	125	.0000050

TABLE I (Continued)

2. POISSON'S RATIO

MATERIAL	AVERAGE VALUES OF $\frac{1}{m}$
Steel, hard	.295
" structural	.299
Iron	.277
Brass	.357
Copper	.340
Lead	.375
Zinc	.205

3. FACTORS OF SAFETY

MATERIAL	STEADY STRESS: BUILDINGS, ETC.	VARYING STRESS: BRIDGES, ETC.	REPEATED OR REVERSED STRESS: MACHINES
Steel, hard	5	8	15
" structural	4	6	10
Iron, wrought	4	6	10
" cast	6	10	20
Timber	8	10	15
Brick and stone	15	25	30

The only rational method of determining the factor of safety is to choose it sufficiently large to bring the working stress well within the elastic limit (see Article 6).

TABLE II
PROPERTIES OF VARIOUS SECTIONS

SHAPE	AREA F	LOCATION OF GRAVITY AXIS x	STATIC MOMENT OF INERTIA (SECOND MOMENT OF AREA) I	SECTION MODULUS $S = \frac{I}{x}$
	bh	$x = \frac{h}{2}$	$\frac{bh^3}{12}$	$\frac{bh^2}{6}$
	bh	$x = \frac{h}{2}$	$\frac{bh^3}{8}$	
	b^2	$x = \frac{b}{2}$	$\frac{b^4}{12}$	$\frac{b^3}{6}$

TABLE II (Continued)
PROPERTIES OF VARIOUS SECTIONS

SHAPE	AREA F	LOCATION OF GRAVITY AXIS x	STATIC MOMENT OF INERTIA (SECOND MOMENT OF AREA) I	SECTION MODULUS $\frac{I}{S = \frac{x}{2}}$
	b^2	$x = \frac{b}{\sqrt{2}} = .707 b$	$\frac{b^4}{12}$	$\frac{b^3}{6\sqrt{2}} = .118 b^2$
	$\frac{1}{2}bh$	$x = \frac{2}{3}h$	$\frac{bh^3}{36}$	$\frac{bh^2}{24}$
	$\frac{1}{2}bh$	$x = \frac{2}{3}h$	$\frac{bh^3}{12}$	

	$\frac{\pi d^2}{4} = .785 d^2$	$x = \frac{d}{2}$	$\frac{\pi d^4}{64} = .049 d^4$	$\frac{\pi d^3}{32} = .008 d^3$
	$\frac{\pi (D^2 - d^2)}{4} = .785 (D^2 - d^2)$	$x = \frac{D}{2}$	$\frac{\pi (D^4 - d^4)}{64} = .049 (D^4 - d^4)$	$\frac{\pi (D^4 - d^4)}{32 D} = .008 \left(D^3 - \frac{d^4}{D} \right)$
	$\frac{\pi b h}{4} = .785 b h$	$x = \frac{h}{2}$	$\frac{\pi b h^3}{64} = .049 b h^3$	$\frac{\pi b h^2}{32} = .008 b h^2$
	$\frac{\pi (BH - bh)}{4} = .785 (BH - bh)$	$x = \frac{H}{2}$	$\frac{\pi (BH^3 - bh^3)}{64} = .049 (BH^3 - bh^3)$	$\frac{\pi (BH^3 - bh^3)}{32 H} = .008 \left(BH^2 - \frac{bh^3}{H} \right)$

TABLE II (Continued)
PROPERTIES OF VARIOUS SECTIONS

SHAPE	AREA F	LOCATION OF GRAVITY AXIS x	STATIC MOMENT OF INERTIA (SECOND MOMENT OF AREA) I	SECTION MODULUS $S = \frac{I}{x}$
	$\frac{3\sqrt{3}}{2} b^2 = 2.6 b^2$	$x = \frac{\sqrt{3}}{2} b = .866 b$	$\frac{5\sqrt{3}}{16} b^4 = .5413 b^4$	$\frac{5 b^3}{8}$
	$\frac{3\sqrt{3}}{2} b^2 = 2.6 b^2$	$x = b$	$\frac{5\sqrt{3}}{16} b^4 = .5413 b^4$	$.5413 b^3$
	$B(H - h) + bh$	$x = \frac{BH^2 - h(B - b)(2H - h)}{2[B(H - h) + bh]}$	$\frac{b(H - x)^3 + Bx^3 - (B - b)(x + h - H)^3}{3}$	$\frac{I}{H - x}$

	$BH - bh$	$x = \frac{H}{2}$	$\frac{BH^3 - bh^3}{12}$	$\frac{BH^3 - bh^3}{6H}$
	$BH + bh$	$x = \frac{H}{2}$	$\frac{BH^3 + bh^3}{12}$	$\frac{BH^3 + bh^3}{6H}$

TABLE III
PROPERTIES OF STANDARD I-BEAMS

DEPTH OF BEAM	WEIGHT PER FOOT	AREA OF SECTION	THICK- NESS OF WEB	WIDTH OF FLANGE	MOMENT OF INERTIA AXIS 1-1	SECTION MODU- LUS AXIS 1-1	RADIUS OF GYRA- TION AXIS 1-1	MOMENT OF INERTIA AXIS 2-2	RADIUS OF GYRA- TION AXIS 2-2
d		A	t	b	I	S	r	I'	r'
Inches	Pounds	Sq. Inches	Inches	Inches	Inches ⁴	Inches ³	Inches	Inches ⁴	Inches
3	5.5	1.63	.17	2.33	2.5	1.7	1.23	.46	.53
"	6.5	1.91	.26	2.42	2.7	1.8	1.19	.53	.52
"	7.5	2.21	.36	2.52	2.9	1.9	1.15	.60	.52
4	7.5	2.21	.19	2.66	6.0	3.0	1.64	.77	.59
"	8.5	2.50	.26	2.73	6.4	3.2	1.59	.85	.58
"	9.5	2.79	.34	2.81	6.7	3.4	1.54	.93	.58
"	10.5	3.09	.41	2.88	7.1	3.6	1.52	1.01	.57
5	9.75	2.87	.21	3.00	12.1	4.8	2.05	1.23	.65
"	12.25	3.60	.36	3.15	13.6	5.4	1.94	1.45	.63
"	14.75	4.34	.50	3.29	15.1	6.1	1.87	1.70	.63
6	12.25	3.61	.23	3.33	21.8	7.3	2.46	1.85	.72
"	14.75	4.34	.35	3.45	24.0	8.0	2.35	2.09	.69
"	17.25	5.07	.47	3.57	26.2	8.7	2.27	2.36	.68
7	15.0	4.42	.25	3.66	36.2	10.4	2.86	2.67	.78
"	17.5	5.15	.35	3.76	39.2	11.2	2.76	2.94	.76
"	20.0	5.88	.46	3.87	42.2	12.1	2.68	3.24	.74
8	17.75	5.33	.27	4.00	56.9	14.2	3.27	3.78	.84
"	20.25	5.96	.35	4.08	60.2	15.0	3.18	4.04	.82
"	22.75	6.69	.44	4.17	64.1	16.0	3.10	4.36	.81
"	25.25	7.43	.53	4.26	68.0	17.0	3.03	4.71	.80
9	21.0	6.31	.29	4.33	84.9	18.9	3.67	5.16	.90
"	25.0	7.35	.41	4.45	91.9	20.4	3.54	5.65	.88
"	30.0	8.82	.57	4.61	101.9	22.6	3.40	6.42	.85
"	35.0	10.29	.73	4.77	111.8	24.8	3.30	7.31	.84
10	25.0	7.37	.31	4.66	122.1	24.4	4.07	6.89	.97
"	30.0	8.82	.45	4.80	134.2	26.8	3.90	7.65	.93
"	35.0	10.29	.60	4.95	146.4	29.3	3.77	8.52	.91
"	40.0	11.76	.75	5.10	158.7	31.7	3.67	9.50	.90
12	32.5	9.26	.35	5.00	215.8	36.0	4.83	9.50	1.01
"	35.0	10.29	.44	5.09	228.3	38.0	4.71	10.07	.99
"	40.0	11.76	.56	5.21	245.9	41.0	4.57	10.95	.96
15	42.0	12.48	.41	5.50	441.8	58.9	5.95	14.62	1.08
"	45.0	13.24	.46	5.55	455.8	60.8	5.87	15.09	1.07
"	50.0	14.71	.56	5.65	483.4	64.5	5.73	16.04	1.04
"	55.0	16.18	.66	5.75	511.0	68.1	5.62	17.06	1.03
"	60.0	17.65	.75	5.84	538.6	71.8	5.52	18.17	1.01
18	55.0	15.93	.46	6.00	795.6	88.4	7.07	21.19	1.15
"	60.0	17.65	.56	6.10	841.8	93.5	6.91	22.38	1.13
"	65.0	19.12	.64	6.18	881.5	97.9	6.79	23.47	1.11
"	70.0	20.59	.72	6.26	921.2	102.4	6.69	24.62	1.09
20	65.0	19.08	.50	6.25	1169.5	117.0	7.83	27.86	1.21
"	70.0	20.59	.58	6.33	1219.8	122.0	7.70	29.04	1.19
"	75.0	22.06	.65	6.40	1268.8	126.9	7.58	30.25	1.17
24	80.0	23.32	.50	7.00	2087.2	173.9	9.46	42.86	1.36
"	85.0	25.00	.57	7.07	2167.8	180.7	9.31	44.35	1.33
"	90.0	26.47	.63	7.13	2238.4	186.5	9.20	45.70	1.31
"	95.0	27.94	.69	7.19	2309.0	192.4	9.09	47.10	1.30
"	100.0	29.41	.75	7.25	2379.6	198.3	8.99	48.55	1.28

TABLE IV
PROPERTIES OF STANDARD CHANNELS

DEPTH OF CHANNEL	WEIGHT PER FOOT	AREA OF SECTION	THICKNESS OF WEB	WIDTH OF FLANGE	MOMENT OF INERTIA AXIS 1-1	SECTION MODULUS AXIS 1-1	RADIUS OF GYRATION AXIS 1-1	MOMENT OF INERTIA AXIS 2-2	SECTION MODULUS AXIS 2-2	RADIUS OF GYRATION AXIS 2-2	DISTANCE OF CENTER OF GRAVITY FROM OUTSIDE OF WEB
d		A	t	b	I	S	r	I'	S'	r'	x
Inches	Pounds	Sq. In.	Inches	Inches	Inches ⁴	Inches ³	Inches	Inches ⁴	Inches ³	Inches	Inches
3	4.00	1.19	.17	1.41	1.6	1.1	1.17	.20	.21	.41	.44
"	5.00	1.47	.26	1.50	1.8	1.2	1.12	.25	.24	.41	.44
"	6.00	1.76	.36	1.60	2.1	1.4	1.08	.31	.27	.42	.46
4	5.25	1.55	.18	1.58	3.8	1.9	1.56	.32	.29	.45	.46
"	6.25	1.84	.25	1.65	4.2	2.1	1.51	.38	.32	.45	.46
"	7.25	2.13	.33	1.73	4.6	2.3	1.46	.44	.35	.46	.46
5	6.50	1.95	.19	1.75	7.4	3.0	1.95	.48	.38	.50	.49
"	9.00	2.65	.33	1.89	8.9	3.5	1.83	.64	.45	.49	.48
"	11.50	3.38	.48	2.04	10.4	4.2	1.75	.82	.54	.49	.51
6	8.00	2.38	.20	1.92	13.0	4.3	2.34	.70	.50	.54	.52
"	10.50	3.09	.32	2.04	15.1	5.0	2.21	.88	.57	.53	.50
"	13.00	3.82	.44	2.16	17.3	5.8	2.13	1.07	.65	.53	.52
"	15.50	4.56	.56	2.28	19.5	6.5	2.07	1.28	.74	.53	.55
7	9.75	2.85	.21	2.00	21.1	6.0	2.72	.98	.63	.59	.55
"	12.25	3.60	.32	2.20	24.2	6.9	2.59	1.19	.71	.57	.53
"	14.75	4.34	.42	2.30	27.2	7.8	2.50	1.40	.79	.57	.53
"	17.25	5.07	.53	2.41	30.2	8.6	2.44	1.62	.87	.56	.55
"	19.75	5.81	.63	2.51	33.2	9.5	2.39	1.85	.96	.56	.58
8	11.25	3.35	.22	2.26	32.3	8.1	3.10	1.33	.79	.63	.58
"	13.75	4.04	.31	2.35	36.0	9.0	2.98	1.55	.87	.62	.56
"	16.25	4.78	.40	2.44	39.9	10.0	2.89	1.78	.95	.61	.56
"	18.75	5.51	.49	2.53	43.8	11.0	2.82	2.01	1.02	.60	.57
"	21.25	6.25	.58	2.62	47.8	11.9	2.76	2.25	1.11	.60	.59
9	13.25	3.89	.23	2.43	47.3	10.5	3.49	1.77	.97	.67	.61
"	15.00	4.41	.29	2.49	50.9	11.3	3.40	1.95	1.03	.66	.59
"	20.00	5.88	.45	2.65	60.8	13.5	3.21	2.45	1.19	.65	.58
"	25.00	7.35	.61	2.81	70.7	15.7	3.10	2.98	1.36	.64	.62
10	15.00	4.46	.24	2.60	66.9	13.4	3.87	2.30	1.17	.72	.64
"	20.00	5.88	.38	2.74	78.7	15.7	3.66	2.85	1.34	.70	.61
"	25.00	7.35	.53	2.89	91.0	18.2	3.52	3.40	1.50	.68	.62
"	30.00	8.82	.68	3.04	103.2	20.6	3.42	3.99	1.67	.67	.65
"	35.00	10.29	.82	3.18	115.5	23.1	3.35	4.66	1.87	.67	.69
12	20.50	6.03	.28	2.94	128.1	21.4	4.61	3.91	1.75	.81	.70
"	25.00	7.35	.39	3.05	144.0	24.0	4.43	4.53	1.91	.78	.68
"	30.00	8.82	.51	3.17	161.6	26.9	4.28	5.21	2.09	.77	.68
"	35.00	10.29	.64	3.30	179.3	29.9	4.17	5.90	2.27	.76	.69
"	40.00	11.76	.76	3.42	196.9	32.8	4.09	6.63	2.46	.75	.72
15	33.00	9.90	.40	3.40	312.6	41.7	5.62	8.23	3.16	.91	.79
"	35.00	10.29	.43	3.43	319.9	42.7	5.57	8.48	3.22	.91	.79
"	40.00	11.76	.52	3.52	347.5	46.3	5.44	9.39	3.43	.89	.78
"	45.00	13.24	.62	3.62	375.1	50.0	5.32	10.29	3.63	.88	.79
"	50.00	14.71	.72	3.72	402.7	53.7	5.23	11.22	3.85	.87	.80
"	55.00	16.18	.82	3.82	430.2	57.4	5.16	12.19	4.07	.87	.82

TABLE V

PROPERTIES OF STANDARD ANGLES, EQUAL LEGS

DIMENSIONS	THICKNESS	WEIGHT PER FOOT	AREA OF SECTION	DISTANCE OF CENTER OF GRAVITY FROM BACK OF FLANGE	MOMENT OF INERTIA AXIS 1-1	SECTION MODULUS AXIS 1-1	RADIUS OF GYRATION AXIS 1-1	DISTANCE OF CENTER OF GRAVITY FROM EXTERNAL APEX ON LINE EXTENDED AT 45° TO FLANGE	LEAST MOMENT OF INERTIA AXIS 2-2	SECTION MODULUS AXIS 2-2	LEAST RADIUS OF GYRATION AXIS 2-2
<i>Inches</i>	<i>Inches</i>	<i>Pounds</i>	<i>Sq. In.</i>	<i>Inches</i>	<i>Inches⁴</i>	<i>Inches³</i>	<i>Inches</i>	<i>Inches</i>	<i>Inches⁴</i>	<i>Inches³</i>	<i>Inches</i>
$\frac{3}{4} \times \frac{3}{4}$	$\frac{3}{16}$	.58	.17	.23	.009	.017	.22	.33	.004	.011	.14
1 × 1	$\frac{1}{8}$	.80	.23	.30	.022	.031	.30	.42	.009	.021	.19
"	$\frac{1}{8}$	1.40	.44	.34	.037	.056	.29	.48	.016	.034	.19
$1\frac{1}{4} \times 1\frac{1}{4}$	$\frac{1}{4}$	1.02	.30	.36	.044	.049	.38	.51	.018	.035	.24
"	$\frac{1}{4}$	1.91	.56	.40	.077	.091	.37	.57	.033	.057	.24
$1\frac{1}{2} \times 1\frac{1}{2}$	$\frac{1}{4}$	2.34	.69	.47	.14	.134	.45	.66	.058	.088	.29
"	$\frac{1}{4}$	3.35	.98	.51	.19	.188	.44	.72	.082	.114	.29
$1\frac{3}{4} \times 1\frac{3}{4}$	$\frac{1}{4}$	2.77	.81	.53	.23	.19	.53	.75	.094	.13	.34
"	$\frac{1}{4}$	3.98	1.17	.57	.31	.26	.51	.81	.133	.16	.34
2 × 2	$\frac{1}{4}$	3.19	.94	.59	.35	.25	.61	.84	.14	.17	.39
"	$\frac{1}{4}$	4.62	1.36	.64	.48	.35	.59	.90	.20	.22	.39
$2\frac{1}{2} \times 2\frac{1}{2}$	$\frac{1}{4}$	4.0	1.19	.72	.70	.39	.77	1.01	.29	.28	.49
"	$\frac{1}{4}$	5.9	1.73	.76	.98	.57	.75	1.08	.41	.38	.48
"	$\frac{1}{4}$	7.7	2.25	.81	1.23	.72	.74	1.14	.52	.46	.48
3 × 3	$\frac{1}{4}$	4.9	1.44	.84	1.24	.58	.93	1.19	.50	.42	.59
"	$\frac{1}{4}$	7.2	2.11	.89	1.76	.83	.91	1.26	.72	.57	.58
"	$\frac{1}{4}$	9.4	2.75	.93	2.22	1.07	.90	1.32	.92	.70	.58
"	$\frac{1}{4}$	11.4	3.36	.98	2.62	1.30	.88	1.38	1.12	.81	.58
$3\frac{1}{4} \times 3\frac{1}{4}$	$\frac{1}{4}$	8.4	2.48	1.01	2.87	1.15	1.07	1.43	1.16	.81	.68
"	$\frac{1}{4}$	11.1	3.25	1.06	3.64	1.49	1.06	1.50	1.50	1.00	.68
"	$\frac{1}{4}$	13.5	3.98	1.10	4.33	1.81	1.04	1.56	1.82	1.17	.68
"	$\frac{1}{4}$	15.9	4.69	1.15	4.96	2.11	1.03	1.62	2.13	1.31	.67
4 × 4	$\frac{1}{4}$	9.7	2.86	1.14	4.36	1.52	1.23	1.61	1.77	1.10	.79
"	$\frac{1}{4}$	12.8	3.75	1.18	5.56	1.97	1.22	1.67	2.28	1.36	.78
"	$\frac{1}{4}$	15.7	4.61	1.23	6.66	2.40	1.20	1.74	2.76	1.59	.77
"	$\frac{1}{4}$	18.5	5.44	1.27	7.66	2.81	1.19	1.80	3.23	1.80	.77
6 × 6	$\frac{1}{4}$	19.6	5.75	1.68	19.91	4.61	1.86	2.38	8.04	3.37	1.18
"	$\frac{1}{4}$	24.2	7.11	1.73	24.16	5.66	1.84	2.45	9.81	4.01	1.17
"	$\frac{1}{4}$	28.7	8.44	1.78	28.15	6.66	1.83	2.51	11.52	4.59	1.17
"	$\frac{1}{4}$	33.1	9.73	1.82	31.92	7.63	1.81	2.57	13.17	5.12	1.16

TABLE V

PROPERTIES OF STANDARD ANGLES, UNEQUAL LEGS

DIMENSIONS	THICKNESS	WEIGHT PER FOOT	AREA OF SECTION	DISTANCE OF CENTER OF GRAVITY FROM BACK OF LONGER FLANGE	MOMENT OF INERTIA AXIS 1-1	SECTION MODULUS AXIS 1-1	RADIUS OF GYRATION AXIS 1-1	DISTANCE OF CENTER OF GRAVITY FROM BACK OF SHORTER FLANGE	MOMENT OF INERTIA AXIS 2-2	SECTION MODULUS AXIS 2-2	RADIUS OF GYRATION AXIS 2-2
Inches	Inches	Pounds	Sq. In.	Inches	Inches ⁴	Inches ³	Inches	Inches	Inches ⁴	Inches ³	Inches
2½ × 2	1/8	3.6	1.06	.54	.37	.25	.59	.79	.65	.38	.78
"	1/4	5.3	1.55	.58	.51	.36	.58	.83	.91	.55	.77
"	3/8	6.8	2.00	.63	.64	.46	.56	.88	1.14	.70	.75
3 × 2½	1/8	4.5	1.31	.66	.74	.40	.75	.91	1.17	.56	.95
"	1/4	6.5	1.92	.71	1.04	.58	.74	.96	1.66	.81	.93
"	3/8	8.5	2.50	.75	1.30	.74	.72	1.00	2.08	1.04	.91
3½ × 2½	1/8	4.9	1.44	.61	.78	.41	.74	1.11	1.80	.75	1.12
"	1/4	7.2	2.11	.66	1.09	.59	.72	1.16	2.56	1.09	1.10
"	3/8	9.4	2.75	.70	1.36	.76	.70	1.20	3.24	1.41	1.09
"	1/2	11.4	3.36	.75	1.61	.92	.69	1.25	3.85	1.71	1.07
3½ × 3	1/8	7.8	2.30	.83	1.85	.85	.90	1.08	2.72	1.13	1.09
"	1/4	10.2	3.00	.88	2.33	1.10	.88	1.13	3.45	1.45	1.07
"	3/8	12.5	3.67	.92	2.76	1.33	.87	1.17	4.11	1.76	1.06
"	1/2	14.7	4.31	.96	3.15	1.54	.85	1.21	4.70	2.05	1.04
4 × 3	1/8	8.5	2.48	.78	1.92	.87	.88	1.28	3.96	1.46	1.26
"	1/4	11.1	3.25	.83	2.42	1.12	.86	1.33	5.05	1.89	1.25
"	3/8	13.6	3.98	.87	2.87	1.35	.85	1.37	6.03	2.30	1.23
"	1/2	15.9	4.69	.92	3.28	1.57	.84	1.42	6.93	2.68	1.22
5 × 3	1/8	9.7	2.86	.70	2.04	.89	.84	1.70	7.37	2.24	1.61
"	1/4	12.8	3.75	.75	2.58	1.15	.83	1.75	9.45	2.91	1.59
"	3/8	15.7	4.61	.80	3.06	1.39	.82	1.80	11.37	3.55	1.57
"	1/2	18.5	5.44	.84	3.51	1.62	.80	1.84	13.15	4.16	1.55
5 × 3½	1/8	10.4	3.05	.86	3.18	1.21	1.02	1.61	7.78	2.29	1.60
"	1/4	13.6	4.00	.91	4.05	1.55	1.01	1.66	9.99	2.99	1.58
"	3/8	16.7	4.92	.95	4.83	1.90	.99	1.70	12.08	3.65	1.56
"	1/2	19.8	5.81	1.00	5.55	2.22	.98	1.75	13.92	4.28	1.55
"	5/8	22.7	6.67	1.04	6.21	2.52	.96	1.79	15.67	4.88	1.53
6 × 3½	1/8	11.6	3.42	.79	3.34	1.23	.99	2.04	12.86	3.24	1.94
"	1/4	15.3	4.50	.83	4.25	1.59	.97	2.08	16.59	4.24	1.92
"	3/8	18.9	5.55	.88	5.08	1.94	.96	2.13	20.08	5.19	1.90
"	1/2	22.3	6.56	.93	5.84	2.27	.94	2.18	23.34	6.10	1.89
"	5/8	25.7	7.55	.97	6.55	2.59	.93	2.22	26.39	6.98	1.87
6 × 4	1/8	12.3	3.61	.94	4.90	1.60	1.17	1.94	13.47	3.32	1.93
"	1/4	16.2	4.75	.99	6.27	2.08	1.15	1.99	17.40	4.33	1.91
"	3/8	19.9	5.86	1.03	7.52	2.54	1.13	2.03	21.07	5.31	1.90
"	1/2	23.6	6.94	1.08	8.68	2.97	1.12	2.08	24.51	6.25	1.88
"	5/8	27.2	7.98	1.12	9.75	3.39	1.11	2.12	27.73	7.15	1.86

TABLE VI
PROPERTIES OF BETHLEHEM GIRDER BEAMS

DEPTH OF BEAM	WEIGHT PER FOOT	AREA OF SECTION	THICK- NESS OF WEB	WIDTH OF FLANGE	NEUTRAL AXIS PERPEN- DICULAR TO WEB AT CENTER			NEUTRAL AXIS COINCIDENT WITH CENTER LINE OF WEB	
Inches	Pounds	Square Inches	Inches	Inches	Moment of Inertia	Radius of Gyration	Section Modulus	Moment of Inertia	Radius of Gyration
30	200.0	58.71	.750	15.00	9150.6	12.48	610.0	630.2	3.28
"	180.0	53.00	.690	13.00	8194.5	12.43	546.3	433.3	2.86
28	180.0	52.86	.690	14.35	7264.7	11.72	518.9	533.3	3.18
"	165.0	48.47	.660	12.50	6562.7	11.64	468.8	371.9	2.77
26	160.0	46.91	.630	13.60	5620.8	10.95	432.4	435.7	3.05
"	150.0	43.94	.630	12.00	5153.9	10.83	396.5	314.6	2.68
24	140.0	41.16	.600	13.00	4201.4	10.10	350.1	346.9	2.90
"	120.0	35.38	.530	12.00	3607.3	10.10	300.6	249.4	2.66
20	140.0	41.19	.640	12.50	2934.7	8.44	293.5	348.9	2.91
"	112.0	32.81	.550	12.00	2342.1	8.45	234.2	239.3	2.70
18	92.0	27.12	.480	11.50	1591.4	7.66	176.8	182.6	2.59
15	140.0	41.27	.800	11.75	1592.7	6.21	212.4	331.0	2.83
"	104.0	30.50	.600	11.25	1220.1	6.32	162.7	213.0	2.64
"	73.0	21.49	.430	10.50	883.4	6.41	117.8	123.2	2.39
12	70.0	20.58	.460	10.00	538.8	5.12	89.8	114.7	2.36
"	55.0	16.18	.370	9.75	432.0	5.17	72.0	81.1	2.24
10	44.0	12.95	.310	9.00	244.2	4.34	48.8	57.3	2.10
9	38.0	11.22	.300	8.50	170.9	3.90	38.0	44.1	1.98
8	32.5	9.54	.290	8.00	114.4	3.46	28.6	32.9	1.86

TABLE VII

PROPERTIES OF BETHLEHEM I-BEAMS

DEPTH OF BEAM	WEIGHT PER FOOT	AREA OF SECTION	THICK- NESS OF WEB	WIDTH OF FLANGE	NEUTRAL AXIS PERPEN- DICULAR TO WEB AT CENTER			NEUTRAL AXIS COINCIDENT WITH CENTER LINE OF WEB	
					Moment of Inertia	Radius of Gyration	Section Modulus	Moment of Inertia	Radius of Gyration
Inches	Pounds	Square Inches	Inches	Inches					
30	120.0	35.30	.540	10.500	5239.6	12.18	349.3	165.0	2.16
28	105.0	30.88	.500	10.000	4014.1	11.40	286.7	131.5	2.06
26	90.0	26.49	.460	9.500	2977.2	10.60	229.0	101.2	1.95
24	84.0	24.80	.460	9.250	2381.9	9.80	198.5	91.1	1.92
"	83.0	24.59	.520	9.130	2240.9	9.55	186.7	78.0	1.78
"	73.0	21.47	.390	9.000	2091.0	9.87	174.3	74.4	1.86
20	82.0	24.17	.570	8.890	1559.8	8.03	156.0	79.9	1.82
"	72.0	21.37	.430	8.750	1466.5	8.28	146.7	75.9	1.88
"	69.0	20.26	.520	8.145	1268.9	7.91	126.9	51.2	1.59
"	64.0	18.86	.450	8.075	1222.1	8.05	122.2	49.8	1.62
"	59.0	17.36	.375	8.000	1172.2	8.22	117.2	48.3	1.66
18	59.0	17.40	.495	7.675	883.3	7.12	98.1	39.1	1.50
"	54.0	15.87	.410	7.590	842.0	7.28	93.6	37.7	1.54
"	52.0	15.24	.375	7.555	825.0	7.36	91.7	37.1	1.56
"	48.5	14.25	.320	7.500	798.3	7.48	88.7	36.2	1.59
15	71.0	20.95	.520	7.500	796.2	6.16	106.2	61.3	1.71
"	64.0	18.81	.605	7.195	664.9	5.95	88.6	41.9	1.49
"	54.0	15.88	.410	7.000	610.0	6.20	81.3	38.3	1.55
"	46.0	13.52	.440	6.810	484.8	5.99	64.6	25.2	1.36
"	41.0	12.02	.340	6.710	456.7	6.16	60.9	24.0	1.41
"	38.0	11.27	.290	6.660	442.6	6.27	59.0	23.4	1.44
12	36.0	10.61	.310	6.300	269.2	5.04	44.9	21.3	1.42
"	32.0	9.44	.335	6.205	228.5	4.02	38.1	16.0	1.30
"	28.5	8.42	.250	6.120	216.2	5.07	36.0	15.3	1.35
10	28.5	8.34	.390	5.990	134.6	4.02	26.9	12.1	1.21
"	23.5	6.94	.250	5.850	122.9	4.21	24.6	11.2	1.27
9	24.0	7.04	.365	5.555	92.1	3.62	20.5	8.8	1.12
"	20.0	6.01	.250	5.440	85.1	3.76	18.9	8.2	1.17
8	19.5	5.78	.325	5.325	60.6	3.24	15.1	6.7	1.08
"	17.5	5.18	.250	5.250	57.4	3.33	14.3	6.4	1.11

TABLE VIII

MOMENTS OF INERTIA AND SECTION MODULI: RECTANGULAR
CROSS SECTION

BREADTH b	HEIGHT h	MOMENT OF INERTIA $I = \frac{bh^3}{12}$	SECTION MODULUS $S = \frac{bh^2}{6}$	BREADTH b	HEIGHT h	MOMENT OF INERTIA $I = \frac{bh^3}{12}$	SECTION MODULUS $S = \frac{bh^2}{6}$	BREADTH b	HEIGHT h	MOMENT OF INERTIA $I = \frac{bh^3}{12}$	SECTION MODULUS $S = \frac{bh^2}{6}$
1	1	.0833	.166	4	4	21.33	10.66	8	8	341.33	85.33
	2	.66	.66		5	41.66	16.66		9	486	108
	3	2.25	1.5		6	72	24		10	666.66	133.33
	4	5.33	2.66		7	114.33	32.66		11	887.33	161.33
	5	10.42	4.16		8	170.66	42.66		12	1152	192
	6	18	6		9	243	54		13	1464.66	225.33
	7	28.58	8.16		10	333.33	66.66		14	1829.33	261.33
	8	42.66	10.66		11	443.66	80.66		15	2250	300
	9	60.75	13.5		12	576	96		16	2730.66	341.33
	10	83.33	16.66								
	11	110.92	20.16		5	52.08	20.83	9	9	546.75	121.5
	12	144	24		6	90	30		10	750	150
2	2	1.33	1.33	5	7	142.92	40.83		11	998.25	181.5
	3	4.5	3		8	213.33	53.33		12	1296	216
	4	10.66	5.33		9	303.75	67.5		13	1647.75	253.5
	5	20.83	8.33		10	416.66	83.33		14	2058	294
	6	36	12		11	554.58	100.83		15	2531.25	337.5
	7	57.16	16.33		12	720	120		16	3072	384
	8	85.33	21.33						17	3684.75	433.5
	9	121.5	27		6	108	36		18	4374	486
	10	166.66	33.33		7	171.5	49	10	10	833.33	166.66
	11	221.85	40.33		8	256	64		11	1109.16	201.66
	12	288	48		9	364.5	81		12	1440	240
3	3	6.75	4.5	6	10	500	100		13	1830.83	281.66
	4	16	8		11	665.5	121		14	2286.66	326.66
	5	31.25	12.5		12	864	144		15	2810	375
	6	54	18						16	3413.33	426.66
	7	85.75	24.5		7	200.08	57.16		17	4094.17	481.66
	8	128	32		8	298.66	74.66		18	4860	540
	9	182.25	40.5		9	425.25	94.5		19	5715.83	601.66
	10	250	50		10	583.33	116.66		20	6666.66	666.66
	11	332.75	60.5		11	776.42	141.16				
	12	432	72		12	1008	168				
					13	1281.58	197.16				
					14	1600.66	228.66				

TABLE IX

MOMENTS OF INERTIA AND SECTION MODULI: CIRCULAR
CROSS SECTION

DIAM- ETER	MOMENT OF INERTIA	SECTION MODULUS	DIAM- ETER	MOMENT OF INERTIA	SECTION MODULUS	DIAM- ETER	MOMENT OF INERTIA	SECTION MODULUS
1	.0491	.0982	35	73,662	4,209	69	1,112,660	32,251
2	.7854	.7854	36	82,448	4,580	70	1,178,588	33,674
3	3.976	2.651	37	91,998	4,973			
4	12.57	6.283	38	102,354	5,387	71	1,247,393	35,138
5	30.68	12.27	39	113,561	5,824	72	1,319,167	36,644
6	63.62	21.21	40	125,664	6,283	73	1,393,995	38,192
7	117.9	33.67				74	1,471,963	39,783
8	201.1	50.27	41	138,709	6,766	75	1,553,156	41,417
9	322.1	71.57	42	152,745	7,274	76	1,637,662	43,096
10	490.9	98.17	43	167,820	7,806	77	1,725,571	44,820
			44	183,984	8,363	78	1,816,972	46,589
11	718.7	130.7	45	201,289	8,946	79	1,911,967	48,404
12	1,018	169.6	46	219,787	9,556	80	2,010,619	50,265
13	1,402	215.7	47	239,531	10,193			
14	1,886	269.4	48	260,576	10,857	81	2,113,051	52,174
15	2,485	331.3	49	282,979	11,550	82	2,219,347	54,130
16	3,217	402.1	50	306,796	12,270	83	2,329,605	56,135
17	4,100	482.3				84	2,443,920	58,189
18	5,153	572.6	51	332,086	13,023	85	2,562,392	60,292
19	6,397	673.4	52	358,908	13,804	86	2,685,120	62,445
20	7,854	785.4	53	387,323	14,616	87	2,812,205	64,648
			54	417,393	15,459	88	2,943,748	66,903
21	9,547	909.2	55	449,180	16,334	89	3,079,853	69,210
22	11,499	1,045	56	482,750	17,241	90	3,220,623	71,569
23	13,737	1,194	57	518,166	18,181			
24	16,286	1,357	58	555,497	19,155	91	3,366,165	73,982
25	19,175	1,534	59	594,810	20,163	92	3,516,586	76,448
26	22,432	1,726	60	636,172	21,206	93	3,671,992	78,968
27	26,087	1,932				94	3,832,492	81,542
28	30,172	2,155	61	679,651	22,284	95	3,998,198	84,173
29	34,719	2,394	62	725,332	23,398	96	4,169,220	86,859
30	39,761	2,651	63	773,272	24,548	97	4,345,671	89,601
			64	823,550	25,736	98	4,527,664	92,401
31	45,333	2,925	65	876,240	26,961	99	4,715,315	95,259
32	51,472	3,217	66	931,420	28,225	100	4,908,727	98,175
33	58,214	3,528	67	989,166	29,527			
34	65,597	3,859	68	1,049,556	30,869			

TABLE X
FOUR-PLACE LOGARITHMS OF NUMBERS

1	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	0	1	2	3	4	5	6	7	8	9

TABLES

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50	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0	1	2	3	4	5	6	7	8	9

TABLE XI

CONVERSION OF LOGARITHMS

REDUCTION OF COMMON LOGARITHMS TO NATURAL LOGARITHMS

Rule for using Table. Divide the given common logarithm into periods of two digits and take from the table the corresponding numbers, having regard to their value as decimals. The sum will be the required natural logarithm.

Example. Find the natural logarithm corresponding to the common logarithm .497149.

COMMON LOGARITHMS

NATURAL LOGARITHMS

.49	1.1282667
.0071	.016348354
.000049	.00011282667
.497149	1.14472788067

COM- MON LOGA- RITHM	NATURAL LOGARITHM	COM- MON LOGA- RITHM	NATURAL LOGARITHM	COM- MON LOGA- RITHM	NATURAL LOGARITHM	COM- MON LOGA- RITHM	NATURAL LOGARITHM
1	2.30259	26	59.86721	51	117.43184	76	174.99647
2	4.60517	27	62.16980	52	119.73442	77	177.29905
3	6.90776	28	64.47238	53	122.03701	78	179.60164
4	9.21034	29	66.77497	54	124.33959	79	181.90422
5	11.51293	30	69.07755	55	126.64218	80	184.20681
6	13.81551	31	71.38014	56	128.94477	81	186.50939
7	16.11810	32	73.68272	57	131.24735	82	188.81198
8	18.42068	33	75.98531	58	133.54994	83	191.11456
9	20.73327	34	78.28789	59	135.85252	84	193.41715
10	23.02585	35	80.59048	60	138.15511	85	195.71973
11	25.32844	36	82.89306	61	140.45769	86	198.02232
12	27.63102	37	85.19565	62	142.76028	87	200.32490
13	29.93361	38	87.49823	63	145.06286	88	202.62749
14	32.23619	39	89.80082	64	147.36545	89	204.93007
15	34.53878	40	92.10340	65	149.66803	90	207.23266
16	36.84136	41	94.40599	66	151.97062	91	209.53524
17	39.14395	42	96.70857	67	154.27320	92	211.83783
18	41.44653	43	99.01116	68	156.57579	93	214.14041
19	43.74912	44	101.31374	69	158.87837	94	216.44300
20	46.05170	45	103.61633	70	161.18096	95	218.74558
21	48.35429	46	105.91891	71	163.48354	96	221.04817
22	50.65687	47	108.22150	72	165.78613	97	223.35075
23	52.95946	48	110.52408	73	168.08871	98	225.65334
24	55.26204	49	112.82667	74	170.39130	99	227.95592
25	57.56463	50	115.12925	75	172.69388	100	230.25851

TABLE XII
FUNCTIONS OF ANGLES

ANGLE	SIN	TAN	SEC	COSEC	COT	COS	
0	0.	0.	.0	∞	∞	1.	90
1	.0175	.0175	1.0001	57.299	57.290	.9998	89
2	.0349	.0349	1.0006	28.654	28.636	.9994	88
3	.0523	.0524	1.0014	19.107	19.081	.9986	87
4	.0698	.0699	1.0024	14.336	14.301	.9976	86
5	.0872	.0875	1.0038	11.474	11.430	.9962	85
6	.1045	.1051	1.0055	9.5668	9.5144	.9945	84
7	.1219	.1228	1.0075	8.2055	8.1443	.9925	83
8	.1392	.1405	1.0098	7.1853	7.1154	.9903	82
9	.1564	.1584	1.0125	6.3925	6.3138	.9877	81
10	.1736	.1763	1.0154	5.7588	5.6713	.9848	80
11	.1908	.1944	1.0187	5.2408	5.1446	.9816	79
12	.2079	.2126	1.0223	4.8097	4.7046	.9781	78
13	.2250	.2309	1.0263	4.4454	4.3315	.9744	77
14	.2419	.2493	1.0306	4.1336	4.0108	.9703	76
15	.2588	.2679	1.0353	3.8637	3.7321	.9659	75
16	.2756	.2867	1.0403	3.6280	3.4874	.9613	74
17	.2924	.3057	1.0457	3.4203	3.2709	.9563	73
18	.3090	.3249	1.0515	3.2361	3.0777	.9511	72
19	.3256	.3443	1.0576	3.0716	2.9042	.9455	71
20	.3420	.3640	1.0642	2.9238	2.7475	.9397	70
21	.3584	.3839	1.0712	2.7904	2.6051	.9336	69
22	.3746	.4040	1.0785	2.6695	2.4751	.9272	68
23	.3907	.4245	1.0864	2.5593	2.3559	.9205	67
24	.4067	.4452	1.0946	2.4586	2.2460	.9135	66
25	.4226	.4663	1.1034	2.3662	2.1445	.9063	65
26	.4384	.4877	1.1126	2.2812	2.0503	.8988	64
27	.4540	.5095	1.1223	2.2027	1.9626	.8910	63
28	.4695	.5317	1.1326	2.1301	1.8807	.8829	62
29	.4848	.5543	1.1434	2.0627	1.8040	.8746	61
30	.5000	.5774	1.1547	2.0000	1.7321	.8660	60
31	.5150	.6009	1.1666	1.9416	1.6643	.8572	59
32	.5299	.6249	1.1792	1.8871	1.6003	.8480	58
33	.5446	.6494	1.1924	1.8361	1.5399	.8387	57
34	.5592	.6745	1.2062	1.7883	1.4826	.8290	56
35	.5736	.7002	1.2208	1.7435	1.4281	.8192	55
36	.5878	.7265	1.2361	1.7013	1.3764	.8090	54
37	.6018	.7536	1.2521	1.6616	1.3270	.7986	53
38	.6157	.7813	1.2690	1.6243	1.2799	.7880	52
39	.6293	.8098	1.2868	1.5890	1.2349	.7771	51
40	.6428	.8391	1.3054	1.5557	1.1918	.7660	50
41	.6561	.8693	1.3250	1.5243	1.1504	.7547	49
42	.6691	.9004	1.3456	1.4945	1.1106	.7431	48
43	.6820	.9325	1.3673	1.4663	1.0724	.7314	47
44	.6947	.9657	1.3902	1.4396	1.0355	.7193	46
45	.7071	1.	1.4142	1.4142	1.	.7071	45
	Cos	COT	COSEC	SEC	TAN	SIN	ANGLE

TABLE XIII
BENDING MOMENT AND SHEAR DIAGRAMS

$$R_1 = R_2 = \frac{P}{2}.$$

$$M_a = M_c = 0.$$

$$M_b = \frac{Pl}{4}.$$

$$D = \frac{Pl^3}{48EI}.$$

$$D_{max} = \frac{Pab(2l-a)}{24EI} \sqrt{\frac{a(2l-a)}{3}}$$

$$R_1 = \frac{Pb}{l}, \quad R_2 = \frac{Pa}{l}.$$

$$M_a = M_c = 0.$$

$$M_b = P \frac{ab}{l}.$$

$$R_1 = R_2 = P.$$

$$M_a = M_c = 0.$$

$$M_b = M_E = Pa.$$

$$M_a = M_c = 0.$$

$$R_1 = R_2 = \frac{wl}{2}.$$

$$M_a = M_c = 0.$$

$$M_b = \frac{wl^2}{8}.$$

$$D = \frac{5wl^4}{384EI}.$$

$$R_1 = \frac{wl}{2} + \frac{Pb}{l}.$$

$$R_2 = \frac{wl}{2} + \frac{Pa}{l}.$$

$$d = \frac{l}{2} - \frac{Pd}{wl}.$$

$$M_{max} = R_1 d - P(d-a).$$

$$R_1 = \frac{11}{16}P.$$

$$R_2 = \frac{5}{16}P.$$

$$M_A = -\frac{3}{16}Pl.$$

$$M_B = \frac{5}{32}Pl.$$

$$d = \frac{3}{11}l.$$

$$R_1 = \frac{5}{8}wl.$$

$$R_2 = \frac{3}{8}wl.$$

$$M_A = -\frac{wl^2}{8}.$$

$$R_1 = R_2 = \frac{wl}{2}.$$

$$M_A = M_C = \frac{wl^2}{12}.$$

$$M_B = \frac{wl^2}{24}.$$

$$D = \frac{wl^4}{384EI}.$$

$$R_1 = R_2 = \frac{P}{2}.$$

$$M_A = M_C = -\frac{Pl}{8}.$$

$$M_B = \frac{Pl}{8}.$$

$$D = \frac{Pl^3}{192EI}.$$

$$R_1 = R_3 = \frac{5}{16}P.$$

$$R_2 = \frac{11}{8}P.$$

$$M_A = 0.$$

$$M_C = 0.$$

$$M_E = \frac{5}{32}Pl.$$

$$M_B = -\frac{3}{16}Pl.$$

$$M_F = \frac{5}{32}Pl.$$

$$R_1 = P + P_1 \frac{b}{l}.$$

$$R_2 = P + P_1 \frac{a}{l}.$$

$$M_b = -Pd.$$

$$M_c = -Pd.$$

$$M_F = P_1 \frac{ab}{l} - Pd.$$

$$R_1 = \frac{P_1 b - Pc}{l}.$$

$$R_2 = \frac{P_1 a + P(l+c)}{l}.$$

$$M_a = 0.$$

$$M_b = -Pc.$$

$$M_E = \frac{(P_1 b - Pc)a}{l}.$$

TABLE XIV

MENSURATION

CIRCULAR MEASURE

Circumference of circle = diameter $\times 3.1416$.

Diameter of circle = circumference $\times 0.3183$.

Side of square of same periphery as circle = diameter $\times 0.7854$.

Diameter of circle of same periphery as square = side $\times 1.2732$.

Side of inscribed square = diameter of circle $\times 0.7071$.

Length of arc = number of degrees $\times$ diameter $\times 0.008727$.

Circumference of circle whose diameter is $1 = \pi$.

$$r = \frac{v^2 + \frac{c^2}{4}}{2v}.$$

$$\text{or very nearly} = \frac{c^2}{8v}.$$

r = radius

c = chord

v = versine

o = ordinate

$$o = \sqrt{r^2 - x^2} - (r - v).$$

$$v = r - \sqrt{r^2 - \frac{c^2}{4}}, \text{ or very nearly } = \frac{c^2}{8r}.$$

NUMBER	COMMON LOGARITHM
$\pi = 3.14159265$	0.4971499
$\sqrt{\pi} = 1.772454.$	0.24857494
$\pi^2 = 9.869604$	0.99429995
$\frac{1}{\pi} = 0.318310.$	9.50285013 - 10
$\frac{1}{\pi^2} = 0.101321.$	9.00570025 - 10
$\frac{1}{\sqrt{\pi}} = 0.564190$	9.75142506 - 10

1 radian = angle subtended by circular arc equal in length to the radius of the circle ; π radians = 180 degrees ;

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = 57.29577951^\circ.$$

Segment of circle = area of sector less triangle ; also for flat segments very

$$\text{nearly} = \frac{4v}{3} \sqrt{0.388 v^2 + \frac{c^2}{4}}.$$

Side of square of same area as circle = diameter $\times 0.8862$; also = circumference $\times 0.2821$.

Diameter of circle of same area as square = side $\times 1.1284$.

Area of parabola = base $\times \frac{2}{3}$ height.

Area of ellipse = long diameter $\times$ short diameter $\times 0.7854$.

Area of regular polygon = sum of sides $\times$ half perpendicular distance from center to sides.

TABLE XV

FRACTIONAL AND DECIMAL EQUIVALENTS

FRACTIONS OF A LINEAL INCH IN DECIMALS

FRACTIONS	DECIMALS OF AN INCH	FRACTIONS	DECIMALS OF AN INCH	FRACTIONS	DECIMALS OF AN INCH	FRACTIONS	DECIMALS OF AN INCH
$\frac{1}{84}$	0.015625	$\frac{17}{64}$	0.265625	$\frac{33}{64}$	0.515625	$\frac{49}{64}$	0.765625
$\frac{1}{32}$	0.03125	$\frac{9}{32}$	0.28125	$\frac{31}{64}$	0.53125	$\frac{47}{64}$	0.78125
$\frac{3}{64}$	0.04687	$\frac{19}{64}$	0.296875	$\frac{35}{64}$	0.546875	$\frac{53}{64}$	0.796875
$\frac{1}{16}$	0.0625	$\frac{5}{16}$	0.3125	$\frac{17}{32}$	0.5625	$\frac{13}{16}$	0.8125
$\frac{5}{64}$	0.078125	$\frac{21}{64}$	0.328125	$\frac{37}{64}$	0.578125	$\frac{55}{64}$	0.828125
$\frac{3}{32}$	0.09375	$\frac{13}{32}$	0.34375	$\frac{19}{32}$	0.59375	$\frac{23}{32}$	0.84375
$\frac{7}{64}$	0.109375	$\frac{23}{64}$	0.359375	$\frac{39}{64}$	0.609375	$\frac{57}{64}$	0.859375
$\frac{1}{8}$	0.125	$\frac{3}{8}$	0.375	$\frac{5}{8}$	0.625	$\frac{7}{8}$	0.875
$\frac{9}{64}$	0.140625	$\frac{55}{64}$	0.390625	$\frac{11}{16}$	0.640625	$\frac{59}{64}$	0.890625
$\frac{5}{32}$	0.15625	$\frac{13}{32}$	0.40625	$\frac{21}{32}$	0.65625	$\frac{29}{32}$	0.90625
$\frac{3}{16}$	0.171875	$\frac{27}{64}$	0.421875	$\frac{43}{64}$	0.671875	$\frac{57}{64}$	0.921875
$\frac{3}{8}$	0.1875	$\frac{7}{8}$	0.4375	$\frac{11}{16}$	0.6875	$\frac{15}{16}$	0.9375
$\frac{13}{64}$	0.203125	$\frac{29}{64}$	0.453125	$\frac{25}{32}$	0.703125	$\frac{61}{64}$	0.953125
$\frac{7}{32}$	0.21875	$\frac{15}{32}$	0.46875	$\frac{23}{32}$	0.71875	$\frac{31}{32}$	0.96875
$\frac{15}{64}$	0.234375	$\frac{31}{64}$	0.484375	$\frac{47}{64}$	0.734375	$\frac{63}{64}$	0.984375
$\frac{1}{4}$	0.25	$\frac{1}{2}$	0.5	$\frac{3}{4}$	0.75	1	1.000

LINEAL INCHES IN DECIMAL FRACTIONS OF A LINEAL FOOT

LINEAL INCHES	LINEAL FOOT	LINEAL INCHES	LINEAL FOOT	LINEAL INCHES	LINEAL FOOT
$\frac{1}{84}$	0.001302083	$\frac{17}{8}$	0.15625	$\frac{63}{4}$	0.5625
$\frac{1}{32}$	0.00260416	2	0.1666	7	0.5833
$\frac{1}{16}$	0.0052083	$\frac{21}{8}$	0.177083	$\frac{71}{4}$	0.60416
$\frac{1}{8}$	0.010416	$\frac{23}{8}$	0.1875	$\frac{73}{4}$	0.625
$\frac{1}{4}$	0.015625	$\frac{25}{8}$	0.197916	$\frac{75}{4}$	0.64583
$\frac{1}{2}$	0.02083	$\frac{27}{8}$	0.2088	8	0.66667
$\frac{3}{8}$	0.0260416	$\frac{29}{8}$	0.21875	$\frac{81}{4}$	0.6875
$\frac{1}{2}$	0.03125	$\frac{31}{8}$	0.22916	$\frac{83}{4}$	0.7083
$\frac{3}{4}$	0.0364583	$\frac{33}{8}$	0.239583	$\frac{85}{4}$	0.72916
$\frac{1}{2}$	0.0416	3	0.25	9	0.75
$\frac{3}{4}$	0.046875	$\frac{31}{4}$	0.27083	$\frac{91}{4}$	0.77083
$\frac{5}{8}$	0.052083	$\frac{33}{4}$	0.2916	$\frac{93}{4}$	0.7916
$\frac{3}{4}$	0.0572916	$\frac{35}{4}$	0.3125	$\frac{95}{4}$	0.8125
$\frac{3}{4}$	0.0625	4	0.33333	10	0.83333
$\frac{7}{8}$	0.0677083	$\frac{41}{4}$	0.35416	$\frac{101}{4}$	0.85416
$\frac{7}{8}$	0.072916	$\frac{43}{4}$	0.375	$\frac{103}{4}$	0.875
$\frac{7}{8}$	0.078125	$\frac{45}{4}$	0.39583	$\frac{105}{4}$	0.89583
1	0.0833	5	0.4166	11	0.9166
$\frac{11}{8}$	0.09375	$\frac{51}{4}$	0.4375	$\frac{111}{4}$	0.9375
$\frac{11}{8}$	0.10416	$\frac{53}{4}$	0.4583	$\frac{113}{4}$	0.9583
$\frac{11}{8}$	0.114583	$\frac{55}{4}$	0.47916	$\frac{115}{4}$	0.97916
$\frac{11}{8}$	0.125	6	0.5	12	1.000
$\frac{11}{8}$	0.135416	$\frac{61}{4}$	0.52083	...	.....
$\frac{11}{8}$	0.14583	$\frac{63}{4}$	0.5416	...	.....

TABLE XVI

WEIGHTS OF VARIOUS SUBSTANCES

MATERIAL		WEIGHT IN LB./FT. ³
Brick and brickwork	Pressed brick	150
	Common hard brick	125
	Soft inferior brick	100
	Good pressed-brick masonry	140
	Ordinary brickwork	125
Stone and masonry	Gneiss, solid	168
	Gneiss, loose piles	96
	Granite	170
	Limestone and marble	165
	Limestone and marble, loose, broken	96
	Sandstone, solid	151
	Sandstone, quarried and piled	86
	Shale	162
	Slate	175
	Granite or limestone masonry, well dressed	165
	Granite or limestone masonry, mortar rubble	154
	Granite or limestone masonry, well-scabbled dry rubble	138
	Sandstone masonry, well dressed	144
Earth, sand, and gravel	Common loam, dry, loose	76
	Common loam, moderately rammed	95
	Soft flowing mud	110
	Dry hard mud	80-110
	Gravel or sand, dry, loose	90-106
	Gravel or sand, well shaken	99-117
	Gravel or sand, wet	120-140
Metals	Aluminium	162
	Brass, cast (copper and zinc)	504
	Brass, rolled	524
	Bronze (copper 8, tin 1)	529
	Copper, cast	542
	Copper, rolled	555
	Iron, cast	450
	Iron, wrought	480
	Lead	710
	Platinum	1342
	Steel	489.6
	Tin, cast	459
	Zinc, spelter	437.5
Coal	Anthracite, solid Pennsylvania	93
	Anthracite, broken, loose (heaped bushel 80 lb.)	54
	Anthracite, broken, shaken	58
	Bituminous, solid	84
	Bituminous, loose (heaped bushel 74 lb.)	49
	Bituminous, broken, shaken	51-56
	Coke, loose (heaped bushel 40 lb.)	26

WEIGHTS OF VARIOUS SUBSTANCES — *Continued*

MATERIAL		WEIGHT IN LB./FT. ³
Lime and hydraulic cement	Quicklime, loose (struck bushel 66 lb.)	53
	Quicklime, well shaken	75
	American Louisville, loose	50
	American Rosendale, loose	56
	American Cumberland, loose	65
	American Cumberland, well shaken	85
	English Portland	90
	American Portland, loose	88
	American Portland, well shaken	110
Dry wood	Ash, American white	38
	Cherry	42
	Chestnut	41
	Cypress	64
	Ebony	76
	Elm	35
	Hemlock	25
	Hickory	53
	Lignum-vitæ	83
	Locust	44
	Mahogany, Spanish	53
	Mahogany, Honduras	35
	Maple	49
	Oak, live	59
	Oak, red or black	32-45
	Oak, white	48
	Pine, white	25
	Pine, yellow Northern	34
	Pine, yellow Southern	45
	Poplar	29
	Sycamore	37
	Spruce	25
	Walnut, black	38

TABLE XVII

STRENGTH OF ROPES AND BELTS

TENSION TESTS OF STEEL WIRE ROPE

CIRCUMFERENCE in.	NUMBER OF STRANDS	WIRES PER STRAND	MEAN DIAMETER OF WIRES in.	CORE	SECTIONAL AREA OF WIRE in. ²	TENSILE STRENGTH	
						Total lb.	Total lb./in. ²
1.5	6	18	.0321	Hemp	.0876	12,898	147,236
1.75	6	18	.0349	Hemp	.1031	15,736	153,836
2	6	18	.0420	Hemp	.1499	20,780	138,900
2.125	6	18	.0456	Hemp	.1766	24,430	138,383
2.25	6	18	.0488	Hemp	.2021	30,960	143,650
2.50	6	18	.0544	Hemp	.2510	33,270	132,500
3	6	18	.0598	Hemp	.3024	46,370	153,340
3.50	6	18	.0718	Hemp	.4380	65,120	148,675
4.50	6	18	.0980	Hemp	.8151	138,625	170,075

STRENGTH OF IRON WIRE ROPE *

(Rope composed of six strands and a hemp center, seven or twelve wires in each strand)

DIAMETER in.	CIRCUMFERENCE in.	APPROXIMATE BREAK- ING STRENGTH lb.	CIRCUMFERENCE IN INCHES OF NEW MANILA ROPE OF EQUAL STRENGTH
1.75	5.50	88,000	11
1.625	5.00	72,000	10
1.50	4.75	64,000	9.5
1.375	4.25	52,000	8.5
1.25	4.00	46,000	8.0
1.125	3.50	36,000	6.5
1.000	3.00	26,000	5.75
.875	2.75	22,000	5.25
.750	2.25	14,600	4.75
.500	1.50	6,400	3.00
.375	1.125	3,600	2.25
.250	.75	1,620	1.50

STRENGTH OF CAST STEEL WIRE ROPE *

(Rope composed of six strands and a hemp center, seven or nineteen wires in each strand)

DIAMETER in.	CIRCUMFERENCE in.	APPROXIMATE BREAK- ING STRENGTH lb.	CIRCUMFERENCE IN INCHES OF NEW MANILA ROPE OF EQUAL STRENGTH
1.25	4.00	106,000	13
2.125	3.50	82,000	11
1.00	3.00	62,000	9
.875	2.75	52,000	8.5
.750	2.25	35,200	7.0
.625	2.00	28,000	6.0
.500	2.50	16,200	4.75
.375	1.125	9,000	3.75

* As given by John A. Roebling.

TABLE XVII (Continued)

TESTS OF MANILA AND SISAL ROPE

MANILA ROPE

SIZE OF ROPE	DIAMETER in.	SECTIONAL AREA in. ²	TENSILE STRENGTH		TOTAL LOAD lb.
			lb./in. ²	Per Yarn lb.	
6-thread	.27	.0567	13,360	126	750
9-thread	.30	.0750	14,180	118	1,064
12-thread	.38	.114	12,920	123	1,473
15-thread	.43	.153	14,250	145	2,180
1.25-in.	.49	.192	11,610	125	2,242
1.50-in.	.56	.259	11,970	148	3,100
1.625-in.	.61	.288	10,800	130	3,120
1.75-in.	.62	.299	11,500	128	3,455
2-in.	.74	.41	9,200	114	3,775
2.25-in.	.79	.478	12,900	148	6,207
2.25-in.	.78	.462	11,900	138	5,509
2.50-in.	.85	.557	12,470	136	6,947
2.75-in.	.96	.715	12,810	153	9,160
3-in.	1.00	.782	13,680	146	10,663
3-in.	.99	.746	13,750	151	10,260
3.25-in.	1.13	.970	12,470	144	12,093
3.50-in.	1.19	1.07	12,190	132	13,050
3.75-in.	1.29	1.27	11,990	134	15,227
3.75-in.	1.28	1.26	11,610	132	14,640
4-in.	1.39	1.46	10,080	117	14,723
4-in.	1.34	1.36	11,790	130	16,017
4.25-in.	1.41	1.51	9,890	113	14,943
4.50-in.	1.59	1.88	10,360	121	19,577
4.50-in.	1.61	1.99	10,480	134	20,873
4.75-in.	1.66	2.04	10,740	128	21,903
5-in.	1.76	2.35	9,940	118	23,360
6-in.	2.25	3.82	8,260	112	31,570
7-in.	2.52	4.86	9,400	125	45,647
8-in.	2.83	6.22	8,600	118	54,000
9-in.	3.35	8.37	7,500	108	62,717
10-in.	3.70	10.06	7,300	102	73,910

SISAL ROPE

SIZE OF ROPE	DIAMETER in.	SECTIONAL AREA in. ²	TENSILE STRENGTH		TOTAL LOAD lb.
			lb./in. ²	Per Yarn lb.	
6-thread	.27	.0567	7,700	72	432
9-thread	.33	.082	7,300	67	605
12-thread	.39	.126	7,500	79	944
1.25-in.	.45	.129	10,810	93	1397
1.50-in.	.56	.254	8,100	99	2067
1.75-in.	.63	.302	7,600	96	2315
2-in.	.70	.395	7,200	97	2925
2.25-in.	.81	.416	9,500	94	3966
2.75-in.	.95	.691	8,300	101	5733
3-in.	1.01	.780	7,500	104	5917
3.50-in.	1.22	1.128	7,200	102	8230

TABLE XVII (Continued)
TESTS OF RUBBER BELTING

DESCRIPTION	DIMENSIONS in.			SECTIONAL AREA in. ²	TENSILE STRENGTH	
	Length	Width	Thick- ness		lb./in. ²	Pound per Inch of Width
2-in., 4-ply . . .	60.17	2.02	.26	.525	3276	851
6-in., 4-ply . . .	60.17	6.08	.26	1.58	3227	839
6-in., 4-ply . . .	60.12	6.13	.26	1.59	3773	979
6-in., 4-ply . . .	60.17	6.05	.26	1.57	2739	711
12-in., 4-ply . . .	60.02	12.08	.27	3.26	3037	819
12-in., 4-ply . . .	60.14	12.24	.26	3.18	2987	776
2-in., 6-ply . . .	60.17	2.14	.36	.770	3104	1116
6-in., 6-ply . . .	59.98	6.26	.37	2.32	2737	1014
6-in., 6-ply . . .	60.08	6.27	.36	2.26	3770	1358
12-in., 6-ply . . .	60.15	12.04	.36	4.33	3436	1236
12-in., 6-ply . . .	60.17	12.16	.34	4.13	3862	1311
24-in., 6-ply . . .	60.13	24.11	.41	9.89	2381	977
30-in., 6-ply . . .	60.04	30.18	.40	12.07	2808	1123

TESTS OF LEATHER BELTING

DESCRIPTION*	DIMENSIONS in.			SECTIONAL AREA in. ²	TENSILE STRENGTH	
	Length	Width	Thick- ness		lb./in. ²	Pounds per Inch of Width
2-in., single . . .	60.00	1.98	.20	.396	5045	1091
6-in., single . . .	60.20	6.07	.22	1.34	2537	560
6-in., single (w) .	60.11	6.08	.24	1.46	2119	533
12-in., single . . .	60.11	12.05	.18	2.17	3917	705
4-in., double . . .	59.55	3.98	.33	1.31	4931	1623
6-in., double . . .	60.18	5.91	.47	2.78	4309	2027
6-in., double (w) .	59.93	6.00	.40	2.40	5166	2066
12-in., double . . .	59.90	11.90	.39	4.64	4090	1595
12-in., double (w) .	60.06	11.93	.36	4.29	4424	1591
24-in., double (w) .	60.00	23.90	.47	11.23	2760	1297
30-in., double . . .	59.90	29.95	.43	12.88	2717	1169

* The letter *w* in the table stands for *waterproofed*.

RESISTANCE OF MATERIALS

SECTION I

STRESS AND DEFORMATION

1. **Elastic resistance, or stress.** The effect of an external force acting upon an elastic body is to produce deformation, or change of shape. For example, if a bar is placed in a testing machine and a tensile load applied, it will be found that the length of the bar is increased and the area of its cross section diminished. Similarly, if a compressive load is applied, the length of the bar is diminished and the area of its cross section increased.

All solid bodies offer more or less resistance to the deformation, or change of shape, produced by external force. This internal resistance, when expressed in definite units, is called **stress**. A body under the action of stress is said to be **strained**.

In general the stress is not the same at all points of a body, but varies from point to point. The intensity of the stress at any particular point is therefore expressed as the force in pounds which would be exerted if the stress were uniform and acted over an area one square inch in extent. That is to say, whatever the actual extent of the area considered, whether finite or infinitesimal, the stress is expressed in pounds per square inch (abbreviated into lb./in.²).

For example, suppose that a wire $\frac{1}{4}$ in. in diameter is pulled with a force of 50 lb. Then for equilibrium the total stress acting on any cross section of the wire must also be 50 lb. But since the area of the cross section is only .049 in.², the intensity of the stress is $\frac{50}{.049}$, or 1000 lb./in.² In other words, if the wire were 1 sq. in. in cross section, the strain under a load of 1000 lb. would be the same as that produced by a load of 50 lb. on a wire $\frac{1}{4}$ in. in diameter.

Taking any plane section of a body under strain, the stress acting on this plane section may in general be resolved, like any force, into two components, one perpendicular to the plane and the other lying in the plane. The perpendicular, or normal, component is called **direct stress** and is either **tension** or **compression**. The tangential component, or that lying in the plane of the cross section, is called **shear**. In what follows, the letter p will always be used to denote normal, or direct, stress, and q to denote tangential stress, or shear.

FIG. 1

The effect of a normal stress is to produce extension or compression, that is, a lengthening or shortening of the fibers, thereby changing the dimensions of the body; whereas shear tends to slide any given cross section over the one adjacent to it, thus producing angular deformation, or change in shape, of the body without altering its dimensions.

2. Varieties of strain. The nature of the deformation produced by external forces acting on an elastic body depends on where and how these forces are applied. Although only two kinds of stress can occur, —namely, normal stress (tension or compression) and

shear, — these may arise in various ways. In general, five different cases of strain may be distinguished, each of which must be considered separately. These are as follows:

1. If the forces act along the same line, toward or away from one another, the strain is called compression or tension (Fig. 1, *a*).

2. If the forces tend to slice off a portion of the body by separating it along a surface, the strain is called shear (Fig. 1, *b*).

FIG. 2

3. If the forces act transverse to the length of the body (usually perpendicular to the long axis of the piece), so as to produce lateral deflection, the strain is called bending, or flexure (Fig. 1, *c*).

4. If one dimension of the body is large as compared with the other two, and the forces act in the direction of the long dimension and toward one another, the strain is called buckling, or column flexure (Fig. 1, *d*).

5. If the forces exert a twist on the body, the strain is called torsion (Fig. 1, *e*).

Two or more of these simple strains may occur in combination, as illustrated in Fig. 1, *f*.

3. Strain diagram. In the case of tension or compression it is easy to show graphically the chief features of the strain. Thus, suppose that a test bar is placed in a testing machine, and that the total load on the bar at any instant is read on the scale beam of the machine, and its corresponding length in inches is measured with an extensometer. Assuming that the stress is uniformly distributed over any cross section of the bar, the unit stress is obtained by dividing the total load in pounds acting on the bar by the area of its cross section in square inches. That is,

$$(1) \quad p = \text{unit stress} = \frac{\text{total load in pounds}}{\text{area of cross section in square inches}}.$$

Also, the total deformation, or change in length, is divided by the original unstrained length of the bar, giving the unit deformation in inches per inch. Let this be denoted by s ; that is, let

$$(2) \quad s = \text{unit deformation} = \frac{\text{change in length}}{\text{original length}}.$$

The unit deformation is therefore an abstract number. Moreover, both the unit stress and the unit deformation are independent of the actual dimensions of the test bar and depend only on the physical properties of the material.

If, now, the unit stresses are plotted as ordinates and the corresponding unit deformations as abscissas, a strain diagram is obtained, as shown in Fig. 2. Such a diagram shows at a glance the physical properties of the material it represents, as explained in what follows.

4. Hooke's law. By inspection of the curves in Fig. 2 it is evident that the strain diagram for each material has certain characteristic features. For instance, in the case of wrought iron the strain diagram from O to A is a straight line; this means that for points between O and A the stress is proportional to the corresponding deformation. That is to say, within certain limits the ratio of p to s is constant, or

$$(3) \quad \frac{p}{s} = E,$$

where the constant E denotes the slope of the initial line. This important property is known as **Hooke's law**, and the constant E is called **Young's modulus of elasticity**.

The upper end *A* of the initial line, or point where the diagram begins to curve, is called the **elastic limit**. The point *B*, where the deformation becomes very noticeable, is called the **yield point**. As these two points occur close together, no distinction is made between them in ordinary commercial testing.

The maximum ordinate to the strain diagram represents the greatest unit stress preceding rupture, and is called the **ultimate strength** of the material.

In the case of shear let q denote the unit shearing stress and ϕ the corresponding angular deformation expressed in circular measure. Then, by Hooke's law,

$$(4) \qquad \frac{q}{\phi} = G,$$

where G is a constant for any given material, called the **modulus of rigidity**, or **shear modulus**. For steel and wrought iron $G = .4 E$, approximately.

Average values of E and G for various materials are given in Table I.

5. Elastic limit. It is found by experiment that as long as the stress does not pass the elastic limit, the deformation disappears when the external forces are removed. If the unit stress (or, more properly, the unit deformation) exceeds the elastic limit, however, then the deformation does not entirely disappear upon removal of the load, but the body retains a permanent set. At the elastic limit, therefore, the body begins to lose its elastic properties, and hence, in constructions which are intended to last for any length of time, the members should be so designed that the actual stresses lie well below the elastic limit.

It has also been found by experiment that, for iron and steel, if the stress lies well within the elastic limit, it can be removed and repeated indefinitely without causing rupture; but if the metal is stressed beyond the elastic limit, and the stress is repeated or alternates between tension and compression, it will eventually cause rupture, the number of changes necessary to produce failure decreasing as the difference between the upper and lower limits of the strain increases. This is known as the **fatigue of metals**,

and indicates that in determining the resistance of any material the elastic limit is much more important than the ultimate strength.

Overstrain of any kind results in a gradual hardening of the material. Where this has already occurred, the elastic properties of the material can be partially or wholly restored by **annealing**; that is, by heating the metal to a cherry redness and allowing it to cool slowly.

6. Working stress. The stress which can be carried by any material without losing its elastic properties is called the **allowable stress** or **working stress**, and must always lie below the elastic limit. The ratio of the assumed working stress to the ultimate strength of the material is called the **factor of safety**; that is,

$$(5) \quad \text{Working stress} = \frac{\text{ultimate strength}}{\text{factor of safety}}.$$

Average values of the ultimate strength, factors of safety, and other elastic constants for the various materials used in construction are given in Table I.

Since for wrought iron and steel the elastic limit can be definitely located, the working stresses for these materials is usually assumed as a certain fraction, say $\frac{1}{2}$ to $\frac{2}{3}$, of the elastic limit.

Materials like cast iron, stone, and concrete have no definite elastic limits; that is, they do not conform perfectly to Hooke's law. For such materials, therefore, the working stress is usually assumed as a small fraction, say from $\frac{1}{5}$ to $\frac{1}{20}$, of the ultimate strength.

Under repeated loads, where the stress varies an indefinite number of times between zero and some large value, the working stress may be assumed as $\frac{2}{3}$ of its value for a static load.

If the stress alternates between large positive and negative values, that is, between tension and compression, the working stress may be assumed as $\frac{1}{3}$ of its value for a static load.

For example, if the elastic limit for mild steel is 35,000 lb./in.², the working stress for a static load may be taken as 18,000 lb./in.²; for repeated loads, either tensile or compressive, as 12,000 lb./in.²; and for loads alternating between tension and compression, as 6000 lb./in.²

ALLOWABLE UNIT STRESSES IN LB./IN.²

(Also called working stress or skin stress)

DEAD LOAD

	ELASTIC LIMIT	TENSION	COMPRES- SION	FLEXURE	SHEAR	TORSION
Structural steel .	32,000	16,000	16,000	16,000	12,000	9,000
Machinery steel .	36,000	18,000	18,000	18,000	13,500	13,500
Crucible O.H. steel	40,000	20,000	20,000	20,000	16,000	16,000
Cast steel . . .	30,000	12,000	18,000	15,000	13,500	10,000
Wrought iron . .	25,000	12,500	12,500	12,500	9,000	6,000
Cast iron . . .		4,500	15,000	7,500	4,500	4,500
Phosphor bronze .	20,000	10,000	10,000	10,000	8,000	8,000
Timber	3,000	1,500	1,000	1,200	120	

LIVE LOAD

Structural steel .		10,600	10,600	10,600	8,000	6,000
Machinery steel .		12,000	12,000	12,000	9,000	9,000
Crucible O.H. steel		13,300	13,300	13,300	10,600	10,600
Cast steel . . .		8,000	12,000	10,000	9,000	6,600
Wrought iron . .		8,400	8,400	8,400	6,000	4,000
Cast iron . . .		3,000	10,000	5,000	3,000	3,000
Phosphor bronze .		6,600	6,600	6,600	5,400	5,400
Timber		1,000	660	800	80	

REVERSIBLE LOAD

Structural steel .		5300	5300	5300	4000	3000
Machinery steel .		6000	6000	6000	4500	4500
Crucible O.H. steel		6600	6600	6600	5300	5300
Cast steel . . .		4000	6000	5000	4500	3300
Wrought iron . .		4200	4200	4200	3000	2000
Cast iron . . .		1500	5000	2500	1500	1500
Phosphor bronze .		3300	3300	3300	2700	2700
Timber		500	330	400	40	

In actual practice the unit working stresses are usually specified, and the designer simply follows his specifications without reference to the factor of safety. Where a large number of men are employed, this method eliminates the personal equation and insures uniformity of results. The student should become familiar with both methods, however, as it is no small part of an engineer's training to know what relation his working stress should bear to the elastic limit and ultimate strength of the material.

7. Resilience. The work done in straining a bar up to the elastic limit of the material is called the **resilience** of the bar. The area under the strain diagram from the origin up to the elastic limit evidently represents the work done on a unit volume of the material, say one cubic inch, in straining it up to the elastic limit. This area therefore represents the resilience per unit volume, and is called the **modulus of elastic resilience** of the material.

Thus, if p denotes the unit stress and s the unit deformation at the elastic limit, then, since the strain diagram up to this point is a straight line, the area subtended by it, or modulus of elastic resilience, is $\frac{1}{2}ps$. Since $\frac{p}{s} = E$, the expression for the modulus may therefore be written

$$(6) \quad \text{Modulus of elastic resilience} = \frac{p^2}{2E}.$$

Hence, if V denotes the volume of the test piece, its total resilience is

$$(7) \quad \text{Total resilience} = \frac{p^2 V}{2E}.$$

The resilience of a bar is a measure of its ability to resist a blow or shock without receiving a permanent set. If a load W is applied gradually, as in a testing machine, the maximum stress when the load is all on is $p = \frac{W}{A}$, where A denotes the area of the cross section of the bar. If, however, the load W is applied suddenly, as in falling from a height h , it produces a certain deformation of the bar, say Δl , and consequently the total external work done on the bar is

$$\text{External work} = (h + \Delta l) W.$$

If, now, the unit stress p produced by the impact lies below the elastic limit, the total internal work of deformation is

$$\text{Internal work} = \frac{1}{2} (Ap) \Delta l.$$

In the case of a suddenly applied, or impact, load, like that due to a train crossing a bridge at high speed, $h = 0$, and, equating the expressions for the internal and external work, the result is

$$\Delta l W = \frac{1}{2} Ap \Delta l,$$

whence

$$p = \frac{2W}{A}.$$

Comparing this with the expression for the stress produced by a static load, namely, $p = \frac{W}{A}$, it is evident that a suddenly applied load produces twice the stress that would be produced by the same load if applied gradually.

8. Poisson's ratio. Experiment shows that when a bar is subjected to tension or compression, its lateral, or transverse, dimensions are changed, as well as its length. Thus, if a rod is pulled, it increases in length and decreases in diameter; if it is compressed, it decreases in length and increases in diameter.

It was found by Poisson that the ratio of the unit lateral deformation to the unit change in length is constant for any given material. This constant is usually denoted by $\frac{1}{m}$, and is called **Poisson's ratio**. Its average value for metals such as steel and wrought iron is .3. Thus, suppose that the load on a steel bar produces a certain unit deformation s lengthways of the bar. Then its unit lateral deformation will be approximately .3 s . Hence, the total lateral deformation is found by multiplying this unit deformation by the width, or diameter, of the bar.

Values of Poisson's ratio for various materials are given in Table I.

9. Temperature stress. A property especially characteristic of metals is that of expansion and contraction with rise and fall of temperature. The proportion of its length which a bar free to move expands when its temperature is raised one degree is called its

coefficient of linear expansion, and will be denoted by C . Values of this constant for various materials are given in Table I.

If a bar is prevented from expanding or contracting, then change in temperature produces stress in the bar, called **temperature stress**. Thus, let l denote the original length of a bar, and suppose its temperature is raised a certain amount, say T degrees. Then, if C denotes the coefficient of linear expansion for the material, and Δl the amount the bar would naturally lengthen if free to move, we have

$$\Delta l = CTl,$$

and consequently the unit deformation is

$$s = \frac{\Delta l}{l} = CT.$$

Therefore, if p denotes the unit temperature stress,

$$(8) \quad p = sE = CTE.$$

APPLICATIONS

1. A 5-in. copper cube supports a load of 75 tons. Find its change in volume.

Solution. Area of one face = $5 \times 5 = 25$ in.² Unit compressive stress p on this area is then $p = \frac{75 \times 2000}{25} = 6000$ lb./in.² Modulus of elasticity for copper $E = 15,000,000$ lb./in.² Therefore unit vertical contraction $s = \frac{p}{E} = \frac{1}{2500}$; total vertical contraction $\Delta l = l \cdot s = 5 \cdot \frac{1}{2500} = \frac{1}{500}$ in. Since Poisson's ratio for copper is $\frac{1}{m} = .340$, the unit deformation laterally is $.340s = \frac{.340}{2500}$, and the total lateral deformation is $5 \times \frac{.340}{2500} = .00068$ in. The three dimensions of the deformed cube are therefore 5.00068, 5.00068, 4.998; its volume is 124.984 in.³, and the decrease in volume is .016 in.³

2. A $\frac{3}{4}$ -in. wrought-iron bolt has a head $\frac{5}{8}$ in. deep. If a load of 4 tons is applied longitudinally, find the factors of safety in tension and shear.

Solution. Area of body of bolt at root of thread = .442 in.² Unit tensile stress in bolt is $p = \frac{4 \times 2000}{.442} = 18,000$ lb./in.² Factor of safety in tension = $\frac{50,000}{18,000} = 2.7$.

Area in shear = $\pi \cdot \frac{3}{4} \cdot \frac{5}{8} = 1.47$ in.² Unit shearing stress is $\frac{4 \times 2000}{1.47} = 5440$ lb./in.²

Factor of safety in shear = $\frac{40,000}{5440} = 7.3$.

3. A steel ring fits loosely over a cylindrical steel pin 3 in. in diameter. How much clearance, or space between them, should there be in order that, when the pin is subjected to a compressive load of 60 tons, the ring shall fit tightly?

Solution. Unit compressive stress in pin is

$$p = \frac{60 \cdot 2000}{\frac{\pi 9}{4}} = 17,000 \text{ lb./in.}^2$$

Unit longitudinal deformation $s = \frac{p}{E} = \frac{17,000}{30,000,000} = .000566$. Unit transverse deformation $= .295 \times .000566 = .000167$. Radial clearance $= 1.5 \times .000167 = .00025$ in.

4. A post 1 ft. in diameter supports a load of 1 ton. Assuming that the stress is uniformly distributed over any cross section, find the unit normal stress.

5. A shearing force of 50 lb. is uniformly distributed over an area 4 in. square. Find the unit shear.

6. A steel rod 500 ft. long and 1 in. in diameter is pulled by a force of 25 tons. How much does it stretch, and what is its unit elongation?

7. A copper wire 10 ft. long and .04 in. in diameter is tested and found to stretch .289 in. under a pull of 50 lb. What is the value of Young's modulus for copper deduced from this experiment?

8. A round cast-iron pillar 18 ft. high and 10 in. in diameter supports a load of 12 tons. How much does it shorten, and what is its unit contraction?

9. A wrought-iron bar 20 ft. long and 1 in. square is stretched .266 in. What is the force acting on it?

10. What is the lateral contraction of the bar in problem 9?

11. A soft-steel cylinder 1 ft. high and 2 in. in diameter bears a weight of 40 tons. How much is its diameter increased?

12. A copper wire 100 ft. long and .025 in. in diameter stretches 2.16 in. when pulled by a force of 15 lb. Find the unit elongation.

13. If the wire in problem 12 was 250 ft. long, how much would it lengthen under the same pull?

14. A vertical wooden post 30 ft. long and 8 in. square shortens .00374 in. under a load of half a ton. What is its unit contraction?

15. How great a pull can a copper wire .2 in. in diameter stand without breaking?

16. How large must a square wrought-iron bar be made to stand a pull of 3000 lb.?

17. A mild-steel plate is $\frac{1}{4}$ in. thick. How wide should it be to stand a pull of 10 tons?

18. A round wooden post is 6 in. in diameter. How great a load will it bear?

19. A wrought-iron bar is 20 ft. long at 32° F. How long will it be at 95° F.?

20. A cast-iron pipe 10 ft. long is placed between two heavy walls. What will be the stress in the pipe if the temperature rises 25°?

21. Steel railroad rails, each 30 ft. long, are laid at a temperature of 40° F. What space must be left between them in order that their ends shall just meet at 100° F.?

22. In the preceding problem, if the rails are laid with their ends in contact, what will be the temperature stress in them at 100° F.?

23. A $\frac{3}{4}$ -in. wrought-iron bolt failed in the testing machine under a pull of 15,000 lb. Find its ultimate tensile strength.

24. Four $\frac{1}{2}$ -in. steel cables are used with a block and tackle on the hoist of a crane whose capacity is rated at 6000 lb. What is the factor of safety? (Use tables for ultimate strength of rope.)

25. A vertical hydraulic press weighing 100 tons is supported by four $2\frac{1}{2}$ -in. round structural-steel rods. Find the factor of safety.

26. A block and tackle consists of six strands of flexible $\frac{1}{2}$ -in. steel cable. What load can be supported with a factor of safety of 5?

27. A vertical wooden bar 6 ft. long and 3 in. in diameter is found to lengthen .013 in. under a load of 2100 lb. hung at the end. Find the value of E for this bar.

FIG. 3

28. A copper wire $\frac{1}{4}$ in. in diameter and 500 ft. long is stretched with a force of 100 lb. when the temperature is 80°F . Find the pull in the wire when the temperature is 0°F ., and the factor of safety.

29. An extended shank is made for a $1\frac{1}{2}$ -in. drill by boring a $1\frac{1}{2}$ -in. hole in the end of a piece of $\frac{3}{4}$ -in. cold-rolled steel, fitting the shank into this, and

putting a steel taper pin through both (Fig. 3). Standard pins taper $\frac{1}{4}$ in. per foot. What size pin should be used in order that the strength of the pin against shear may equal the strength of the drill shank in compression around the hole?

30. The head of a steam cylinder of 12-in. inside diameter is held on by ten wrought-iron bolts. How tight should these bolts be screwed up in order that the cylinder may be steam tight under a pressure of 180 lb./in.²?

31. Find the depth of head of a wrought-iron bolt in terms of its diameter in order that the tensile strength of the bolt may equal the shearing strength of the head.

32. The pendulum rod of a regulator used in an astronomical observatory is made of nickel steel in the proportion of 35.7 per cent nickel to 64.3 per cent steel. The coefficient of expansion of this alloy is approximately 0.0000005.

The rod carries two compensation tubes, A and B (Fig. 4), one of copper and the other of alloy, the length of the two together being 10 cm. If the length of the rod to the top of tube A is 1 m., find the lengths of the two compensation tubes so that a change in temperature shall not affect the length of the pendulum.

FIG. 4

33. Refer to the *Watertown Arsenal Reports* (*United States Government Reports on Tests of Metals*), and from the experimental results there tabulated draw typical strain diagrams for mild steel, wrought iron, cast iron, and timber, and compute E in each case.

34. A steel wire $\frac{1}{4}$ in. in diameter and a brass wire $\frac{1}{2}$ in. in diameter jointly support a load of 1200 lb. If the wires were of the same length when the load was applied, find the proportion of the load carried by each.

35. An engine cylinder is 10 in. inside diameter and carries a steam pressure of 80 lb./in.² Find the number and size of the bolts required for the cylinder head for a working stress in the bolts of 2000 lb./in.²

36. Find the required diameter for a short piston rod of hard steel for a piston 20 in. in diameter and steam pressure of 125 lb./in.² Use factor of safety of 8.

37. A rivet $\frac{1}{2}$ in. in diameter connects two wrought-iron plates each $\frac{3}{8}$ in. thick. Compare the shearing strength of the rivet with the crushing strength of the plates around the rivet hole.

38. In the United States government tests of rifle-barrel steel it was found that for a certain sample the unit tensile stress at the elastic limit was 71,000 lb./in.², and that the ultimate tensile strength was 118,000 lb./in.² What must the factor of safety be in order to bring the working stress within the elastic limit?

39. In the United States government tests of concrete cubes made of Atlas cement in the proportions of 1 part of cement to 3 of sand and 6 of broken stone, the ultimate compressive strength of one specimen was 883 lb./in.², and of another specimen was 3256 lb./in.² If the working stress is determined from the ultimate strength of the first specimen by using a factor of safety of 5, what factor of safety must be used to determine the same working stress from the other specimen?

40. An elevator cab weighs 3 tons. With a factor of safety of 5, how large must a steel cable be to support the cab?

41. A hard-steel punch is used to punch holes in a wrought-iron plate $\frac{3}{8}$ in. thick. Find the diameter of the smallest hole that can be punched.

42. A mild-steel plate 10 in. square and $\frac{1}{2}$ in. thick is stretched 0.002 in. in one direction by a certain pull. What pull must be applied at right angles to reduce the first stretch to 0.0014 in.?

43. A structural steel tie rod of a bridge is to be 25 ft. long when the bridge is completed. What should its original length be if the maximum stress in it when loaded is 18,000 lb./in.²?

44. A hard-steel punch is used to punch a circular hole $\frac{1}{2}$ in. in diameter in a wrought-iron plate $\frac{1}{4}$ in. thick. Find the factor of safety for the punch when in use.

45. A cast-iron flanged shaft coupling is bolted together with 1-in. wrought-iron bolts, the distance from the axis of each bolt to the axis of the shaft being 6 in. If the shaft transmits a maximum torque of 12,000 ft.-lb., find the number of bolts required.

46. A steam cylinder of 16 in. inside diameter carries a steam pressure of 150 lb./in.² Find the proper size for the hard-steel piston rod, and the number of $\frac{3}{4}$ -in. wrought-iron bolts required to hold on the cylinder head.

47. A horizontal beam 10 ft. long is suspended at one end by a wrought-iron rod 12 ft. long and $\frac{1}{4}$ in. in diameter, and at the other end by a copper rod 12 ft. long and 1 in. in diameter. At what point on the beam should a load be placed if the beam is to remain horizontal; that is, if each rod is to stretch the same amount?

48. When a bolt is screwed up by means of a wrench, the tension T in the bolt in terms of the pull P on the handle of the wrench is found to be given approximately by the empirical formula

$$T = 75 P$$

for a wrench of maximum length of from 15 to 16 times the diameter of the bolt. What is the largest wrench that should be used on a $\frac{3}{4}$ -in. wrought-iron bolt, and what is the maximum pull that should be exerted on the handle?

49. For a steam-tight joint the pitch (or distance apart) of studs or bolts in cylinder heads is determined by the empirical formulas

High-pressure cylinders,	pitch = $3.5 d$,
Intermediate-pressure cylinders,	pitch = $4.5 d$,
Low-pressure cylinders,	pitch = $5.5 d$;

or, in general,

$$\text{pitch} = \sqrt{\frac{100 t}{w}},$$

where

d = diameter of studs or bolts,

t = thickness of head or cover in sixteenths of an inch,

w = steam pressure in lb./in.²

Calculate the number, size, and pitch of steel studs for a steam cylinder 20 in. inside diameter under a pressure of 150 lb./in.² (high pressure).

FIG. 5

50. The standard proportions for a cottered joint with wrought-iron rods and steel cotter of the type shown in Fig. 5 are as indicated on the figure. Show that these relative proportions make the joint practically of uniform strength in tension, compression, and shear.

SECTION II

FIRST AND SECOND MOMENTS

10. Static moment. If a force acts upon a body having a fixed axis of rotation, it will in general tend to produce rotation of the body about this axis. This tendency to rotate becomes greater as the magnitude of the force increases, and also as its distance from the axis of rotation increases. The numerical amount of this tendency to rotate is thus measured by the product of the force by its perpendicular distance from the given axis, or center of rotation. This product is called the **first moment**, or **static moment**, of the force with respect to the given axis, or point.

Thus, let F denote any force, P the fixed axis of rotation, assumed to be at right angles to the plane of the paper, and d the perpendicular distance of F from P . Then d is called the **lever arm** of the force, and its moment about P is defined as

$$\text{Moment} = \text{force} \times \text{lever arm},$$

or, if the moment is denoted by M ,

$$(9) \quad M = Fd.$$

It is customary to call the moment positive if it tends to produce rotation in a clockwise direction, and negative if its direction is counter-clockwise (Fig. 6).

11. Fundamental theorem of moments. When two concurrent forces act on a body simultaneously, their joint effect is the same as that of a single force, given in magnitude and direction by the diagonal of the parallelogram formed on the two given

FIG. 6

forces as adjacent sides (Fig. 7). This single force, which is equivalent to the two given forces, is called their **resultant**.

FIG. 7

Any number of concurrent forces may be thus combined by finding the resultant of any two, combining this with the third, etc. Or, what amounts to the same thing, the given forces may be placed end to end, forming a polygon, and their final resultant will then be the closing side of this polygon (Fig. 8).

Now, in Fig. 9, let F_1 and F_2 be any two concurrent forces, and F their resultant. Also let O be any given point, and θ_1, θ_2, ϕ , the angles between OA and the forces F_1, F_2, F , respectively. Then, taking moments about O ,

Moment of F_1 about O

$$= F_1 \times OA \sin \theta_1.$$

Moment of F_2 about O

$$= F_2 \times OA \sin \theta_2.$$

The sum of these moments is

$$\sum M = F_1 \times OA \sin \theta_1 + F_2 \times OA \sin \theta_2 = OA (F_1 \sin \theta_1 + F_2 \sin \theta_2).$$

FIG. 9

FIG. 8

But, since F is the resultant of F_1 and F_2 ,

$$F \sin \phi = F_1 \sin \theta_1 + F_2 \sin \theta_2,$$

and consequently

$$\sum M = OA \times F \sin \phi.$$

The right member, however, is the moment of the resultant F with respect to O . Therefore, since O is arbitrary, the sum

of the moments of any two concurrent forces with respect to a given point is equal to the moment of their resultant with respect to this point.

If the forces F_1 and F_2 are parallel, introduce two equal and opposite forces $H, -H$, as shown in Fig. 10, and combine the H 's with F_1 and F_2 into resultants F'_1, F'_2 . Transferring these resultants F'_1, F'_2 to their point of intersection O , they may now be resolved into their original components, giving two equal and opposite forces, $+H$ and $-H$, which cancel, and a resultant $F_1 + F_2$ parallel to F_1 and F_2 .

Moreover, applying the theorem of moments proved above to the concurrent forces F'_1, F'_2 at O , the sum of their moments about any point is equal to the moment of their resultant $F_1 + F_2$ about the same point. But the moment of F'_1 is equal to the sum of the moments of F_1 and $-H$, and, similarly, the moment of F'_2 is equal to the sum of the moments of F_2 and $+H$. Since the forces $+H$ and $-H$ have the same line of action, their moments about any point cancel, and therefore the theorem of moments is also valid for parallel forces.

FIG. 10

This theorem may obviously be extended to any number of forces by combining the moments of any two of them into a resultant moment, combining this resultant moment with the moment of the third force, etc. Hence,

The sum of the moments of any number of forces lying in the same plane with respect to a given point in this plane is equal to the moment of their resultant with respect to this point.

12. Center of gravity. An important application of the theorem of moments arises in considering a system of particles lying in the same plane and rigidly connected. The weights $w_1, w_2, \dots, w_n$ of the particles are forces directed toward the center of the earth. Since this is relatively at an infinite distance as compared with the distances between the particles, their weights may be regarded as a system of parallel forces.

The total weight W of all the particles is

$$W = w_1 + w_2 + \dots + w_n = \sum w;$$

that is, W is the resultant of the n parallel forces $w_1, w_2, \dots, w_n$. The location of this resultant W may be determined by applying the theorem of moments. Thus, let $x_1, x_2, \dots, x_n$ denote the perpendicular distances of $w_1, w_2, \dots, w_n$ from

any fixed point O (Fig. 11). Then, if x_0 denotes the perpendicular distance of the resultant W from O , by the theorem of moments

$$Wx_0 = w_1x_1 + w_2x_2 + \dots + w_nx_n = \sum wx,$$

$$\text{whence} \quad x_0 = \frac{\sum wx}{W};$$

FIG. 11

or, since $W = \sum w$, this may also be written

$$(10) \quad x_0 = \frac{\sum wx}{\sum w}.$$

This relation determines the line of action of W for the given position of the system. If, now, the system is turned through any angle in its plane, and the process repeated, a new line of action for W will be determined. The point of intersection of two such lines is called the **center of gravity** of the system. From the method of determining this point it is evident that if the entire weight of the system was concentrated at its center of gravity, this single weight, or force, would be equivalent to the given system of forces, no matter what the position of the system might be.

If the particles do not all lie in the same plane, a reference plane must be drawn through O instead of a reference line. In this case

the equation $x_0 = \frac{\sum wx}{\sum w}$ determines the position of a plane in which

the resultant force W must lie. The intersection of three such planes corresponding to different positions of the system of particles will then determine a point which is the required center of gravity.

If $m_1, m_2, \dots, m_n$ denote the masses of the n particles, and M their sum, then, since

$$W = Mg, \quad w_1 = m_1g, \quad w_2 = m_2g, \quad \dots, \quad w_n = m_ng,$$

where g denotes the acceleration due to gravity, the above relations for determining the center of gravity become

$$Mg = \sum mg, \quad x_0 = \frac{\sum mgx}{\sum mg};$$

or, since g is constant,

$$(11) \quad M = \sum m, \quad x_0 = \frac{\sum mx}{\sum m}.$$

The point determined from these relations by taking the system of particles in two or more positions is called the **center of inertia** or **center of mass**. Since these relations are identical with those given above, it is evident that the center of mass is identical with the center of gravity.

13. Centroid. It is often necessary to determine the point called the center of gravity or center of mass without reference to either the mass or weight of the body, but simply with respect to its geometric form.

For a solid body let Δv denote an element of volume, Δm its mass, and D the density of the body. Then, since mass is jointly proportional to volume and density,

$$\Delta m = D\Delta v.$$

Therefore the formulas given above may be written

$$DV = \sum D\Delta v, \quad x_0 = \frac{\sum xD\Delta v}{\sum D\Delta v};$$

or, since the density D is constant, these become

$$(12) \quad V = \sum \Delta v, \quad x_0 = \frac{\sum x\Delta v}{\sum \Delta v}.$$

Since the point previously called the center of gravity or center of mass is now determined simply from the geometric form of the body, it is designated by the special name **centroid**.

Evidently it is also possible to determine the centroid of an area or line, although neither has a center of gravity or center of mass, since mass and weight are properties of solids.

For a plane area the centroid is determined by the equations

$$(13) \quad A = \sum \Delta a, \quad x_0 = \frac{\sum x \Delta a}{\sum \Delta a},$$

where Δa denotes an element of area and A the total area of the figure.

Similarly, for a line or arc the centroid is given by

$$(14) \quad L = \sum \Delta l, \quad x_0 = \frac{\sum x_0 \Delta l}{\sum \Delta l},$$

where Δl denotes an element of length and L the total length of the line or arc.

14. Centroid of triangular area. To find the centroid of a triangle, divide it up into narrow strips parallel to one side AC (Fig. 12).

FIG. 12

Since the centroid of each strip PQ is at its middle point, the centroid of the entire figure must lie somewhere on the line BD joining these middle points; that is, on the median of the triangle. Similarly, by dividing the triangle up into strips parallel to another side BC , it is proved that the centroid must also lie on the median AE . The point of intersection G of these two medians must therefore be the centroid of the triangle.

Since the triangles DEG and ABG are similar,

$$\frac{DG}{GB} = \frac{DE}{AB};$$

and since $DE = \frac{1}{2} AB$, this gives

$$DG = \frac{1}{2} GB = \frac{1}{3} BD.$$

The centroid of a triangle therefore lies on a median to any side at a distance of one third the length of the median from the opposite vertex. From this it also follows that the perpendicular distance

of the centroid G from any side is one third the distance of the opposite vertex from that side.

15. Centroid of circular arc. For a circular arc CD (Fig. 13) the centroid G must lie on the diameter OF bisecting the arc. Now suppose the arc divided into small segments, and from the ends of any segment PQ draw PR parallel to the chord CD , and QR perpendicular to this chord. Since the moment of the entire arc with respect to a line AB drawn through O perpendicular to OF must be equal to the sum of the moments of the small segments PQ with respect to this line, the equation determining the centroid is

$$x_0 \text{ arc } CD = \sum PQ \times x.$$

FIG. 13

But from the similarity of the triangles PQR and OQE we have

$$\frac{PQ}{PR} = \frac{OQ}{QE} = \frac{r}{x}.$$

Therefore $PQ \cdot x = PR \cdot r$, and consequently $\sum PQ \cdot x = \sum PR \cdot r$, or, since the radius r is constant,

$$\sum PQ \cdot x = r \sum PR = r \cdot \text{chord } CD.$$

The position of the centroid is therefore given by

$$(15) \quad x_0 = \frac{\text{chord}}{\text{arc}} \cdot \text{radius}.$$

If the central angle COD is denoted by 2α , then arc $= 2r\alpha$ and chord $= 2r \sin \alpha$, and therefore the expression for the centroid may be written

$$x_0 = \frac{r \sin \alpha}{\alpha}.$$

For a semicircle $2\alpha = \pi$, and consequently

$$x_0 = \frac{2r}{\pi}.$$

16. Centroid of circular sector and segment. To determine the centroid of a circular sector OCB (Fig. 14), denote the radius by r and the central angle COB by 2α .

FIG. 14

Then any small element OPQ of the sector may be regarded as a triangle the centroid of which is on its median at a distance of $\frac{2}{3}r$ from O . The centroids of all these elementary triangles therefore lie on a concentric arc DEF of radius $\frac{2}{3}r$, and the centroid of the entire sector coincides with

the centroid of this arc DEF . Therefore, from the results of the preceding article, the centroid of the entire sector OCB is given by

$$(16) \quad x_0 = \frac{2}{3} r \frac{\sin \alpha}{\alpha}.$$

For a semicircular area of radius r the distance of the centroid from the diameter, or straight side, is

$$x_0 = \frac{4r}{3\pi}.$$

To determine the centroid of a circular segment CBD (Fig. 15),

let G denote the centroid of the entire sector $OCBD$, G_0 of the segment CBD , and G_1 of the triangle OCD . Then the position of G_0 may be determined by noting that the sum of the moments of the triangle OCD and the segment CBD about any point, say O , is equal to the moment of the entire sector about this point. Thus,

FIG. 15

if A_0 , A_1 , A denote the areas of the segment, triangle, and sector, respectively, and x_0 , x_1 , x , the distances of their centroids from O , then

$$A_0 x_0 + A_1 x_1 = Ax,$$

whence

$$x_0 = \frac{Ax - A_1x_1}{A_0}.$$

Now let c denote the length of the chord CD and a the length of the arc CBD . Then, from the results of this and the preceding articles,

$$x = \frac{2}{3} \frac{rc}{a}, \quad x_1 = \frac{2}{3} \sqrt{r^2 - \frac{c^2}{4}},$$

and also, from geometry,

$$A = \frac{1}{2} ar, \quad A_1 = \frac{1}{2} c \sqrt{r^2 - \frac{c^2}{4}}.$$

Inserting these values in the expression for x_0 , the result is

$$(17) \quad x_0 = \frac{c^3}{12 A_0}.$$

For a semicircle, $A_0 = \frac{\pi r^2}{2}$ and $c = 2r$. Therefore, in this case, as also shown above,

$$x_0 = \frac{4}{3} \pi.$$

17. Centroid of parabolic segment. For a parabolic segment with vertex at A (Fig. 16) the position of the centroid G is given by

$$(18) \quad x_0 = \frac{3}{5} a, \quad y_0 = \frac{3}{8} b,$$

where a and b denote the sides of the circumscribing rectangle. Also,

$$(19) \quad \text{Area } ABC = \frac{2}{3} ab.$$

For the external segment ABD (Fig. 16) the centroid is given by

$$(20) \quad x_0 = \frac{3}{10} a, \quad y_0 = \frac{3}{4} b,$$

and the area of the external segment is

$$(21) \quad \text{Area } ABD = \frac{1}{3} ab.$$

FIG. 16

18. Axis of symmetry. If a figure has an axis of symmetry, then to any element of the figure on one side of the axis there must

FIG. 17

correspond an equidistant element on the opposite side, and since the moments of these equal elements about the axis of symmetry are equal in amount and opposite in sign, their sum is zero (Fig. 17). Since the moment of each pair of elements with respect to the axis of symmetry is identically

zero, the total moment is also zero, and hence the centroid of the figure must lie on the axis of symmetry.

When a figure has two or more axes of symmetry, their intersection completely determines the centroid.

19. Centroid of composite figures. To determine the centroid of a figure made up of several parts, the centroid of each part may first be determined separately. Then, assuming that the area of each part is concentrated at its centroid, the centroid of the entire figure may be determined by equating its moment to the sum of the moments of the several parts.

To illustrate this method, let it be required to find the centroid of the I-shape shown in Fig. 18. Since the figure has an axis of symmetry MN , the centroid must lie somewhere on this line. To find its position, divide the I into three rectangles, as indicated by the dotted lines in the figure. The

FIG. 18

centroids of these rectangles are at their centers a , b , c . Therefore, denoting these three areas of the rectangles by A , B , C , respectively, and taking moments with respect to the base line, the distance of the

centroid of the entire figure from the base is found to be

$$x_0 = \frac{A \times \overline{ad} + B \times \overline{bd} + C \times \overline{cd}}{A + B + C}.$$

As another example, consider the circular disk with a circular hole cut in it, shown in Fig. 19. Here also the centroid must lie somewhere on the axis of symmetry C_1, C_2 . Therefore, denoting

FIG. 19

the radii of the circles by R, r , as shown, and taking moments about the tangent perpendicular to the line of centers, the distance x_0 of the centroid from this tangent is found to be

$$x_0 = \frac{\pi R^2 x_2 - \pi r^2 x_1}{\pi R^2 - \pi r^2},$$

or, since $x_2 = R$ and $x_1 = R - e$, where e denotes the eccentricity of the hole, or distance between centers,

$$x_0 = \frac{R^3 - r^2(R - e)}{R^2 - r^2}.$$

20. Moment of inertia. In the analysis of beams, shafts, and columns it will be found necessary to compute a factor, called the moment of inertia, which depends only on the shape and size of the cross section of the member.

This shape factor is usually denoted by I , and is defined as the sum of the products obtained by multiplying each element of area of the cross section by the square of its distance from a given line or point. Thus, in

FIG. 20

Fig. 20, if ΔA denotes an element of area and y its distance from any given axis OO' , then the moment of inertia of the figure with respect to this axis is defined as

$$I = \sum y^2 \Delta A.$$

Since an area is not a solid and therefore does not possess inertia, the shape factor I should not be called moment of inertia, but rather the **second moment of area**, since the distance y occurs squared.

FIG. 21

To compute I for any plane area, divide the area up into small elements ΔA (Fig. 21). Then the first (or static) moment of each element with respect to any axis OO is $y\Delta A$, where y denotes the distance of this element from the given axis. Now erect on ΔA as base a prism of

height y . If this is done for every element of the plane area, the result will be a solid, or truncated cylinder, as shown in Fig. 21, the planes of the upper and lower bases intersecting in the axis OO at an angle of 45° .

Let V denote the volume of this **moment solid**, as it will be called, and y_0 the distance of its centroidal axis from OO . Then, by the theorem of moments,

$$Vy_0 = \sum y\Delta V.$$

Since $\Delta V = y\Delta A$, the right member becomes

$$\sum y\Delta V = \sum y^2\Delta A = I.$$

Hence

$$(22) \quad I = Vy_0.$$

21. I for rectangle.

Let it be required to find I for a rectangle of breadth b and height h with respect to an axis through its centroid; or middle point, and parallel to the base (Fig. 22). The moment solid in this case consists of a double wedge,

FIG. 22

as shown in Fig. 22, the base of each wedge being $\frac{bh}{2}$, its height $\frac{h}{2}$, and its volume

$$V = \frac{1}{2} \text{ base} \times \text{altitude} = \frac{bh^2}{8}.$$

Since the centroid of a triangular wedge, like that of a triangle, is at a distance of $\frac{2}{3}$ its altitude from the vertex,

$$y_0 = \frac{2}{3} \times \frac{h}{2} = \frac{h}{3}.$$

Therefore

$$I = 2 V y_0 = \frac{bh^3}{12}.$$

For any plane area the I 's with respect to two parallel axes are related as follows:

Let OO denote an axis through the centroid of the figure, AA any parallel axis, and d their distance apart (Fig. 23). Also let I_o denote the I of the figure with respect to the axis OO , and I_A with respect to the axis AA . Then, from the definition of I ,

FIG. 23

$$I_A = \sum \Delta A (y + d)^2 = \sum y^2 \Delta A + 2d \sum y \Delta A + d^2 \sum \Delta A.$$

But since OO is a centroidal axis, $\sum y \Delta A = 0$ for this axis. Therefore, since $\sum y^2 \Delta A = I_o$, the above expression becomes

$$(23) \quad I_A = I_o + d^2 A.$$

From this relation it is evident that the I for a centroidal axis is less than for any parallel axis.

As an application of this formula, find the I for a rectangle with respect to its base. From what precedes, $I_o = \frac{bh^3}{12}$. Also, $d = \frac{h}{2}$ and $A = bh$. Hence the I for a rectangle with respect to its base is

$$I_B = \frac{bh^3}{12} + bh \left(\frac{h}{2} \right)^2 = \frac{bh^3}{3}.$$

22. I for triangle. Consider a triangle of base b and altitude h and compute first its I with respect to an axis AA through its vertex and parallel to the base (Fig. 24). The moment solid in this case is a pyramid of base bh and altitude h , the volume of which is

$$V = \frac{1}{3} \text{ base} \times \text{altitude} = \frac{bh^2}{3}.$$

Since the centroid of this pyramid is at a distance $y_0 = \frac{3}{4}h$ from the vertex, we have

$$I_A = Vy_0 = \frac{bh^3}{4}.$$

FIG. 24

To find I for the triangle with respect to an axis OO through its centroid and parallel to AA , apply the theorem

$$I_A = I_O + d^2A.$$

Since in the present case $I_A = \frac{bh^3}{4}$, $A = \frac{bh}{2}$, and $d = \frac{2}{3}h$, we have therefore

$$I_O = I_A - d^2A = \frac{bh^3}{36}.$$

Similarly, for the axis BB we have

$$I_B = I_O + d^2A = \frac{bh^3}{12}.$$

23. I for circle. In computing the I for a circle, it is convenient to determine it first with respect to an axis through the center of the circle and perpendicular to its plane (the so-called polar moment of inertia of the circle).

Consider the circle as made up of a large number of elementary triangles OAB with common vertex at O (Fig. 25). Since the altitude of each of these triangles is the radius R of the circle, from the preceding article the I for each with respect to the point O is $\frac{AB \times R^3}{4}$. For the entire circle, therefore,

$$I_o = \frac{R^3}{4} \sum AB;$$

or, since $\sum AB = \text{circumference} = 2\pi R$, this becomes

$$(24) \quad I_o = \frac{\pi R^4}{2}.$$

If D denotes the diameter of the circle, then $R = \frac{D}{2}$ and we also have

$$(25) \quad I_o = \frac{\pi D^4}{32}.$$

If XX and YY are two rectangular diameters of the circle, and r is the distance of any element of area ΔA from their point of intersection O (Fig. 25), then

$$r^2 = x^2 + y^2.$$

Hence $\sum r^2 \Delta A = \sum x^2 \Delta A + \sum y^2 \Delta A$, or

$$(26) \quad I_o = I_Y + I_X.$$

Since a circle is symmetrical about all diameters, we have $I_X = I_Y$. Therefore the I of a circle with respect to any diameter is

$$I_D = I_X = I_Y = \frac{I_o}{2},$$

or

$$(27) \quad I_D = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}.$$

FIG. 25

24. I for composite figures. When a plane figure can be divided into several simple figures, such as triangles, rectangles, and circles, the I of the entire figure with respect to any axis may be found by adding together the I 's for the several parts with respect to this axis. Thus, in Fig. 26 each area may be regarded as the difference

of two rectangles — a large rectangle of base B and height H , and a smaller rectangle of base b and height h . Consequently the I for

FIG. 26

FIG. 27

either figure, with respect to its centroidal axis GG , is given by

$$I = \frac{BH^3}{12} - \frac{bh^3}{12}.$$

Similarly, the figures shown in Fig. 27 may each be regarded as the sum of two rectangles, and hence the I for either

of these figures with respect to its centroidal axis GG is given by

$$I = \frac{BH^3}{12} + \frac{bh^3}{12}.$$

For the angle, or tee, shown in Fig. 28 the I about the base line OO is the sum of the I 's for the two rectangles into which the figures are divided by the dotted lines; that is,

$$I_o = \frac{BH^3}{3} + \frac{bh^3}{3}.$$

FIG. 28

The position of the centroidal axis GG may then be determined by taking moments about the base OO . That is to say, since the

total area is $A = BH + bh$, we have, by the principle of moments,

$$x_0(BH + bh) = BH \times \frac{H}{2} + bh \times \frac{h}{2}, \text{ whence}$$

$$x_0 = \frac{BH^2 + bh^2}{2A}.$$

Having found x_0 , the I for the centroidal axis GG is determined by the relation

$$I_G = I_O - Ax_0^2.$$

APPLICATIONS

51. A uniform rod 18 in. long weighs 8 lb. and has weights of 2 lb., 3 lb., 4 lb., and 5 lb. strung on it at distances of 6 in. apart. Find the point at which the rod will balance.

Solution. Since the rod is uniform, its weight may be assumed to be concentrated at its center. If, then, x_0 denotes the distance of the center of gravity from the end at which the 2-lb. weight is hung, by taking moments about this end

$$x_0 = \frac{2 \times 0 + 3 \times 6 + 8 \times 9 + 4 \times 12 + 5 \times 18}{2 + 3 + 8 + 4 + 5} = 10\frac{1}{4} \text{ in.}$$

52. A section like that shown in Fig. 29 has the dimensions $b = 3$ in., $d = 5$ in., $t = \frac{1}{2}$ in. Locate its center of gravity, or centroid.

FIG. 29

Solution. To locate the gravity axis 1-1, take moments about any parallel line as a base, say AB . Then, dividing the figure into two rectangles, since the center of gravity of each rectangle is at its center, we have

$$y_0 = \frac{(5 \times \frac{1}{2}) 2\frac{1}{2} + (2\frac{1}{2} \times \frac{1}{2}) \frac{1}{4}}{5 \times \frac{1}{2} + 2\frac{1}{2} \times \frac{1}{2}} = 1\frac{3}{4} \text{ in.}$$

Similarly, to determine the gravity axis 2-2, by taking moments about CD we have

$$x_0 = \frac{(3 \times \frac{1}{2}) 1\frac{1}{2} + (4\frac{1}{2} \times \frac{1}{2}) \frac{1}{4}}{3 \times \frac{1}{2} + 4\frac{1}{2} \times \frac{1}{2}} = \frac{3}{4} \text{ in.}$$

53. The section shown in Fig. 30 is made up of two 10-in. channels 30 lb./ft. and a top plate 9 in. $\times$ $\frac{1}{2}$ in. Locate its gravity axes and determine its moment of inertia with respect to the axis 1-1.

Solution. From Table IV the area of each channel is 8.82 in.² To determine the gravity axis 1-1, take moments about the lower edge of the section. Then

$$y_0 = \frac{2 \times 8.82 \times 5 + 9 \times \frac{1}{2} \times 10\frac{1}{4}}{2 \times 8.82 + 9 \times \frac{1}{2}} = 6.07 \text{ in.}$$

From the table, the moment of inertia of each channel with respect to an axis perpendicular to the web at center is 103.2 in.^4 , and the distance from this axis to

FIG. 30

FIG. 31

the gravity axis of the entire section is $6.07 - 5 = 1.07 \text{ in.}$ Also, the moment of inertia of the top plate with respect to its gravity axis is $\frac{bh^3}{12} = \frac{9 \times (\frac{1}{2})^3}{12} = \frac{3}{32} \text{ in.}^4$,

and the distance of this axis from the gravity axis of the entire section is 4.18 in. Therefore

$$I_{1-1} = 2[103.2 + 8.82 \times (1.07)^2] + [\frac{3}{32} + 4.5 \times (4.18)^2] = 305 \text{ in.}^4$$

For the net section the rivet holes must be deducted from this value. Assuming two $\frac{3}{4}$ -in. rivets, the amount to be deducted is approximately 24 in.^4 , giving for the net section

$$I_{1-1} = 281 \text{ in.}^4$$

54. In problem 53 determine the moment of inertia of the net section with respect to the gravity axis 2-2.

55. The section shown in Fig. 31 is made up of four angles $4 \times 3 \times \frac{1}{2} \text{ in.}$, with the longer leg horizontal, and a web plate $12 \times \frac{3}{4} \text{ in.}$, with $\frac{3}{4}$ -in. rivets. Find the moment of inertia for its net section with respect to the gravity axis 1-1.

56. The section shown in Fig. 32 is built up of two 8-in. channels 18.75 lb./ft. and two plates $9 \times \frac{3}{4} \text{ in.}$ Find the moment of inertia of its net section about the gravity axis 1-1, deducting the area of four $\frac{3}{4}$ -in. rivet holes.

FIG. 32

57. In problem 56 find the moment of inertia of the net section with respect to the gravity axis 2-2.

58. The section shown in Fig. 33 has the dimensions $b = 10$ in., $d = 4$ in., $t = 1$ in.

FIG. 33

Locate the gravity axis 1-1.

59. In problem 58 find the moment of inertia of the section with respect to the gravity axis 2-2.

60. The section shown in Fig. 34 has the dimensions $B = 10$ in., $b = 6$ in., $d = 8$ in., $t_1 = t_2 = t_3 = 1$ in. Locate the gravity axis 1-1.

61. The section shown in Fig. 32 is composed of two 12-in. channels, 20.5 lb./ft., and two $\frac{1}{2}$ -in. plates. How wide must the plates be in order that the

moments of inertia of the section shall be the same about both gravity axes?

62. The section shown in Fig. 35 has the dimensions $b = 8$ in., $h = 10$ in., $b' = 5$ in., $h' = 6$ in. Find its moments of inertia about both gravity axes.

63. Two 6-in. channels 10.5 lb./ft. are connected by latching. How far apart should they be placed, back to back, in order that the moments of inertia may be the same about both gravity axes?

64. A hollow cast-iron column is 6 in. external diameter and 1 in. thick. Find the moment of inertia of its cross section with respect to a diameter.

65. The section shown in Fig. 31 is made up of a web plate $9 \times \frac{3}{8}$ in. and four angles $3 \times 3 \times \frac{3}{8}$ in. Find its moment of inertia with respect to both gravity axes.

66. The top chord of a bridge truss has a section like that shown in Fig. 36, with top plate $20 \times \frac{5}{8}$ in., two web plates each $18 \times \frac{5}{8}$ in., and four angles $3 \times 3 \times \frac{5}{8}$ in. Find the eccentricity of the section; that is, the distance from center of figure to gravity axis 1-1.

FIG. 35

FIG. 34

67. In problem 66 find the moment of inertia of the net section with respect to the axis 1-1, deducting for four $\frac{3}{4}$ -in. rivets.

68. A circular table rests on three legs placed at the edge and forming an equilateral triangle. Find the least weight which will upset the table when hung from its edge.

69. A horizontal beam 20 ft. long and weighing 120 lb. rests on two supports 10 ft. apart. A load of 75 lb. is hung at one end of the beam and 150 lb. at the other end. How must the beam be placed so that the pressure on the supports may be equal?

FIG. 36

70. Explain how a clock hand on a smooth pivot can be made to show the time by means of clockwork concealed in the hand and carrying a weight around.

71. A brick wall is 12 in. thick and 40 ft. high. What uniform wind pressure will cause it to tip over? Weight of ordinary brick masonry is 125 lb./ft.³

72. A masonry dam is 30 ft. high, 6 ft. wide at top, and 30 ft. wide at bottom, with upstream face vertical. Assuming the masonry to weigh 160 lb./ft.³, compute the moment of the weight of the dam about the toe of the base.

73. In problem 72 the resultant water pressure for a vertical strip 1 ft. wide is 46,800 lb. and is applied at a point 12 ft.

above the base of the dam. Determine its stability against overturning.

74. The casting for a gas-engine piston is a hollow cylinder of uniform thickness, with one end closed. The external diameter is 5 in., length over all 6 in., thickness of cylinder shell $\frac{1}{2}$ in., thickness of end $1\frac{1}{2}$ in. Find the distance of its center of gravity from the closed end.

75. A cast-iron pulley weighs 50 lb. and its center of gravity is 0.1 in. out of center. To balance the pulley, a hole is drilled in the light side, 6 in. from the center of the pulley and in line with its center of gravity, and filled with lead. How much iron must be removed, the specific gravity of lead being 11.35 and of iron 7.5?

SECTION III

BENDING-MOMENT AND SHEAR DIAGRAMS

25. Conditions of equilibrium. In order that any structure may be in equilibrium, the external forces acting on it must satisfy two conditions:

1. *The sum of the forces acting in any given direction must be zero.*
2. *The sum of the moments of the forces about any point must be zero.*

If force and moment are denoted by F and M , respectively, these conditions are expressed more briefly in the form

$$(28) \quad \text{For equilibrium} \quad \begin{cases} \sum F = 0; \\ \sum M = 0. \end{cases}$$

If the forces all lie in one plane, the condition $\sum F = 0$ is expressed more conveniently in the form

$$(29) \quad \sum \text{vertical forces} = 0; \quad \sum \text{horizontal forces} = 0.$$

To illustrate the application of these conditions, consider a simple beam AB of length l , supported at the ends and bearing a single concentrated load P at a distance d from one end A (Fig. 37). Let the reactions of the supports at A and B be denoted by R_1 , R_2 , and, to find the value of these reactions, apply the condition $\sum M = 0$; that is, equate to zero the sum of the moments of all the external forces with respect to any convenient moment center, say A . Then

$$Pd - R_2 l = 0,$$

whence

$$R_2 = \frac{Pd}{l}.$$

FIG. 37

Now, applying the condition $\sum \text{vertical forces} = 0$, we have

$$R_1 + R_2 - P = 0,$$

and inserting in this equation the value just found for R_2 and then solving for R_1 , the result is

$$R_1 = \frac{P(l-d)}{l}.$$

26. Vertical shear. By applying the conditions $\sum F = 0$, $\sum M = 0$, as just explained, all the external forces acting on the beam may be found. The beam may then be supposed to be cut in two at any point and these conditions applied to the portion on either side of the section.

In general, the sum of the external forces on one side of any arbitrary cross section will not be identically zero. If, then, the condition of equilibrium $\sum F = 0$ is satisfied for the portion of the beam on one side of the section, the stress in the material at this point must supply a force equal in amount and opposite in direction to the resultant of the external forces on one side of this point. This resisting force, or resultant of the vertical stresses in the plane of the cross section, which balances the external forces on one side of the section, is called the **vertical shear**. Therefore

The vertical shear on any cross section = the algebraic sum of the external vertical forces on either side of the section.

For instance, suppose that a beam 10 ft. long bears a uniform load of 300 lb./ft., and it is required to find the vertical shear on a section 4 ft. from the left support. In this case the total load on the beam is 3000 lb., and, since the load is uniform, each reaction is 1500 lb. The load on the left of the given section is then $4 \times 300 = 1200$ lb., and therefore the shear at the section is $1500 - 1200 = 300$ lb.

27. Bending moment. In applying the condition $\sum M = 0$ to the portion of a beam on either side of any cross section, the center of moments is taken at the centroid of the section. Since the position of the cross section is arbitrary, it is obvious that the sum of the moments of the forces on one side of the section about its centroid will not in general be zero. Therefore, to satisfy the condition $\sum M = 0$, the normal stresses in the beam at the section considered must

supply a moment which balances the sum of the moments of the external forces on either side of the point. This resisting moment in the beam is called the **stress couple** or **bending moment**, and is evidently equal to the resultant external moment at the point in question. Consequently

The bending moment at any cross section of a beam is equal to the sum of the moments of the external forces on one side of this point, about the centroid of the section.

For example, in Fig. 38, consider a cross section mn at an arbitrary distance x from the left support. Then for the portion of the beam on the left of mn the moment of R_1 about the centroid of the section is R_1x , and the moment of P_1 about the same point is $P_1(x - d_1)$. Therefore the total bending moment at the section mn is

$$M = R_1x - P_1(x - d_1).$$

As another example, consider a beam of length l

bearing a uniform load of amount w per unit of length. Then the total load on the beam is wl , and each reaction is $\frac{wl}{2}$. Therefore, taking a section at a distance x from the left support and considering only the forces on the left of the section, the total bending moment at this point is

$$M = \frac{wl}{2} \cdot x - wx \cdot \frac{x}{2} = \frac{wx}{2} (l - x).$$

From this relation it is evident that $M = 0$ when $x = 0$ or $x = l$, and attains its maximum value when $x = \frac{l}{2}$; that is, the bending moment is zero at each end of the beam and a maximum at the center.

28. Bending-moment and shear diagrams. Since in general the bending moment and shear vary from point to point along a beam, it is desirable to show graphically the moment and shear at

FIG. 38

each point of the beam. This may be done by means of a bending-moment diagram and a shear diagram, obtained by plotting the general expressions for the moment and shear, such as those given in the examples in the preceding paragraph. Thus, the shear diagram is obtained by plotting the shear at any arbitrary section mn as ordinate and the distance x of this section from a fixed origin as abscissa. Similarly, the moment diagram is obtained by plotting the moment at any arbitrary section mn as ordinate and the distance x as abscissa.

FIG. 39

The following simple applications illustrate the method of drawing the diagrams.

1. *Simple beam bearing a single concentrated load P at its center* (Fig. 39). From symmetry, the reactions R_1 and R_2 are each equal to $\frac{P}{2}$. Let mn denote any section of the beam at a distance x from the left support, and consider the portion of the beam on the left of this section. Then the moment at mn is $R_1x \left(= \frac{P}{2}x \right)$ and the shear is $R_1 \left(= \frac{P}{2} \right)$. For a section on the

right of the center the bending moment is $R_2(l-x)$ and the shear is R_2 . Consequently, the bending moment varies as the ordinates of a triangle, being zero at either support and attaining a maximum value of $\frac{Pl}{4}$ at the center, while the shear is constant from A to B , and also constant, but of opposite sign, from B to C .

The diagrams in Fig. 39 represent these variations in bending moment and shear along the beam under the assumed loading. Consequently, if the ordinates vertically beneath B are laid off to scale to represent the bending moment and shear at this point, the bending moment and shear at any other point D of the beam are found at once from the diagram by drawing the ordinates EF and HK vertically beneath D .

2. *Beam bearing a single concentrated load P at a distance c from one support.*

The reactions in this case are

$$R_1 = \frac{P(l-c)}{l}$$

and
$$R_2 = \frac{Pc}{l}.$$

Hence, the bending moment at a distance x from the left support is

$$R_1 x = \frac{P(l-c)x}{l},$$

provided $x < c$, and

$$R_2(l-x) = \frac{Pc(l-x)}{l},$$

if $x > c$. If $x = c$, each of these moments becomes

$$\frac{Pc(l-c)}{l},$$

and consequently the bending-moment and shear diagrams are as shown in Fig. 40.

3. *Beam bearing several separate loads.*

In this case the bending-moment diagram may be obtained by constructing the diagrams for each load separately and then adding their ordinates, as indicated in Fig. 41.

FIG. 40

FIG. 41

FIG. 42

4. Beam bearing a continuous uniform load.

Let the load per unit of length be denoted by w . Then the total load on the beam is wl , and the reactions are

$$R_1 = R_2 = \frac{wl}{2}.$$

Hence, at a distance x from the left support the bending moment M_x is

$$\begin{aligned} M_x &= \frac{wl}{2}x - wx \cdot \frac{x}{2} \\ &= \frac{w}{2}(lx - x^2). \end{aligned}$$

The bending-moment diagram is therefore a parabola. When $x = \frac{l}{2}$,

$$M_x = \frac{wl^2}{8}, \text{ which is its}$$

maximum value. The bending-moment and shear diagrams are therefore as represented in Fig. 42.

5. Beam bearing uniform load over part of the span.

Let the load extend over a distance c and be of amount w per unit of length. Then the total load is wc . The reactions of the supports are the same as though the load were concentrated

FIG. 43

at its center of gravity G . Therefore, if d denotes the distance of G from the left support,

$$R_2 = \frac{wcd}{l} \quad \text{and} \quad R_1 = \frac{wc(l-d)}{l}.$$

Also, the bending-moment diagrams for the portions AB and CD are the same as though the load were concentrated at G , and are therefore the straight lines $A'H$ and $D'K$, intersecting in the point T vertically beneath G (Fig. 43).

From B to C there is an additional bending moment due to the uniform load on this portion of the beam. Thus, if LMN is the parabolic moment diagram for a beam of length LN or c , the ordinates to the line HK must be increased by those to the parabola LMN , giving as a complete moment diagram the line $A'HJKD'$.

Analytically, if x denotes the distance of any section from the left support, the equations of the three portions $A'H$, HJK , and KD' of the moment diagram are

$$M_{AB} = R_1 x = \frac{wc(l-d)x}{l}, \quad \text{for} \quad 0 \equiv x \equiv d - \frac{c}{2};$$

$$M_{BC} = R_1 x - \frac{w\left(x - d + \frac{c}{2}\right)^2}{2} = \frac{wc(l-d)x}{l} - \frac{w\left(x - d + \frac{c}{2}\right)^2}{2},$$

$$\text{for} \quad d - \frac{c}{2} \equiv x \equiv d + \frac{c}{2};$$

$$M_{CD} = R_2 x = \frac{wcd(l-x)}{l}, \quad \text{for} \quad d + \frac{c}{2} \equiv x \equiv l.$$

29. Relation between shear and moment diagrams. Consider a beam bearing any number of concentrated loads $P_1, P_2, \dots, P_n$, at distances $d_1, d_2, \dots, d_n$ from one end A (Fig. 44). Then the moment M at any section mn , distant x from the origin A , is

$$M = R_1 x - \sum P(x-d),$$

FIG. 44

where the summation includes only the loads on the left of the section. For an adjacent section distant Δx from mn , that is, at a distance $x + \Delta x$ from the origin, the moment is

$$M' = R_1(x + \Delta x) - \sum P(x + \Delta x - d).$$

Let ΔM denote the difference between these two moments. Then

$$\Delta M = M' - M = R_1 \Delta x - \sum P \Delta x,$$

or

$$\frac{\Delta M}{\Delta x} = R_1 - \sum P.$$

But, by definition, the shear S at the given section mn is

$$S = R_1 - \sum P.$$

Consequently,

$$(30) \quad \frac{\Delta M}{\Delta x} = S.$$

This relation also holds for a beam uniformly loaded. Thus, if w denotes the uniform load per foot of length, and l is the span in feet, the moment in this case at any section distant x from the left support is

$$M = \frac{wl}{2}x - \frac{wx^2}{2},$$

and at a section distant Δx from this it is

$$M' = \frac{wl}{2}(x + \Delta x) - \frac{w(x + \Delta x)^2}{2}.$$

Therefore the change in the moment is now

$$\Delta M = M' - M = \frac{wl}{2}\Delta x - wx \cdot \Delta x - \frac{w}{2}(\Delta x)^2.$$

If Δx is assumed to be small, its square may be neglected in comparison with the other terms. In this case, dropping the last term, we have

$$(31) \quad \frac{\Delta M}{\Delta x} = \frac{wl}{2} - wx = S.$$

Evidently the same relation holds for any combination of uniform and concentrated loads. The general fundamental relation between the shear and moment diagrams is therefore

$$(32) \quad \frac{\Delta M}{\Delta x} = S.$$

Since $\frac{\Delta M}{\Delta x}$ represents the rate at which the ordinate to the moment diagram is changing, this relation may be expressed in words by saying that

The rate of change of the moment is equal to the shear.

From this result important properties of the two diagrams may be deduced, as explained in the next paragraph.

30. Properties of shear and moment diagrams. Consider the highest point of any given moment diagram—for instance, of those shown in Figs. 39–43.

Since the moment increases up to this point and decreases after it passes it, the change in the moment ΔM , corresponding to an increase Δx in the abscissa, must be positive on one side of the point and negative on the other. Since Δx is positive in both cases, the ratio $\frac{\Delta M}{\Delta x}$ changes sign in passing the point. But since $\frac{\Delta M}{\Delta x} = S$, this means that the shear changes from positive to negative in passing the given point, and therefore must pass through zero at the point in question.

The same reasoning evidently holds for the lowest point of the moment diagram. Therefore, *at the section where the moment is greatest or least the shear is either zero or passes through zero in passing the point.*

By referring to the diagrams in the preceding article or in Table XIII it will be observed that this is true in each case.

If the moment is constant, then $\Delta M = 0$ and consequently $S = 0$. That is to say, *where the moment is constant the shear is zero.*

For a system of concentrated loads the equations for moment and shear, as shown in article 29, are

$$M = R_1 x - \sum P(x - d),$$

$$S = R_1 - \sum P.$$

The first of these represents an inclined straight line, and the second a horizontal straight line. Therefore, *for concentrated loads the moment diagram is a broken line and the shear diagram is a series of horizontal lines or steps.*

For a uniform load the expressions for moment and shear, as shown in article 29, are

$$M = \frac{wl}{2}x - \frac{wx^2}{2},$$

$$S = \frac{wl}{2} - wx.$$

The first of these equations evidently represents a parabola, and the second an inclined straight line of slope $= w$.

From these results it follows that *for any combination of uniform and concentrated loads the moment diagram is a connected series of parabolic arcs, and the shear diagram is a succession of inclined lines or sloping steps.*

Since $S\Delta x$ represents an elementary vertical strip of the shear diagram, the area subtended by the shear diagram between any two given points is $\sum S\Delta x$. Making use of the relation $\Delta M = S\Delta x$, and summing between two points P_1 and P_2 , we have

$$\sum_{P_1}^{P_2} S\Delta x = \sum_{P_1}^{P_2} \Delta M = M_1 - M_2,$$

where M_1 and M_2 denote the moments at the two points in question. Hence *the difference between the moments at any two given points is equal to the area of the shear diagram between these points.*

At the ends of a simple beam the moment is always zero. Therefore, by the theorem just proved, *for a simple beam the area of the shear diagram from one end to any point is equal to the moment at this point.*

31. General directions for sketching diagrams. To economize time and effort it is important to follow a definite program in drawing the diagrams and determining the expressions for shear and moment. The following outline of procedure for either cantilever beams or simple beams resting on two supports is therefore suggested.

1. Find each reaction by summing the moments of all the external forces about a point on the opposite reaction as moment center. Check this calculation by noting that the sum of the reactions must equal the sum of the loads.

2. Note that the expressions for moment and shear both change whenever a concentrated load is passed. Consequently, there will

FIG. 45

be as many different segments of the moment and shear diagrams as there are segments of the beam between concentrations.

3. For a simple beam, take the origin at the left end of the beam. For a cantilever beam, take the origin at the free, or unsupported, end of the beam. Keep the origin at this point throughout the calculations.

4. Take a section between the origin and the first concentration, let x denote the distance of this section from the origin, and find the general expressions for the moment and shear at this section in terms of x .

5. Proceed in the same way for a section between each pair of consecutive concentrations.

6. Plot these equations, checking the work by means of the general relations stated in the preceding article.

7. Plot the shear diagram first. In plotting this diagram it is convenient to follow the direction in which the forces act. Thus, in Fig. 45 the shear at the left end is equal to the reaction and may be laid off in the same direction, that is, upwards. Proceeding to the right, drop the shear diagram by an amount equal to each load as it is met, until the reaction at the right end is reached, which will bring the shear diagram back to the base line. By following this method the shear diagram will always begin and end on the base line, which serves as a check on the work.

8. Note that as long as the shear diagram lies above the base line the shear is positive and therefore ΔM is also positive; that is to say, the moment is increasing. Where the shear diagram crosses the axis, the moment diagram must attain its highest or lowest point. When the shear diagram lies below the base line, the moment is decreasing.

9. Compute numerical values of the moment and shear at the critical points of the diagrams, and indicate these numerical values on the diagrams.

A sample set of diagrams as they should be drawn by the student is shown in Fig. 45.

APPLICATIONS

76. A beam 16 ft. long is supported at the left end and at a point 4 ft. from the right end, and carries a uniform load of 200 lb./ft. over its entire length and a concentrated load of 1 ton at a point 4 ft. from the left end. Sketch the shear and moment diagrams and note the maximum shear and maximum moment.

Solution. On cross-section paper indicate the loading as shown in Fig. 45.

To find either reaction, take moments about the other point of support. Thus, for the left reaction R_1 we have

$$R_1 \cdot 12 - 3200 \cdot 4 + 2000 \cdot 8 = 0, \quad \text{whence} \quad R_1 = 2400 \text{ lb.}$$

Similarly, for R_2 , $R_2 \cdot 12 - 3200 \cdot 8 - 2000 \cdot 4 = 0$, whence $R_2 = 2800$ lb.

As a check on the correctness of these results, sum of loads is $3200 + 2000 = 5200$, and sum of reactions is $2400 + 2800 = 5200$.

To obtain the shear diagram, start at the left end and lay off the reaction of 2400 lb. upward. Since the load is 200 lb./ft., at 4 ft. from the left end the shear will be $2400 - 4 \times 200 = 1600$ lb. As we pass this point the concentrated load of 1 ton will cause the shear to drop to $1600 - 2000 = -400$ lb. The shear then continues to drop 200 lb./ft., until at the right support it becomes $-400 - 8 \times 200 = -2000$ lb. As this point is passed, the reaction, which is equivalent to a concentrated load of 2800 lb. upward, causes the shear to change suddenly to $-2000 + 2800 = 800$ lb. It then gradually drops again and becomes zero at the end of the beam.

On account of the uniform load the moment diagram will be segments of parabolas. To plot these parabolas the values of the moment at a number of points along the beam may be calculated. Thus, at points 2, 4, 10, 12, and 14 ft. from the left end the moments are

$$M_2 = 2400 \cdot 2 - 400 \cdot 1 = 4400 \text{ ft.-lb.}$$

$$M_4 = 2400 \cdot 4 - 800 \cdot 2 = 8000 \text{ ft.-lb.}$$

$$M_{10} = 2400 \cdot 10 - 2000 \cdot 5 - 2000 \cdot 6 = 2000 \text{ ft.-lb.}$$

$$M_{12} = 2400 \cdot 12 - 2400 \cdot 6 - 2000 \cdot 8 = -1600 \text{ ft.-lb.}$$

$$M_{14} = 2400 \cdot 14 - 2800 \cdot 7 - 2000 \cdot 10 = -400 \text{ ft.-lb.}$$

The maximum moment is evidently at the 1-ton load, and the maximum shear at the left support.

77. A simple beam 10 ft. long is supported at the ends and carries a load of 800 lb. at a point 4 ft. from the left end. Draw the shear and moment diagrams.

78. A simple beam 20 ft. long, supported at the ends, carries a uniform load of 50 lb./ft. and a concentrated load of 600 lb. at 5 ft. from the right end. Draw the shear and moment diagrams.

79. A simple beam of 15 ft. span is supported at the ends and carries a uniform load of 100 lb./ft. and concentrated loads of 500 lb. at 4 ft. from the left end and 1000 lb. at 8 ft. from the left end. Plot the shear and moment diagrams.

80. A simple beam of 16 ft. span carries concentrated loads of 200 lb., 400 lb., and 100 lb. at distances of 4 ft., 8 ft., and 12 ft., respectively, from the left support. Neglecting the weight of the beam itself, sketch the shear and moment diagrams.

81. A simple beam of 9 ft. span carries a total uniform load of 400 lb. over the middle third of the span. Neglecting the weight of the beam, draw the shear and moment diagrams for this loading.

82. The total load on a car axle is 8 tons, equally divided between the two wheels. Distance between centers of wheels is $4\frac{2}{3}$ ft., and distance between centers of journals is $5\frac{1}{2}$ ft. Draw the shear and moment diagrams for the axle so loaded.

83. Draw the shear and moment diagrams for a simple beam 10 ft. long, bearing a total uniform load of 100 lb./ft. and concentrated loads of 1 ton at 4 ft. from the left end and 2 tons at 3 ft. from the right end.

84. A beam 12 ft. long is supported at the ends and carries loads of 4000 lb. and 1000 lb. at 2 ft. and 4 ft., respectively, from the left end. No uniform load. Sketch the shear and moment diagrams.

85. A beam 20 ft. long, supported at the ends, bears a uniform load of 100 lb./ft. extending from the left end to the center, and a concentrated load of 1000 lb. at 5 ft. from the right end. Plot the shear and moment diagrams.

86. A beam 16 ft. long, supported at the ends, carries a uniform load of 200 lb./ft. extending 10 ft. from the left end, and concentrated loads of 1 ton and $\frac{1}{2}$ ton at 8 ft. and 12 ft., respectively, from the left end. Draw the shear and moment diagrams.

87. A simple beam of 8 ft. span bears a distributed load which varies linearly from zero at one end to a maximum at the other. The total load on the beam is 1200 lb. Plot the shear and moment diagrams.

88. A cantilever beam extends 9 ft. from a wall and bears a uniform load of 60 lb./ft. and a concentrated load of 175 lb. at the free end. Draw the shear and moment diagrams.

89. A cantilever beam projects 6 ft. and supports a uniform load of 100 lb./ft. and concentrated loads of 90 lb. and 120 lb. at points 2 ft. and 4 ft., respectively, from the free end. Draw the shear and moment diagrams.

90. A cantilever beam projects 10 ft. and carries a concentrated load of 100 lb. at the free end and also concentrated loads of 90 lb. and 50 lb. at 3 ft. and 5 ft., respectively, from the free end. Sketch the shear and moment diagrams.

91. A cantilever beam projects 6 ft. from its support and bears a concentrated load of 50 lb. upward at the free end and 50 lb. downward at 2 ft. from the free end. Draw the shear and moment diagrams, neglecting the weight of the beam.

92. A cantilever beam projects 8 ft. from its support and bears a distributed load which varies linearly from zero at the free end to a maximum at the fixed end. The total load is $\frac{1}{2}$ ton. Draw the shear and moment diagrams.

93. Sketch the shear and moment diagrams for a cantilever 12 ft. long, carrying a total uniform load of 50 lb./ft. and concentrated loads of 200 lb., 150 lb., and 400 lb. at distances of 2 ft., 4 ft., and 7 ft., respectively, from the fixed end.

94. An overhanging beam of length 30 ft. carries concentrated loads of 1 ton at the left end, 1.5 tons at the center, and 2 tons at the right end, and rests on two supports, one 4 ft. from the left end and the other 6 ft. from the right end. Draw the shear and moment diagrams.

95. An overhanging beam 20 ft. in length bears a uniform load of 100 lb./ft. and rests on two supports 10 ft. apart and 5 ft. from the ends of the beam. Sketch the shear and moment diagrams.

96. An overhanging beam 25 ft. in length carries a uniform load of 200 lb./ft. over its entire length, and rests on two supports, one at the right end and the other at 10 ft. from the left end. Plot the shear and moment diagrams.

97. An overhanging beam 40 ft. in length is supported at points 4 ft. from the left end and 8 ft. from the right end. It carries concentrated loads of 4 tons at the left end, 3 tons at 6 ft. from the left end, 2 tons at 14 ft. from the left end, and 1 ton at the right end. Draw the shear and moment diagrams.

98. Draw the shear and moment diagrams for an overhanging beam 18 ft. in length, supported at points 4 ft. from each end, and carrying a uniform load of 50 lb./ft. over its entire length and a concentrated load of 800 lb. at the middle.

99. Draw the shear and moment diagrams for an overhanging beam 20 ft. in length, supported at points 3 ft. from the left end and 5 ft. from the right end, which carries a uniform load of 80 lb./ft. between the supports and concentrated loads of 600 lb. at each end.

100. Draw the shear and moment diagrams for an overhanging beam 16 ft. in length, supported at points 2 ft. from the left end and 4 ft. from the right end, which carries a load of 200 lb./ft. distributed uniformly over 12 ft. from the left end, and a concentrated load of 1600 lb. at the right end.

SECTION IV

STRENGTH OF BEAMS

32. Nature of bending stress. For a horizontal beam carrying a set of vertical loads the method just explained for drawing the moment and shear diagrams is to combine the forces on one side of any cross section into a single force, and the moments of these forces about the centroid of the section into a single moment. For equilibrium the stresses in the beam at the given section must therefore also reduce to a single force and moment, called the **shear and bending moment**, respectively, equal in amount and opposite in direction to the external resultant force and moment.

By considering a few simple cases the nature of the shearing and bending stresses will be apparent. Thus, in Fig. 46, suppose that a small vertical slice is cut out of the beam, as shown;

FIG. 46

there will evidently be a tendency for the top of the cut to close up and for the lower side to spread apart. This might be prevented by placing a small block in the upper edge of the cut and connecting the lower edges with a link. Supposing this to be done, there will, in general, still be a tendency for the part on one side of the cut to slide up or down past the part on the other side. To prevent this vertical motion, it would be necessary to introduce a vertical support, as shown in the lower diagram of Fig. 46.

From this illustration it is evident that the resisting stress in a beam required to equilibrate any system of external forces is of two kinds:

1. A compressive stress on one side, normal (that is, perpendicular) to the plane of the cross section.

A tensile stress on the opposite side, also normal to the plane of the cross section.

2. A vertical shearing stress in the plane of the cross section.

33. Distribution of stress. The effect of the external bending moment on a beam originally straight is to cause its axis to become bent into a curve, called the **elastic curve**. Considering the beam to be composed of single fibers parallel to its axis, it is found by experiment that when a beam is bent, the fibers on one side are lengthened and those on the other side are shortened. Between these there must evidently be a layer of fibers which are neither lengthened nor shortened, but retain their original length. The line in which this unstrained layer of fibers intersects any cross section is called the **neutral axis** (Fig. 47).

FIG. 47

It is also found by experiment that a cross section of the beam which was plane before flexure (bending) is plane after flexure. This is known as **Bernoulli's assumption**.^{*} As a consequence of Bernoulli's assumption it is evident from Fig. 47 that the lengthening or shortening of any longitudinal fiber is proportional to its distance from the neutral axis. But by Hooke's law the stress is proportional to the deformation produced. Therefore the normal stress at any point in the cross section is likewise proportional to the distance of this point from the neutral axis. If, then, the normal stresses are plotted for every point of any vertical strip *MN* (Fig. 48), their ends will all lie in a straight line. This distribution of stress is therefore called the **straight-line law**.

^{*} St. Venant has shown that Bernoulli's assumption is rigorously true only for certain forms of cross section. If the bending is slight, however, as is the case in all structural work, no appreciable error is introduced by assuming it to be true whatever the form of cross section.

Since the normal, or bending, stresses are the only horizontal forces acting on the portion of the beam considered, in order to satisfy the condition of equilibrium $\sum \text{horizontal forces} = 0$ we must have

Resultant tensile stress = resultant compressive stress.

Therefore, since the tensile and compressive stresses act in opposite directions (that is, are of opposite sign), the algebraic sum of all the normal stresses acting on the section must be zero. Thus, if ΔA denotes an element of area of the section, and p the intensity of the normal stress acting on it, the total stress on this area is $p\Delta A$, and consequently

$$\sum p\Delta A = 0.$$

FIG. 48

Now, if the normal stress at a variable distance y from the neutral axis is denoted by p , and that at some fixed distance y' is denoted by p' , then, from the straight-line law, $\frac{p}{p'} = \frac{y}{y'}$, or

$$p = \frac{p'}{y'} y.$$

Inserting this value of p in the above condition of equilibrium, it becomes

$$\frac{p'}{y'} \sum y\Delta A = 0.$$

Therefore, since p' and y' are definite quantities different from zero, we have $\sum y\Delta A = 0$. But, from article 13, the distance of the centroid from the neutral axis is given by

$$y_0 = \frac{\sum y\Delta A}{\sum \Delta A},$$

and if $\sum y\Delta A = 0$, then also $y_0 = 0$. Therefore the neutral axis passes through the centroid of the cross section; that is, **the neutral axis coincides with the horizontal centroidal axis.**

34. Fundamental formula for beams. For equilibrium the resultant moment of the normal stresses acting on any cross section must be equal to the resultant moment of the external forces on one side of the section, taken with respect to the neutral axis of the

section. Now, if ΔA denotes an element of area of the cross section, and p' the intensity of the normal stress acting on it, the total stress on this area is $p'\Delta A$. If, then, y is the distance of this stress, or internal force, from the neutral axis of the section, and M denotes the resultant moment of the external forces about this axis, for equilibrium

$$M = \sum (p'\Delta A)y.$$

Now let p denote the stress on the extreme fiber and e the distance of this fiber from the neutral axis. Then, by the straight-line law,

$$\frac{p'}{y} = \frac{p}{e},$$

and, inserting this value of p' in the above equation, it becomes

$$M = \frac{p}{e} \sum y^2 \Delta A.$$

The quantity $\sum y^2 \Delta A$, however, is the moment of inertia, I , of the cross section (article 20). Therefore

$$(33) \quad M = \frac{pI}{e}.$$

The right member of this equation, $\frac{pI}{e}$, is the resultant internal stress couple, and is called the **moment of resistance** of the beam.

Since e denotes the distance of the extreme fiber of the beam from the neutral axis, the ratio $\frac{I}{e}$ is also a function of the shape and size of the cross section, and is therefore called the **section modulus**. Let this section modulus be denoted by Z . Then $Z = \frac{I}{e}$, and the fundamental formula becomes

$$(34) \quad M = pZ.$$

Since this is an equality between the resultant external moment M and the product of the working stress p by the section modulus Z , it expresses the fact that the strength of a beam depends jointly on the shape and size of the cross section and the allowable stress for the material.

35. Calculation and design of beams. For a beam of given size and loading the maximum external moment M , acting at any point along the beam, is first determined by the methods explained in

Section III. The section modulus Z is then calculated from the given dimensions. For ordinary rolled shapes of structural steel the section moduli are given in Tables III-VII. The stress in the extreme fiber (or *skin stress*, as it is called) is then found by substituting these numerical values of M and Z in the equation

$$p = \frac{M}{Z}.$$

By comparing this calculated value of p with the allowable unit stress for the material, it is determined whether or not the beam is safe.

For a beam of given size and shape the maximum external moment it can carry safely is found by calculating its moment of resistance. Thus, if p denotes the allowable, or working, stress for the material in lb./in.², and the section modulus Z is calculated from the given dimensions of the cross section, the maximum external moment M which this beam can carry with safety is found by inserting these numerical values in the equation

$$M = pZ.$$

In designing a beam to carry a given loading, the maximum external moment M due to this loading is first calculated. Then, for any specified unit working stress p , the required section modulus is found from the relation

$$Z = \frac{M}{p}.$$

This section modulus Z may then be looked up in Tables III-VI, thus determining the exact dimensions of the beam.

APPLICATIONS

101. Find the safe moment of resistance for an oak beam 8 in. deep and 4 in. wide.

Solution. In this case $I = 170.7$ in.⁴ and $e = 4$ in. Therefore the section modulus is

$$Z = \frac{I}{e} = \frac{170.7}{4} = 42.7 \text{ in.}^3$$

From Table I the safe stress for timber may be assumed as $p = 1000$ lb./in.² Consequently, the moment of resistance for this beam is

$$M = pZ = 42,700 \text{ in.-lb.}$$

102. In an inclined railway the angle of inclination with the horizontal is 30° . The stringers are 10 ft. 6 in. apart, inside measurement, and the rails are placed 1 ft. inside the stringers. The ties are 8 in. deep and 6 in. wide, and the maximum load transmitted by each rail to one tie is 10 tons. Calculate the maximum normal stress in the tie.

FIG. 49

Solution. The bending moment is the same for all points of the tie between the rails, and is 20,000 ft.-lb. The components of the moment with respect to the axes of the section

(Fig. 49) are $M_z = 240,000 \left(\frac{\sqrt{3}}{2} \right)$ in.-lb.

and $M_y = 240,000 \left(\frac{1}{2} \right)$ in.-lb., and the section moduli with respect to these axes are $Z_x = 64$ in.³ and $Z_y = 48$ in.³ Therefore the maximum normal stress is

$$p_{\max} = \frac{240,000 \left(\frac{\sqrt{3}}{2} \right)}{64} + \frac{240,000 \left(\frac{1}{2} \right)}{48} = 5744 \text{ lb./in.}^2$$

103. A rectangular cantilever projects a distance l from a brick wall and bears a single concentrated load P at its end. How far must the inner end of the cantilever be embedded in the wall in order that the pressure between this end and the wall shall not exceed the crushing strength of the brick?

Solution. Let b denote the width of the beam and x the distance it extends into the wall. For equilibrium the reaction between the beam and the wall must consist of a vertical force and a moment. If p_a denotes the intensity of the vertical stress, and it is assumed to be uniformly distributed over the area bx ,

then $p_a bx = P$; whence $p_a = \frac{P}{bx}$ (see Fig. 50, *a*).

Similarly, let p_b denote the maximum intensity of the stress forming the stress couple. Then, taking moments about the center C of the portion AB , since the stress forming the couple is also distributed over the area bx , we have

$$I = \frac{bx^3}{12}, \quad e = \frac{x}{2}, \quad \text{and} \quad M = P \left(l + \frac{x}{2} \right).$$

FIG. 50

Therefore, substituting in the formula $p = \frac{Me}{I}$, we have

$$p_b = \frac{P \left(l + \frac{x}{2} \right) \frac{x}{2}}{\frac{bx^3}{12}} = \frac{6P \left(l + \frac{x}{2} \right)}{bx^2}.$$

Consequently,

$$p_{\max}^{\min} = p_b \pm p_a = \frac{6P \left(l + \frac{x}{2} \right)}{bx^2} \pm \frac{P}{bx};$$

whence

$$p_{\max} = \frac{2P}{bx} \left(2 + \frac{3l}{x} \right),$$

and

$$p_{\min} = \frac{2P}{bx} \left(1 + \frac{3l}{x} \right).$$

As a numerical example, let $l = 5$ ft., $P = 200$ lb., $b = 4$ in., and $p = 600$ lb./in.² (for ordinary brickwork). Then, solving the above equation by the formula for quadratics,

$$x = \frac{2P \pm \sqrt{4P^2 + 6bpPl}}{bp};$$

whence, by substituting the above numerical values,

$$x = 5.6 \text{ in.}$$

104. Find the required dimensions for the arms of a cast-iron pulley of external diameter D for a tension in the belt of T_1 on the tight side and T_2 on the slack side.

Arms assumed to be elliptical in cross section, of dimensions $h = 2b$ (Fig. 51).

Solution. According to Bach the load may be assumed to be carried by one third of the spokes, and the working stress taken as 4500 lb./in.² Let n denote the number of spokes. Then the maximum moment on one spoke is approximately

$$M_{\max} = \frac{(T_1 - T_2) \frac{D}{2}}{\frac{n}{3}}.$$

The moment of inertia of an ellipse about its minor axis is $\frac{\pi b h^3}{64}$, and its section modulus is $Z = \frac{h^2}{16}$. Therefore, substituting these values in the formula $p = \frac{M}{Z}$, we have

$$4500 = \frac{3(T_1 - T_2) D 16}{2 n h^2}.$$

whence

$$h = \frac{1}{13.7} \sqrt{\frac{(T_1 - T_2) D}{n}}.$$

FIG. 51

105. Derive a formula for the pitch of a cast-iron gear to carry safely a driving force F .

Solution. Circular pitch is defined as the distance between corresponding points on two successive teeth, measured along the pitch circle. Let P denote the circular pitch for the case in question (Fig. 52). Then, if h denotes the depth of the tooth, b its breadth, and t its thickness at the root, the relative proportions ordinarily used are

FIG. 52

$$h = .7 P, \quad t = .5 P, \quad b = 2 P \text{ to } 3 P,$$

$$BC = .47 P, \quad AB = .53 P,$$

Height above pitch circle (called addendum) = $.3 P$,

Depth within pitch circle = $.4 P$.

The driving force F is ordinarily applied tangent to the pitch circle. Assume, however, that by reason of the gear being worn, or from some other cause, it reaches the tip of the tooth, as shown in the figure. Then, considering the tooth as a cantilever beam, the maximum moment is

$$M = Fh,$$

and its section modulus at the root is

$$Z = \frac{bt^2}{6}.$$

Therefore, assuming a working stress for cast iron of $p = 4500 \text{ lb./in.}^2$, we have

$$4500 \frac{bt^2}{6} = Fh,$$

and, inserting the values

$$b = 2 P, \quad t = \frac{1}{2} P, \quad h = .7 P,$$

this becomes

$$F = 536 P^2.$$

106. Find the moment of resistance for the section given in problem 53, assuming the working stress for structural steel as $16,000 \text{ lb./in.}^2$

107. Find the section modulus and moment of resistance for the section given in problem 55.

108. Find the section modulus and moment of resistance for the section given in problem 56.

109. Find the moment of resistance of a circular cast iron beam 6 in. in diameter.

110. Find the moment of resistance of a 24-in. steel I-beam weighing 80 lb./ft.

111. Compare the moments of resistance of a rectangular beam 8 in. $\times$ 14 in. in cross section, when placed on edge and when placed on its side.

FIG. 53

112. Find the section moduli for the sections given in problems 58, 60, and 62.

113. Design a steel I-beam, 10 ft. long, to bear a total uniform load of 1500 lb./ft., including its own weight.

FIG. 54

114. A built beam is to be composed of two steel channels placed on edge and connected by latticing. What must be the size of the channels if the beam is to be 18 ft. long and bear a load of 10 tons at its center, for a working stress of 16,000 lb./in.²?

115. Compare the strength of a pile of 10 boards, each 14 ft. long, 1 ft. wide, and 1 in. thick, when the boards are piled horizontally and when they are placed close together on edge.

116. Design a rectangular wooden cantilever to project 4 ft. from a wall and bear a load of 500 lb. at its end, the factor of safety being 8.

117. A wooden girder supporting the bearing partitions in a dwelling is made up of four 2-in. by 10-in. joists set on edge and spiked together. Find the size of a steel I-beam of equal strength.

118. A factory floor is assumed to carry a load of 200 lb./ft.² and is supported by steel I-beams of 16-ft. span and spaced 4 ft. apart on centers. What size I-beam is required for a working stress of 16,000 lb./in.²?

FIG. 55

119. Find the required size of a square wooden beam of 14-ft. span to carry an axial tension of 2 tons and a uniform load of 100 lb./ft.

120. A floor designed to carry a uniform load of 200 lb./ft.² is supported by 10-in. steel I-beams weighing 30 lb./ft. How far apart may they be placed for a span of 16 ft. and a working stress of 16,000 lb./in.²?

FIG. 56

121. A floor is supported by wooden joists 2 in. $\times$ 12 in. in section and 16 ft. span, spaced 16 in. apart on centers. Find the safe load per sq. ft. of floor area for a working stress of 800 lb./in.²

122. A floor is required to support a uniform load of 150 lb./ft.² and is supported by steel I-beams, 18 ft. span and spaced 5 ft. apart on centers. What size I-beam

is required for a working stress of 16,000 lb./in.²?

123. A structural-steel built beam is 20 ft. long and has the cross section shown in Fig. 53. Compute its moment of resistance and find the safe uniform load it can carry per linear foot for a factor of safety of 5.

124. The cast-iron bracket shown in Fig. 54 has at the dangerous section the dimensions shown in the figure. Find the maximum concentrated load it can carry with a factor of safety of 15.

125. Find the proper dimensions for a wrought-iron crank of dimensions shown in Fig. 55 for a crank thrust of 1500 lb. and a factor of safety of 6.

126. A wrought-iron pipe 1 in. in external diameter and $\frac{1}{8}$ in. thick projects 6 ft. from a wall. Find the maximum load it can support at the outer end.

FIG. 57

127. The yoke of a hydraulic press used for forcing gears on shafts is of the form and dimensions shown in Fig. 56. The yoke is horizontal, with groove up,

FIG. 58

dangerous section AB are as given in the figure. Find the maximum stress at this section.

130. The load on a car axle is 8 tons, equally distributed between the two wheels (Fig. 58). The axle is of cast steel. Find its diameter for a factor of safety of 15.

131. The floor of an ordinary dwelling is assumed to carry a load of 50 lb./ft.² and is supported by wooden joists 2 in. by 10 in. in section, spaced 16 in. apart on centers. Find the greatest allowable span for a factor of safety of 10.

132. An engine shaft of machinery steel rests in bearings 6 ft. apart between centers and carries a 12-ton flywheel midway between the bearings. Find the required size of shaft.

133. A cast-iron flange coupling is connected with ten wrought-iron bolts. Distance from axis of each bolt to axis of shaft is 6 in. Total torque (twisting moment) transmitted is 12,000 ft.-lb. If the flanges are accidentally separated 2 in. and the bolts are a drive fit, find the bending stress produced in each bolt.

134. In the carriage clamp shown in Fig. 59 the screw is of wrought iron, $\frac{5}{8}$ in. diameter, square thread, 5 threads per inch, and the casting has the dimensions given in the figure. Find what load on the screw will cause failure by shearing the threads, and find the maximum stress in the casting under this load, due to combined bending and tension.

FIG. 60

shown in Fig. 60 the bar is of carbon steel, $1\frac{1}{4}$ in. $\times$ $\frac{1}{4}$ in., tensile strength 70,000 lb./in.², and the screw is steel, $\frac{5}{8}$ in. diameter, square threads, 5 threads to the inch. Find the dimensions of the cast-iron handle so that it shall be light enough to act as the breaking piece.

so that the shaft to be fitted lies in the groove, as shown in plan in the figure. The ram is 32 in. in diameter and under a water pressure of 250 lb./in.² Find the dangerous section of the yoke and the maximum stress at this section.

128. A 10-in. I-bar weighing 40 lb./ft. is supported on two trestles 15 ft. apart. A chain block carrying a 1-ton load hangs at the center of the beam. Find the factor of safety.

129. The hydraulic punch shown in Fig. 57 is designed to punch a $\frac{5}{8}$ -in. hole in a $\frac{5}{8}$ -in. plate. The dimensions of the

FIG. 59

Find what load on the screw will cause failure by shearing the threads, and find the maximum stress in the casting under this load, due to combined bending and tension.

135. In the joiner's clamp

SECTION V

DEFLECTION OF CANTILEVER AND SIMPLE BEAMS

36. General deflection formula. By a **simple beam** is meant one which is simply supported at the ends. The only external forces acting on it in addition to the loads are, then, the two vertical reactions at the supports. A **cantilever** is a beam which overhangs, or projects outward from the support, the loads on it being equilibrated by the moment at the support and by the vertical reaction

FIG. 61

at this point. The results of applying the general deflection formula, derived below, to these two classes of beams will be made the basis of the treatment of continuous and restrained beams in the sections which follow.

Taking a vertical longitudinal section of a beam, the line in which this plane intersects the neutral-fiber surface is called the **elastic curve**. Any small segment,

Δx , of the elastic curve may be considered as a circular arc with center at some point O (Fig. 61). This point O is therefore called the **center of curvature** for the arc Δx . The radius of curvature is not constant, but changes from point to point along the beam. Evidently the radius of curvature is least where the beam is curved most sharply.

Any two adjacent plane sections, AB and DH (Fig. 61), originally parallel, intersect after flexure in the center of curvature O . Let $KC = \Delta x$ denote the original length of the fibers, and draw

through C a line EF parallel to AB . Then DF denotes the shortening of the extreme fiber on one side, and EH the lengthening of the extreme fiber on the other. Now, since the triangles KOC and ECH are similar, we have the proportion

$$\frac{EH}{KC} = \frac{CH}{OK} = \frac{e}{r}.$$

Moreover, the left member, $\frac{EH}{KC}$, is the change in length of the extreme fiber divided by its original length, which is by definition the unit deformation s of this fiber. Also, by Hooke's law, $\frac{p}{s} = E$, where p denotes the unit normal stress on this fiber. Hence the above proportion becomes

$$\frac{p}{E} = \frac{e}{r};$$

or, since $p = \frac{M}{Z} = \frac{Me}{I}$,
as shown in the preceding section,

$$\frac{Me}{EI} = \frac{e}{r};$$

whence

$$(35) \quad r = \frac{EI}{M}.$$

FIG. 62

Now let AB denote any segment of the elastic curve, and AA' , BB' the tangents at A and B respectively (Fig. 62). If AB is divided up into small segments Δx , and $\Delta\phi$ denotes the angle which each subtends at the center of curvature O , as shown in Fig. 62, then $\Delta x = r\Delta\phi$, or $\Delta\phi = \frac{\Delta x}{r}$, and, inserting in this the value of r obtained above, it becomes

$$\Delta\phi = \frac{M\Delta x}{EI}.$$

Hence, by summation, the total angular deflection ϕ is

$$\phi = \sum \Delta\phi = \frac{1}{EI} \sum M\Delta x.$$

Now for any small arc, Δx , the deflection Δd at any point at a distance x , measured from the tangent to the arc at the initial point (Fig. 62), is

$$\Delta d = x \Delta \phi.$$

Hence the total deflection for any finite portion of the arc AB , measured from one end A to the tangent at the other end B , is

$$d = \sum \Delta d = \sum x \Delta \phi = \frac{1}{EI} \sum (M \Delta x) x.$$

But $M \Delta x$ denotes the area of a small vertical strip of the moment diagram of altitude M and base Δx , and $\sum (M \Delta x) x$ is the sum of the static moments of all these elements of area with respect to the point A . From the results of Section II, however, this is equal to the area of the moment diagram between A and B multiplied by the distance of its centroid from A . That is, if A_{ab} denotes the area of the moment diagram between the points A and B , and x_0 is the distance of the centroid of A_{ab} from A , then

$$\sum (M \Delta x) x = A_{ab} \cdot x_0.$$

Therefore

$$d = \frac{1}{EI} A_{ab} \cdot x_0,$$

or, in general,

$$(36) \quad d = \frac{1}{EI} (\text{static moment of the moment diagram}).$$

The angular deflection ϕ between any two points A and B , that is, the angle between the tangents to the elastic curve at these two points, is given by

$$\phi = \sum \Delta \phi = \frac{1}{EI} \sum M \Delta x.$$

Therefore, since $\sum M \Delta x$ denotes the area A_{ab} of the moment diagram between the two points in question, and since for the small deflections which actually occur in practice we may assume $\phi = \tan \phi$ without introducing any appreciable error,

$$(37) \quad \tan \phi = \frac{A_{ab}}{EI}.$$

37. Cantilever bearing concentrated load. For a cantilever bearing a concentrated load at the end, the moment diagram is a triangle, as shown in Fig. 63. The area of the moment diagram is therefore

FIG. 63

is $A = \frac{Pl^2}{2}$, and the distance of its centroid from the free end is $x_0 = \frac{2}{3}l$. Therefore the deflection at the free end is

$$(38) \quad d = \frac{1}{EI} Ax_0 = \frac{Pl^3}{3EI}.$$

The deflection d may also be expressed in terms of the stress on the extreme fiber. Thus, since $\frac{pI}{e} = M = Pl$, by substituting this value of Pl in the expression for d , it becomes

$$(39) \quad d = \frac{pl^3}{3Ee}.$$

Also, the angular deflection at the load is found to be

$$(40) \quad \tan \phi = \frac{A}{EI} \\ = \frac{Pl^2}{2EI}.$$

FIG. 64

If the load is at a distance a from the fixed end and b from the free end (Fig. 64), then the deflection at the load, as shown above, is

$$(41) \quad d = \frac{Pa^3}{3EI},$$

and similarly, from equation (40), the angular deflection at the load is

$$(42) \quad \tan \phi = \frac{Pa^2}{2EI}.$$

Consequently, the additional deflection d' from the load to the free end of the cantilever is

$$(43) \quad d' = b \tan \phi = \frac{Pa^2b}{2EI}.$$

The total deflection D at the free end is therefore

$$(44) \quad D = a + d' = \frac{Pa^2}{EI} \left(\frac{a}{3} + \frac{b}{2} \right).$$

It is often convenient to let $b = kl$, where k denotes a proper fraction. Then in the present case $a = l - b = l(1 - k)$, and the expression for the deflection at the end becomes

$$(45) \quad D = \frac{Pl^3}{6EI} (2 - 3k + k^3).$$

FIG. 65

For instance, if the load is at the middle of the cantilever, then $k = \frac{1}{2}$, and the deflection at the free end becomes

$$(46) \quad D = \frac{5}{48} \frac{Pl^3}{EI}.$$

38. Cantilever bearing uniform load. For a uniformly loaded cantilever the moment diagram is a parabola and the moment at

the support is $M = wl \cdot \frac{l}{2} = \frac{wl^2}{2}$ (Fig. 65). Also, from article 17, the area of the moment diagram is $A = \frac{1}{3} \frac{wl^2}{2} \cdot l = \frac{wl^3}{6}$, and the distance of its centroid from the free end is $x_0 = \frac{3}{4}l$. Therefore the deflection at the free end is

$$(47) \quad d = \frac{1}{EI} Ax_0 = \frac{wl^4}{8EI}.$$

From the relation $\frac{pI}{e} = M = \frac{wl^2}{2}$ the expression for the deflection in terms of the maximum fiber stress p is found to be

$$(48) \quad d = \frac{pl^3}{4Ee}.$$

Also, the total angular deflection at the end of the beam is

$$(49) \quad \tan \phi = \frac{A}{EI} = \frac{wl^3}{6EI}.$$

39. Cantilever under constant moment. If a cantilever is subjected to a couple, that is, a pair of equal and opposite parallel forces, as shown in Fig. 66, the moment is constant for the entire length of the beam. The moment diagram is therefore a rectangle of area Ml , and the deflection at the free end is

$$(50) \quad d = \frac{1}{EI} Ax_0 \\ = \frac{Ml^2}{2EI}.$$

The angular deflection at the free end in this case is

$$(51) \quad \tan \phi = \frac{A}{EI} \\ = \frac{Ml}{EI}.$$

FIG. 66

40. Simple beam bearing concentrated load. To apply the deflection formula to a simple beam, the deflection must be measured at one end A from a tangent at the middle C . For a concentrated load P at the middle (Fig. 67), the area of the moment diagram from A to C is $A = \frac{Pl^2}{16}$, and $x_0 = \frac{2}{3} \cdot \frac{l}{2} = \frac{l}{3}$. Hence, in this case

$$(52) \quad d = \frac{1}{EI} Ax_0 = \frac{Pl^3}{48EI}.$$

Also, the total angular deflection for half the beam is found to be

$$(53) \quad \tan \phi = \frac{A}{EI} = \frac{Pl^2}{16EI}.$$

The deflection may also be obtained by considering the moment diagram as representing the load on the beam, and then taking

FIG. 67

moments about the point at which the deflection is measured, say the center C (Fig. 67). Since the total area of the moment diagram is $\frac{1}{2} \cdot \frac{Pl}{4} \cdot l = \frac{Pl^2}{8}$, if it is regarded as the load on the beam each reaction will be $\frac{Pl^2}{16}$. Then, taking

moments about the center, the result is

$$(54) \quad d = \frac{1}{EI} \left(\frac{Pl^2}{16} \cdot \frac{l}{2} - \frac{Pl^2}{16} \cdot \frac{l}{6} \right) = \frac{Pl^3}{48EI}.$$

From the relation $\frac{pI}{e} = M = \frac{Pl}{4}$, the deflection at the center may

be expressed in terms of the maximum fiber stress p . Thus, replacing $\frac{Pl}{4}$ by its equal $\frac{pI}{e}$ in the expression for d , the result is

$$(55) \quad d = \frac{pl^2}{12Ee}.$$

FIG. 68

If the concentrated load P is not at the center but divides the span l into two unequal segments a and b , the reactions are $\frac{Pb}{l}$, $\frac{Pa}{l}$,

and the moment at the load is $\frac{Pab}{l}$. Also, the area of the moment diagram between the load and one end is $\frac{a}{2} \cdot \frac{Pab}{l}$, and the distance of the centroid of this segment from the end is $x_0 = \frac{2}{3}a$. Hence the deflections of the ends from the tangent at the point of application C of the load are

$$\frac{Pa^3b}{3EI} \quad \text{and} \quad \frac{Pab^3}{3EI},$$

and the deflection of C below the level of the supports is

$$(56) \quad d = \frac{Pa^2b^2}{3EI} = \frac{pab}{3Ee},$$

where p denotes the maximum fiber stress.

41. Simple beam bearing uniform load. For a simple beam uniformly loaded the moment diagram is a parabola, the maximum ordinate being $\frac{wl^2}{8}$.

From article 17, the area of this parabola is

$$A = \frac{2}{3} \cdot \frac{wl^2}{8} \cdot l = \frac{wl^3}{12}.$$

To apply the general formula for deflection, consider d as measured from one end A to the tangent at the center C (Fig. 69). Then,

since the area of one half the moment diagram is $\frac{wl^3}{24}$, and the distance of the centroid of this half from a vertical through A is $x_0 = \frac{5}{8} \cdot \frac{l}{2} = \frac{5l}{16}$, the deflection is

$$(57) \quad d = \frac{1}{EI} Ax_0 = \frac{5wl^4}{384EI}.$$

To express the deflection in terms of the maximum fiber stress p , make use of the relation $\frac{pI}{e} = M = \frac{wl^2}{8}$. Then, replacing $\frac{wl^2}{8}$, in the

FIG. 69

expression for d , by its equal $\frac{pI}{e}$, it becomes

$$(58) \quad d = \frac{5pl^2}{48Ee}.$$

The total angular deflection for half the beam in this case is

$$(59) \quad \tan \phi = \frac{A}{EI} = \frac{wl^3}{24EI}.$$

The deflection may also be obtained by regarding the moment diagram as representing the load on the beam. Since the total area of the moment diagram is $\frac{wl^3}{12}$, each reaction will then be $\frac{wl^3}{24}$, and therefore, taking moments about the center to find the deflection at this point, the result is, as before,

$$(60) \quad d = \frac{1}{EI} \left(\frac{wl^3}{24} \cdot \frac{l}{2} - \frac{wl^3}{24} \cdot \frac{3}{16} l \right) = \frac{5wl^4}{384EI}.$$

APPLICATIONS

136. A 15-in. I-beam weighing 60 lb./ft. carries a 25-ton load at the center of a 12-ft. span. Find the maximum deflection.

137. In building construction the maximum allowable deflection for plastered ceilings is $\frac{1}{360}$ of the span. A floor is supported on 2 in. $\times$ 10 in. wooden joists of 14-ft. span and spaced 16 in. apart on centers. Find the maximum load per square foot of floor surface, in order that the deflection may not exceed the amount specified.

138. Determine the proper spacing, center to center, for 12-in. steel I-beams weighing 35 lb./ft., for a span of 20 ft. and a uniform floor load of 100 lb./ft.², in order that the deflection shall not exceed $\frac{1}{360}$ of the span.

139. A structural steel shaft 8 in. in diameter and 5 ft. long between centers of bearings carries a 25-ton flywheel midway between the bearings. Find the maximum deflection of the shaft, considering it as a simple beam.

140. A wrought-iron bar 2 in. square is bent to a right angle 4 ft. from one end. The other end is then embedded in a concrete block so that it stands upright with the 4 ft. length horizontal. If the upright projects 12 ft. above the concrete, and a load of 300 lb. is hung at the end of the horizontal arm, find the deflection at the end of this arm.

141. A wooden cantilever 2 in. $\times$ 10 in. in section, with the longer side vertical, projects 10 ft. from the face of a wall and carries a concentrated load of 600 lb. at a point 6 ft. from the wall. Find the deflection at the free end of the beam.

142. A 10-in. steel I-beam weighing 40 lb./ft. spans an opening 16 ft. wide and supports a total load of 40 tons. Find how much greater the maximum deflection of the beam is when this load is concentrated at its center than when it is distributed uniformly over the beam.

143. A built beam is composed of two 10-in. steel channels, 40 lb./ft., placed on edge and connected with latticing. The span is 20 ft. Find what uniform load per linear foot the beam can carry under the condition that the maximum deflection shall not exceed $\frac{3}{4}$ in.

144. A 15-in. steel I-beam, 42 lb./ft., spans an 18-ft. opening. Find the maximum deflection for a maximum fiber stress of 16,000 lb./in.²

145. The total load on a car axle is 10 tons, equally distributed between the two wheels. Distance between centers of wheels is 56 in., and between centers of bearings is 68 in. Find the maximum deflection of the axle measured from a horizontal line joining the centers of bearings.

146. A 10-in. steel I-beam weighing 30 lb./ft. rests on two supports 16 ft. apart and carries a uniform load of 200 lb./ft. in addition to its own weight. A third support just touches the beam at the center. How much must this central support be raised so that it shall carry all the weight, and the beam just touch the two end supports?

147. A cast-iron pipe 20 in. internal diameter and 1 in. thick rests on supports 30 ft. apart. Find the maximum deflection when the pipe is full of water.

148. A beam of uniform section, carrying a concentrated load at the center, has a maximum deflection equal to 1 per cent of the span. Find the slope of the beam at its ends.

149. Three beams of the same material are laid side by side across an opening of 12-ft. span, and a load of 1000 lb. rests across them at the center of the span so that they must all bend together. The beams are each 2 in. wide, but two of them are 6 in. deep, while the third is 12 in. deep. How much of the weight is carried by each beam?

150. A steel bar 2 in. square rests on knife edges 5 ft. apart, and its maximum deflection under a central load of 1000 lb. is found to be .1125 in. Calculate from this experiment the modulus of elasticity of the bar.

SECTION VI

CONTINUOUS BEAMS

42. Theorem of three moments for uniform loads. A continuous beam, or girder, is one which is supported at several points of its length. The reactions and moments in this case are **statically indeterminate**; that is to say, the ordinary static conditions of equilibrium, $\sum F = 0$, $\sum M = 0$, are insufficient to determine them. To solve the problem it is necessary also to take into account the deflections of the beam.

FIG. 70

The simplest method of finding the reactions and moments at the supports for a continuous beam is by applying what is known as the **theorem of three moments**. This theorem establishes a relation between the moments at three consecutive supports of a continuous beam and the loads on the two included spans, and was first published by **Clapeyron** in 1857. The following proof of the theorem, however, is very much simpler than any previously given.

For a continuous beam bearing a uniform load let A, B, C denote any three consecutive points of support, assumed to be in the same line, and let $M_1, M_2, M_3; R_1, R_2, R_3$ denote the moments and reactions at these three points respectively. Also let l_1, l_2 denote the lengths of the two spans considered, w_1, w_2 the unit loads on them, and S_1^R, S_1^L the shears on the right and left of R_1 respectively (Fig. 70), with a similar notation for the other points of support.

Now consider a portion of the beam cut off by planes just inside the supports at A and C , as shown in Fig. 71. Then, considering the end B as fixed, the deflection at A from the

FIG. 71

tangent at B consists of three parts: that due to the moment M_1 , to the shear S_1^R considered as a load, and to the uniform load w_1 . Calling these deflections d_1, d_2, d_3 , respectively, we have

$$d_1 = -\frac{M_1 l_1^2}{2 EI}, \quad (\text{Eq. (50), Art. 39})$$

$$d_2 = \frac{S_1^R l_1^3}{3 EI}, \quad (\text{Eq. (38), Art. 37})$$

$$d_3 = -\frac{w_1 l_1^4}{8 EI}. \quad (\text{Eq. (47), Art. 38})$$

Hence the total deflection D_A of the point A measured from a tangent at the point B is

$$(61) \quad D_A = -\frac{M_1 l_1^2}{2 EI} + \frac{S_1^R l_1^3}{3 EI} - \frac{w_1 l_1^4}{8 EI}.$$

To eliminate the shear S_1^R , form a moment equation by taking moments about the point B . Then

$$M_2 = M_1 - S_1^R l_1 + \frac{w_1 l_1^2}{2};$$

whence

$$(62) \quad S_1^R = \frac{1}{l_1} (M_1 - M_2) + \frac{w_1 l_1}{2},$$

and, substituting this value of S_1^R in the above expression for D_A , it reduces to

$$(63) \quad D_A = -\frac{M_1 l_1^2}{6 EI} - \frac{M_2 l_1^2}{3 EI} + \frac{w_1 l_1^4}{24 EI}.$$

Similarly, by considering the span BC and calculating the deflection D_C of the point C measured from the same tangent at B , we obtain the equation

$$(64) \quad D_C = -\frac{M_3 l_2^2}{2 EI} + \frac{S_3^L l_2^3}{3 EI} - \frac{w_2 l_2^4}{8 EI}.$$

Also, forming a moment equation with C as center of moments, we have

$$M_2 = M_3 - S_3^L l_2 + \frac{w_2 l_2^2}{2},$$

and, eliminating S_3^L between these relations, the result is

$$(65) \quad D_C = -\frac{M_3 l_2^2}{6 EI} - \frac{M_2 l_2^2}{3 EI} + \frac{w_2 l_2^4}{24 EI}.$$

Now, since these deflections lie on opposite sides of the tangent at B , we have, from similar triangles,

$$\frac{D_A}{l_1} = -\frac{D_C}{l_2}.$$

Therefore, substituting the expressions for D_A and D_C in this relation, combining terms, and transposing, we obtain the relation

$$(66) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = \frac{w_1 l_1^3 + w_2 l_2^3}{4}.$$

In this relation, M_1 , M_2 , and M_3 are stress couples acting *on the beam*. The external moments at the supports are equal in amount but opposite in sign to the stress couples, or internal moments. Therefore, calling M_1 , M_2 , M_3 the *external moments at the supports*, the sign of the expression is changed; that is

$$(67) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = -\frac{w_1 l_1^3 + w_2 l_2^3}{4}.$$

This is the required theorem of three moments for uniform loads.

43. Theorem of three moments for concentrated loads. Consider a continuous beam bearing a single concentrated load in each span.

The distance of the load in any span from the adjoining support on the left will be denoted by kl , where l is the length of the span and k is a proper fraction; that is, kl is some fractional part of the span (Fig. 72). Thus, if the load is at the middle of the span, $k = \frac{1}{2}$; if it is at the quarter point, $k = \frac{1}{4}$, etc.

Now consider a portion of the beam extending over three consecutive supports A , B , and C , and let M_1 , M_2 , M_3 denote the moments, and R_1 , R_2 , R_3 the reactions, at these supports. Then, to obtain the theorem of three moments, calculate the deflections of A and C measured from the tangent to the elastic curve at B . To calculate the deflection of A , suppose the beam to be cut by a plane just inside the sup-

FIG. 72

port at A , and call the shear on the section S_1^R . Then, considering the end B as fixed, calculate the deflection of A by treating the part AB as a cantilever subjected to the moment M_1 , the shear S_1^R regarded as a load, and the concentrated load P_1 . Calling these three partial deflections d_1 , d_2 , d_3 , respectively, we have

$$d_1 = -\frac{M_1 l_1^2}{2 EI}, \quad (\text{Eq. (50), Art. 39})$$

$$d_2 = \frac{S_1^R l_1^3}{3 EI}, \quad (\text{Eq. (38), Art. 37})$$

$$d_3 = -\left(\frac{P_1 a^3}{3 EI} + \frac{P_1 a^2 b}{2 EI}\right). \quad (\text{Eq. (44), Art. 37})$$

In the present notation the quantities a and b in the expression for d_3 are

$$a = \text{distance from fixed end} = l_1 - k_1 l_1,$$

$$b = \text{distance from free end} = k_1 l_1.$$

Substituting these values of a and b , the equation for d_3 becomes

$$(68) \quad d_3 = -\frac{P_1 l_1^3}{6 EI} (2 - 3 k_1 + k_1^3).$$

Therefore, by addition, the total deflection of the end A with respect to the tangent at B is

$$(69) \quad D_A = -\frac{M_1 l_1^2}{2 EI} + \frac{S_1^R l_1^3}{3 EI} - \frac{P_1 l_1^3}{6 EI} (2 - 3 k_1 + k_1^3).$$

Now, forming a moment equation for the portion AB , taking center of moments at B , we have

$$M_2 = M_1 - S_1^R l_1 + P_1 l_1 (1 - k_1);$$

whence
$$S_1^R = \frac{1}{l_1} (M_1 - M_2) + P_1 (1 - k_1),$$

and eliminating S_1^R between this equation and the expression for D_A , the result is

$$(70) \quad D_A = -\frac{M_1 l_1^2}{6 EI} - \frac{M_2 l_1^2}{3 EI} + \frac{P_1 l_1^3}{6 EI} (k_1 - k_1^3).$$

Similarly, to find the deflection at C , measured from the tangent to the elastic curve at B , treat the portion BC as a cantilever fixed at B and subjected to the moment M_3 , the shear S_3^L considered as a load, and the concentrated load P_2 . Then, calling these partial deflections d_1 , d_2 , d_3 , we have

$$d_1 = -\frac{M_3 l_2^2}{2 EI}, \quad (\text{Eq. (50), Art. 39})$$

$$d_2 = \frac{S_3^L l_2}{3 EI}, \quad (\text{Eq. (38), Art. 37})$$

$$d_3 = -\left(\frac{P_2 a^3}{3 EI} + \frac{P_2 a^2 b}{2 EI} \right), \quad (\text{Eq. (44), Art. 37})$$

or, since in the present case $a = k_2 l_2$, $b = l_2 (1 - k_2)$, the expression for d_3 becomes

$$(71) \quad d_3 = -\frac{P_2 l_2^3}{6 EI} (3 k_2^2 - k_2^3).$$

Therefore the total deflection D_C from the tangent at B is

$$(72) \quad D_C = -\frac{M_3 l_2^2}{2 EI} + \frac{S_3^L l_2^3}{3 EI} - \frac{P_2 l_2^3}{6 EI} (3 k_2^2 - k_2^3).$$

Now, forming a moment equation for the portion BC , taking center of moments at B , we have

$$M_2 = M_3 - S_3^L l_2 + P_2 k_2 l_2;$$

whence

$$(73) \quad S_3^L = \frac{1}{l_2} (M_3 - M_2) + P_2 k_2,$$

and, eliminating S_3^L between this equation and the expression for D_C , the result is

$$(74) \quad D_C = -\frac{M_3 l_2^2}{6 EI} - \frac{M_2 l_2^2}{3 EI} + \frac{P_2 l_2^3}{6 EI} (2 k_2 - 3 k_2^2 + k_2^3).$$

Since the deflections at A and C lie on opposite sides of the tangent at B , we have, from similar triangles,

$$\frac{D_A}{l_1} = -\frac{D_C}{l_2}.$$

Substituting in this relation the values of D_A and D_C just found, combining like terms, and transposing, we obtain the relation

$$(75) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = P_1 l_1^2 (k_1 - k_1^3) + P_2 l_2^2 (2 k_2 - 3 k_2^2 + k_2^3).$$

In this relation M_1 , M_2 , M_3 are the stress couples acting on the beam. The external moments at the supports are equal in amount but opposite in sign to the stress couples. Therefore, calling M_1 , M_2 , M_3 the external moments at the supports, the sign of the expression is changed; that is,

$$(76) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = -P_1 l_1^2 (k_1 - k_1^3) - P_2 l_2^2 (2 k_2 - 3 k_2^2 + k_2^3),$$

which is the required theorem of three moments for a single concentrated load in each span.

For a single concentrated load at the center of each span, each $k = \frac{1}{2}$. In this case the theorem becomes

$$(77) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = -\frac{3}{8} (P_1 l_1^2 + P_2 l_2^2).$$

If there are a number of concentrated loads in each span, an equation like (76) can be written for each load separately. By

adding these equations the general theorem of three moments for any number of concentrated loads is found to be

$$(78) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = - \sum P_1 l_1^2 (k_1 - k_1^3) - \sum P_2 l_2^2 (2 k_2 - 3 k_2^2 + k_2^3).$$

44. Effect of unequal settlement of supports. In deriving the theorem of three moments the supports were assumed to be at a fixed elevation in the same line. If their relative elevation changes, owing to unequal settlement of the supports or to other causes, the effect in general is to increase the stress in the member. To take account of this effect in applying the theorem, suppose that the supports were originally in line, and denote the settlement of three consecutive supports A, B, C from their original level by h_1, h_2, h_3 , respectively. Then the difference in elevation between A and B is $h_1 - h_2$, and between B and C is $h_3 - h_2$. Thus, if A settles more than B , $h_1 - h_2$ is positive and the deflection at A is increased by this amount. If A settles less than B , $h_1 - h_2$ is negative and the deflection at A is decreased by this amount, etc. In general, then, equations (70) and (74) for the deflections at A and C become

$$D_A = -\frac{M_1 l_1^2}{6 EI} - \frac{M_2 l_1^2}{3 EI} + \sum \frac{P_1 l_1^3}{6 EI} (k_1 - k_1^3) + (h_1 - h_2)$$

$$D_C = -\frac{M_3 l_2^2}{6 EI} - \frac{M_2 l_2^2}{3 EI} + \sum \frac{P_2 l_2^3}{6 EI} (2 k_2 - 3 k_2^2 + k_2^3) + (h_3 - h_2).$$

Substituting these values of D_A and D_C in the relation

$$\frac{D_A}{l_1} = -\frac{D_C}{l_2},$$

the result, after combining terms and changing the signs of M_1, M_2, M_3 , is

$$(79) \quad M_1 l_1 + 2 M_2 (l_1 + l_2) + M_3 l_2 = - \sum P_1 l_1^2 (k_1 - k_1^3) - \sum P_2 l_2^2 (2 k_2 - 3 k_2^2 + k_2^3) - 6 EI \left(\frac{h_1 - h_2}{l_1} + \frac{h_3 - h_2}{l_2} \right).$$

This relation is therefore the most general form of the theorem of three moments for any number of concentrated loads, including the effect of unequal settlement of the supports, or other change in their relative elevation.

APPLICATIONS

151. A continuous beam of four equal spans is uniformly loaded. Find the bending moments at the supports.

Solution. The system of simultaneous equations to be solved in this case is

$$M_1 = M_5 = 0,$$

$$M_1 + 4M_2 + M_3 = -\frac{wl^2}{2},$$

$$M_2 + 4M_3 + M_4 = -\frac{wl^2}{2},$$

$$M_3 + 4M_4 + M_5 = -\frac{wl^2}{2},$$

the solution of which gives

$$M_2 = M_4 = -\frac{3}{8}wl^2, \quad M_3 = -\frac{1}{4}wl^2.$$

152. A continuous beam of n equal spans carries a uniform load of the same amount in each span. Check the moments at the supports given in the following table for values of n from 2 to 7. The tabular values here given are the numerical coefficients of $-wl^2$.

MOMENTS AT SUPPORTS FOR EQUAL SPANS AND UNIFORM LOAD

SHEARS AT SUPPORTS FOR EQUAL SPANS AND UNIFORM LOAD

153. Calculate the reactions of the supports in problem 151.

Solution. The reaction at any support may be found by finding the shears close to the support on each side. The sum of these two shears is then equal to the reaction. Thus, in the present case, to find any given reaction, say R_2 , consider the portion of the beam between R_1 and R_2 , as shown in Fig. 70, and form the moment equation for this segment. Then

$$M_1 = M_2 - S_2^L l + \frac{wl^2}{2},$$

and therefore, since $M_1 = 0$ and $M_2 = \frac{3}{8} wl^2$, $S_2^L = \frac{1}{2} \frac{7}{8} wl$. Similarly, the moment equation for the segment of the beam between R_2 and R_3 is

$$M_3 = M_2 - S_2^R l + \frac{wl^2}{2};$$

whence, by substituting $M_3 = \frac{2}{8} wl^2$ and $M_2 = \frac{3}{8} wl^2$, we have

$$S_2^R = \frac{1}{2} \frac{5}{8} wl.$$

Consequently, $R_2 = S_2^L + S_2^R = \frac{17 + 15}{28} wl = \frac{32}{28} wl$.

This method applies when it is required to find one reaction only, independently of the others. If all the reactions are required, it is simpler to calculate them in succession, starting at one end, without reference to the shears. For instance, to find R_1 , take a section through R_2 and consider the loads on the left of the section. Then the moment equation for this portion is

$$M_2 = \frac{wl^2}{2} - R_1 l;$$

whence

$$R_1 = \frac{1}{2} \frac{1}{8} wl.$$

To find R_2 , take a section through R_3 and consider all the loads on the left of the section. Then the moment equation is

$$M_3 = \frac{w(2l)^2}{2} - R_1 \cdot 2l - R_2l,$$

and inserting the value of R_1 just obtained, it is found that

$$R_2 = \frac{3}{2} \frac{2}{8} wl.$$

By this method each reaction may be obtained in terms of those already found, without calculating the shears.

154. In problem 152 determine the shears and reactions at the supports and check the results with the values tabulated on page 78. The tabular values are the numerical coefficients of wl .

155. An 18-in. steel I-beam, 60 lb./ft., is continuous over four supports, the lengths of the three spans, beginning at the left, being 25 ft., 40 ft., and 35 ft., respectively. What uniform load per foot run would produce a maximum fiber stress in the beam of 16,000 lb./in.²?

SECTION VII

RESTRAINED, OR BUILT-IN, BEAMS

45. Uniformly loaded beam fixed at both ends. By a restrained, or built-in, beam is meant one which is fixed in direction at certain points of its length, usually the ends—as, for example, beams built into a wall or forming a part of monolithic concrete construction. The simplest form of restrained beam is a cantilever, which can be treated by ordinary methods, as explained in articles 37, 38, and 39.

FIG. 73

Consider first a uniformly loaded beam fixed at both ends, as shown in Fig. 73. Let B, F denote the points of inflection of the elastic curve; that is, the points at which the bending moment is zero. Then the central portion BF may be considered as a simple beam of length $2x$ bearing a total uniform load of amount $2wx$, and each of the ends, AB and FE ,

as a cantilever uniformly loaded and carrying a concentrated load wx at the end, equal to one of the reactions for the portion BF .

If, then, d denotes the deflection of the point F with respect to A or E , assumed to be at the same level, the value of d , computed from the segment AF , is, from (38) and (47), articles 37 and 38,

$$(80) \quad d = \frac{w \left(\frac{l}{2} + x \right)^4}{8 EI} + \frac{wx \left(\frac{l}{2} + x \right)^3}{3 EI},$$

and, computed from the segment FE , is

$$(81) \quad d = \frac{w \left(\frac{l}{2} - x \right)^4}{8 EI} + \frac{wx \left(\frac{l}{2} - x \right)^3}{3 EI}.$$

Equating these two values of the deflection d and solving for x , the result is

$$x = \frac{l}{2\sqrt{3}}.$$

The length of the central portion BF is therefore $2x = \frac{l}{\sqrt{3}}$, and the maximum moment, which occurs at the center C , is

$$(82) \quad M_C = \frac{wl^2}{24}.$$

Similarly, the negative moment at the support A or E is

$$M_A = M_E = wx \left(\frac{l}{2} - x \right) + \frac{w \left(\frac{l}{2} - x \right)^2}{2},$$

and, since $x = \frac{l}{2\sqrt{3}}$, this reduces to

$$(83) \quad M_A = -\frac{wl^2}{12}.$$

The maximum deflection for the central portion BF , considered as a simple beam, is, from (57), article 41,

$$(84) \quad d_{BF} = \frac{5w \left(\frac{l}{\sqrt{3}} \right)^4}{384 EI} = \frac{5wl^4}{384 EI},$$

and for one end, say AB , considered as a cantilever, is, from (38) and (47), articles 37 and 38,

$$(85) \quad d_{AB} = \frac{w \left(\frac{l}{2} - \frac{l}{2\sqrt{3}} \right)^4}{8 EI} + \frac{wl}{2\sqrt{3}} \left(\frac{l}{2} - \frac{l}{2\sqrt{3}} \right)^3 = \frac{4}{9} \frac{wl^4}{384 EI}.$$

Therefore, since the total deflection of the center C below the supports at A or E is the sum of these two, we have

$$(86) \quad D_{\max} = \frac{wl^4}{384 EI}.$$

46. Beam fixed at both ends and bearing concentrated load at center. Following the method of the preceding article, let B and D

FIG. 74

denote the points of inflection of the elastic curve, or positions of zero moment (Fig. 74). Then the equilibrium would not be disturbed if the beam was hinged or jointed at B and D , and it may therefore be considered as a simple beam of length BD suspended from the ends of two cantilevers AB and DE .

Now consider the segment AD and compute

the deflection of D below A . Then, from (44), article 37, the deflection at D due to the load P is

$$(87) \quad d_P = \frac{Pa^2}{EI} \left(\frac{a}{3} + \frac{b}{2} \right),$$

where in the present case $a = \frac{l}{2}$ and $b = x$, and consequently

$$(88) \quad d_P = \frac{Pl^2}{8EI} \left(\frac{l}{3} + x \right).$$

But from (38), article 37, the load $\frac{P}{2}$, acting upward at D , produces a deflection upward of amount

$$(89) \quad d_{\frac{P}{2}} = \frac{\frac{P}{2} \left(\frac{l}{2} + x \right)^3}{3EI}.$$

Consequently the total deflection of D below A is

$$(90) \quad d_{AD} = \frac{Pl^2}{8EI} \left(\frac{l}{3} + x \right) - \frac{P}{6EI} \left(\frac{l}{2} + x \right)^3.$$

Similarly, for the portion DE , the deflection of D below E is

$$(91) \quad d_{DE} = \frac{P \left(\frac{l}{2} - x \right)^3}{3 EI}.$$

Equating these two values of the deflection and solving for x , the result is

$$x = \frac{l}{4},$$

and consequently the length of the central portion BD is

$$2x = \frac{l}{2}.$$

Therefore the moment at the center C and also at each end is of

numerically the same amount, namely, $\frac{P \left(\frac{l}{2} \right)}{4}$, or

$$(92) \quad M = \frac{Pl}{8}.$$

The maximum deflection for the central portion BD is the same as for a simple beam of span $\frac{l}{2}$, namely,

$$(93) \quad d_{BD} = \frac{P \left(\frac{l}{2} \right)^3}{48 EI} = \frac{Pl^3}{384 EI},$$

and for either end AB or DE is the same as for a cantilever of length $\frac{l}{4}$ carrying a load $\frac{P}{2}$ at the end, namely,

$$(94) \quad d_{AB} = \frac{P \left(\frac{l}{4} \right)^3}{3 EI} = \frac{Pl^3}{384 EI}.$$

Therefore the total deflection of the center C below the level of the supports at A and D is the sum of these two, that is,

$$(95) \quad D_{\max} = \frac{Pl^3}{192 EI}.$$

47. Single eccentric load. For a beam fixed at both ends and bearing a single concentrated eccentric load the simplest method of computing the unknown reactions and moments at the sup-

FIG. 75

ports is as follows:

Consider the beam as fixed at one end E only (Fig. 75) and carrying, in addition to the concentrated load P , the shear R_1 at the left support and the restraining moment M_1 at this point. Then, from (50) and (51), article 39, the deflection from the tangent at E due to the moment M_1 is

$$d_1 = -\frac{M_1 l^2}{2 EI}, \quad \tan \phi_1 = -\frac{M_1 l}{EI}.$$

From (38) and (40), article 37, that due to the shear R_1 is

$$d_2 = \frac{R_1 l^3}{3 EI}, \quad \tan \phi_2 = \frac{R_1 l^2}{2 EI};$$

and from (42) and (44), article 37, that due to the load P is

$$d_3 = -\frac{Pl_2^3}{3 EI} - \frac{Pl_1 l_2^2}{2 EI}, \quad \tan \phi_3 = -\frac{Pl_2^2}{2 EI}.$$

Since the total vertical deflection of the point A with respect to the point E is zero, and since the total angular deflection is also zero, these two conditions furnish the equations

$$(96) \quad \begin{cases} -\frac{M_1 l^2}{2 EI} + \frac{R_1 l^3}{3 EI} - \frac{Pl_2^3}{3 EI} - \frac{Pl_1 l_2^2}{2 EI} = 0, \\ -\frac{M_1 l}{EI} + \frac{R_1 l^2}{2 EI} - \frac{Pl_2^2}{2 EI} = 0. \end{cases}$$

From the second equation,

$$M_1 = \frac{R_1 l}{2} - \frac{Pl_2^2}{2 l},$$

and, inserting this value in the first equation, we have

$$(97) \quad R_1 = P \frac{l_2^2(l_2 + 3l_1)}{l^3}.$$

Also, inserting this value of R_1 in the expression for the moment, it is found that

$$(98) \quad M_1 = P \frac{l_1 l_2^2}{l^2}.$$

Similarly, by forming the expressions for the total deflection of the point E with respect to the tangent at A , we obtain the equations

$$(99) \quad \begin{cases} -\frac{M_2 l^2}{2EI} + \frac{R_2 l^3}{3EI} - \frac{Pl_1^3}{3EI} - \frac{Pl_1^2 l_2}{2EI} = 0, \\ -\frac{M_2 l}{EI} + \frac{R_2 l^2}{2EI} - \frac{Pl_1^2}{2EI} = 0; \end{cases}$$

and, solving these equations simultaneously for M_2 and R_2 , as above, the results are

$$(100) \quad R_2 = P \frac{l_1^2(l_1 + 3l_2)}{l^3},$$

$$(101) \quad M_2 = P \frac{l_1^2 l_2}{l^2}.$$

If AC is the longer of the two segments, the maximum deflection will occur somewhere between A and C . Also, since the tangent at this point must be horizontal, the total angular deflection from one end, say A , to the point of maximum deflection is zero. Let x denote the distance of this point of maximum deflection from A . Then, computing the total angular deflection up to this point and equating it to zero, we have

$$-\frac{M_1 x}{EI} + \frac{R_1 x^2}{2EI} = 0;$$

whence

$$x = \frac{2M_1}{R_1}.$$

Inserting the values just obtained for M_1 and R_1 , this becomes

$$(102) \quad x = \frac{2l_1}{l_2 + 3l_1}.$$

The maximum deflection D is then found to be

$$D = -\frac{M_1 x^2}{2EI} + \frac{R_1 x^3}{3EI},$$

and, inserting in this expression the values of M_1 , R_1 , and x , just obtained, it reduces to

$$(103) \quad D_{\max} = \frac{2Pl_1^3l_2^2}{3EI(l_2 + 3l_1)^2}.$$

48. Uniformly loaded beam fixed at one end. In this case (Fig. 76) the deflection of the end A with respect to the fixed end B consists of two parts: that due to the reaction R is

$$d_1 = \frac{Rl^3}{3EI},$$

and that due to the total uniform load wl is

$$d_2 = -\frac{wl^4}{8EI}.$$

Since the ends A and B are assumed to be at the same level, the total deflection of A

FIG. 76

from the tangent at B must be zero; that is,

$$\frac{Rl^3}{3EI} - \frac{wl^4}{8EI} = 0;$$

whence the reaction at the unrestrained end is

$$(104) \quad R = \frac{3wl}{8}.$$

The reaction R' at the restrained end B is therefore

$$(105) \quad R' = wl - R = \frac{5wl}{8}.$$

The maximum moment, which in this case occurs at the fixed end B , is then

$$(106) \quad M = Rl - \frac{wl^2}{2} = -\frac{wl^2}{8}.$$

At the point where the maximum deflection occurs the tangent is horizontal. Let the distance of this point from the fixed end B be denoted by x . Then, from (40), (49), and (51), articles 37, 38, and 39, the condition that the total angular deflection for this length x shall be zero is

$$-\frac{Mx}{EI} + \frac{R'x^2}{2EI} - \frac{wx^3}{6EI} = 0.$$

Inserting in this expression the values of M and R' obtained above, and solving for x , the result is

$$(107) \quad x = \frac{l}{16} (15 - \sqrt{33}) = .578 l.$$

The maximum deflection is then found by finding the total deflection for the length x with respect to the tangent at B . Hence, from (38), (47), and (50), articles 37, 38, and 39,

$$D_{\max} = -\frac{Mx^2}{2EI} + \frac{R'x^3}{3EI} - \frac{wx^4}{8EI},$$

or, inserting the values of M and R' ,

$$(108) \quad D_{\max} = -\frac{wx^2}{48EI} (3l^2 - 10lx + 6x^2).$$

The numerical value of the deflection is most easily found by first calculating the numerical value of x and then substituting in this formula.

49. Beam fixed at one end and bearing concentrated load at center. The deflection of the end A (Fig. 77) with

FIG. 77

respect to the fixed end B in this case consists of two parts: from (38), article 37, that due to the reaction R is

$$d_1 = \frac{Rl^3}{3EI},$$

and from (44), article 37, that due to the load P is

$$d_2 = -\frac{P\left(\frac{l}{2}\right)^3}{3EI} - \frac{P\left(\frac{l}{2}\right)^2 \frac{l}{2}}{2EI}.$$

Since A and B are assumed to be at the same level, the total deflection of the end A with respect to the tangent at B must be zero. Consequently,

$$\frac{Rl^3}{3EI} - \frac{Pl^3}{24EI} - \frac{Pl^3}{16EI} = 0;$$

whence

$$(109) \quad R = \frac{5}{16} P.$$

Therefore

$$(110) \quad R' = P - R = \frac{11}{16} P.$$

The maximum moment, which in this case occurs at B , is then

$$(111) \quad M = Rl - P \frac{l}{2} \\ = -\frac{3}{16} Pl.$$

FIG. 78

The position and amount of the maximum deflection may be found as in the preceding article.

50. Beam fixed at one end and bearing a concentrated eccentric load. The deflection of the end A

with respect to the fixed end B (Fig. 78) also consists of two parts in this case. From (38), article 37, that due to the reaction R is

$$d_1 = \frac{Rl^3}{3EI},$$

and from (44), article 37, that due to the load P is

$$d_2 = -\frac{Pl_2^3}{3EI} - \frac{Pl_2^2 l_1}{2EI}.$$

Since the supports are assumed to be at the same level, the total deflection of A with respect to the tangent at B is zero. Consequently,

$$\frac{Rl^3}{3EI} - \frac{Pl_2^3}{3EI} - \frac{Pl_2^2l_1}{2EI} = 0;$$

whence

$$(112) \quad R = \frac{Pl_2^2}{2l^3}(2l_2 + 3l_1) = \frac{Pl_2^2}{2l^3}(2l + l_1).$$

Also

$$(113) \quad R' = P - R = \frac{Pl_1}{2l^3}[2l(l + l_2) - l_2^2].$$

The moment at the fixed end B is then

$$(114) \quad M_B = Rl - Pl_2 = -\frac{Pl_1l_2}{2l^2}(l + l_1),$$

and the moment at the load C is

$$(115) \quad M_C = Rl_1 = \frac{Pl_1l_2^2}{2l^3}(2l + l_1).$$

The maximum deflection may be found by the method explained in article 48.

APPLICATIONS

156. One end of a beam is built into a wall, and the other end is supported at the same level by a post 12 ft. from the wall. The beam carries a uniform load of 100 lb. per linear foot. Find the position and amount of the maximum moment and also of the maximum deflection.

157. One end of a beam is built into a wall, and the other end rests on a prop 20 ft. from the wall at the same level. The beam bears a concentrated load of 1 ton at a point 8 ft. from the wall. Find the position and amount of the maximum moment and also of the maximum deflection.

158. A cantilever of length l is loaded uniformly. At what point of its length should a prop be placed, supporting the beam at the same level as the fixed end, in order to reduce the bending stress as much as possible, and what proportion of the load is then carried by the prop?

159. A 20-in. steel I-beam weighing 65 lb./ft. is built into a wall at one end and rests on a support 20 ft. from the wall at the other end. A load of 25 tons rests on the beam at a point distant 15 ft. from the wall. Find the reaction of the support and the maximum deflection.

160. A beam of uniform section is built into a wall at one end, projecting 16 ft. from the face of the wall, and rests on a column at 12 ft. from the wall. The beam carries a uniform load of 5 tons per foot run. Find the load on the column.

161. A beam of uniform section is built into walls at both ends, the distance between walls being 25 ft. Two concentrated loads, each of 5 tons, rest on the beam at points 5 ft. from each wall. Find the maximum bending moment in the beam, and also the position of zero bending moment.

162. A continuous beam of two spans, each of 40 ft., carries a uniform load of 1 ton per foot run. Find the reactions of the supports by the method of article 48, and also the maximum moment and maximum deflection.

163. A continuous beam of two equal spans is uniformly loaded. Find the bending moment over the middle support when the three supports are at the same level, and also when the middle support is raised or lowered an amount h .

164. A uniform beam of 20 ft. span is fixed at both ends and carries a load of 4 tons at the center and two loads of 3 tons each at 5 ft. from each end. Find the maximum moment and the position of zero moment.

165. A beam of length $2l$ is supported at the center, one end being anchored down to a fixed abutment and the other end carrying a concentrated load W . Neglecting the weight of the beam, find the deflection of the free end.

SECTION VIII

COLUMNS AND STRUTS

51. Nature of compressive stress. When a prismatic piece of length equal to several times its breadth is subjected to axial compression, it is called a **column**, or **strut**, the word *column* being used to designate a compression member placed vertically and bearing a static load, all other compression members being called struts.

If the axis of a column or strut is not perfectly straight, or if the load is not applied exactly at the centers of gravity of its ends, a bending moment is produced which tends to make the column deflect sideways, or "buckle." The same is true if the material is not perfectly homogeneous, causing certain parts to yield more than others. Such lateral deflection increases the bending moment and consequently increases the tendency to buckle. A compression member is therefore in a different condition of equilibrium from one subjected to tension, for in the latter any deviation of the axis from a straight line tends to be diminished by the stress instead of increased.

The oldest theory of columns is due to Euler, and his formula is still the standard for comparison. Euler's theory, however, is based upon the assumptions that the column is perfectly straight, the material perfectly homogeneous, and the load exactly centered at the ends — assumptions which are never exactly realized. For practical purposes, therefore, it has been found necessary to modify Euler's formula in such a way as to bring it into accord with the results of actual experiments, as explained in the following articles.

52. Euler's theory of long columns. Consider a long column subjected to axial loading, and assume that the column is perfectly straight and homogeneous and that the load is applied exactly at the centers of gravity of its ends. Assume also that the ends of the column are free to turn about their centers of gravity, as would be the case, for example, in a column with round or pivoted ends.

Now suppose that the column is bent sideways by a lateral force, and let P be the axial load which is just sufficient to cause the column to retain this lateral deflection when the lateral force is removed. Let OX and OY be the axes of X and Y respectively (Fig. 79). Then it can be shown that the elastic curve OCX is a sine curve. For simplicity, however, it will be assumed to be a parabola. Since the deflection at any point C is the lever arm of the load P , the moment at C is Py . The moment at any point is therefore P times as great as the deflection at that point, and consequently the moment diagram will also be a parabola (Fig. 80).

FIG. 79

FIG. 80

Now let d denote the maximum deflection, which in this case is at the center. Then the maximum ordinate to the moment diagram is Pd . Therefore, from article 17, the area of one half the diagram is $A = \frac{2}{3} (Pd) \frac{l}{2} = \frac{Pdl}{3}$, and the distance of its centroid from one end is $x_0 = \frac{5}{8} \cdot \frac{l}{2} = \frac{5l}{16}$. Hence, from the general deflection formula, the deflection at the center will be

$$(116) \quad d = \frac{1}{EI} A x_0 = \frac{1}{EI} \left(\frac{Pdl}{3} \right) \frac{5l}{16} = \frac{5 P l^2 d}{48 EI}.$$

Canceling the common factor d and solving for P , the result is

$$(117) \quad P = \frac{48 EI}{5 l^2} = \frac{9.6 EI}{l^2}.$$

If the elastic curve had been assumed to be a sine curve instead of a parabola, the result would have been the well-known equation

$$(118) \quad P = \frac{\pi^2 EI}{l^2} = \frac{9.87 EI}{l^2},$$

which is **Euler's formula for long columns** in its standard form.

Under the load P given by this formula the column is in neutral equilibrium; that is to say, the load P is just sufficient to cause it to retain any lateral deflection which may be given to it. For this reason P is called the *critical load*. If the load is less than this critical value, the column is in stable equilibrium, and any lateral deflection will disappear when its cause is removed. If the load exceeds this critical value, the column is in unstable equilibrium, and the slightest lateral deflection will rapidly increase until rupture occurs.

53. Effect of end support. The above deduction of Euler's formula is based on the assumption that the ends of the column are free to turn, and therefore formula (118) applies only to long columns with round or pivoted ends.

If the ends of a column are rigidly fixed against turning, the elastic curve has two points of inflection, say B and D (Fig. 81). From symmetry, the tangent to the elastic curve at the center C must be parallel to the original position of the axis of the column AE , and therefore the portion AB of the elastic curve must be symmetrical with BC , and CD with DE . Consequently, the points of inflection, B and D , occur at one fourth the length of the column from either end. The critical load for a column with fixed ends is therefore the same as for a column with free ends of half the length; whence, for *fixed ends*, Euler's formula becomes

$$(119) \quad P = \frac{4\pi^2 EI}{l^2}.$$

Columns with flat ends, fixed against lateral movement, are usually regarded as coming under formula (119), the terms *fixed ends* and *flat ends* being used interchangeably.

FIG. 81

If one end of the column is fixed and the other end is free to turn, the elastic curve is approximately represented by the line $BCDE$ in Fig. 81. Therefore the critical load in this case is approximately the same as for a column with both ends free, of length BCD , that is, of length equal to $\frac{2}{3} BE$ or $\frac{2}{3} l$; whence, for a column with *one end fixed and the other free*, Euler's formula becomes

$$(120) \quad P = \frac{9 \pi^2 EI}{4 l^2}, \text{ approximately.}$$

If the lower end is fixed in direction but the upper end is entirely free (that is, if there is no horizontal reaction to prevent it from bending out sideways), it may be regarded as half of a column with round or pin ends and of length $2l$. Consequently, in this case Euler's formula becomes

$$(121) \quad P = \frac{\pi^2 EI}{(2l)^2} \\ = \frac{1}{4} \cdot \frac{\pi^2 EI}{l^2}.$$

The general expression for Euler's formula is then

$$(122) \quad P = k \frac{\pi^2 EI}{l^2},$$

FIG. 82

where the constant k is determined by the way in which the ends of the column are supported. The values of k corresponding to various end conditions are given in Fig. 82.

54. Modification of Euler's formula. It has been found by experiment that Euler's formula applies correctly only to very long columns, and that for short columns or those of medium length it gives a value of P considerably too large.

Very short columns or blocks fail solely by crushing, the tendency to buckle in such cases being practically zero. Therefore, if p denotes the crushing strength of the material and A the area

of a cross section, the breaking load for a very short column is $P = pA$.*

For columns of ordinary length, therefore, the load P must lie somewhere between pA and the value given by Euler's formula. Consequently, to obtain a general formula which shall apply to columns of any length, it is only necessary to express a continuous relation between pA and $\frac{\pi^2 EI}{l^2}$. Such a relation is furnished by the equation

$$(123) \quad P = \frac{pA}{1 + pA \left(\frac{l^2}{\pi^2 EI} \right)}.$$

For when $l=0$, $P=pA$, and when l becomes very large, P approaches the value $\frac{\pi^2 EI}{l^2}$. Moreover, for intermediate values of l this formula gives values of P considerably less than those given by Euler's formula, thus agreeing more closely with experiment.

55. Rankine's formula. Although the above modification of Euler's formula is an improvement on the latter, it does not yet agree closely enough with experiment to be entirely satisfactory. The reason for the discrepancy between the results given by this formula and those obtained from actual tests is that the assumptions upon which the formula is based, namely, that the column is perfectly straight, the material perfectly homogeneous, and the load applied exactly at the centers of gravity of the ends, are never actually realized in practice.

To obtain a more accurate formula, two empirical constants will be introduced into equation (123). Thus, for fixed ends, let

$$(124) \quad P = \frac{gA}{1 + f \left(\frac{l}{t} \right)^2},$$

where f and g are arbitrary constants to be determined by experiment, and t is the least radius of gyration of a cross section of the column. This formula has been obtained in different ways by

* As Euler's formula is based upon the assumption that the column is of sufficient length to buckle sideways, it is evident a priori that it cannot be applied to very short columns, in which this tendency is practically zero. Thus, in formula (118), as l approaches zero P approaches infinity, which of course is inadmissible.

Gordon, Rankine, Navier, and Schwarz.* Among German writers it is known as Schwarz's formula, but in English and American textbooks it is called **Rankine's formula**.

For $l' = 0$, $P = gA$, and, since short blocks fail by crushing, g is therefore the ultimate compressive strength of the material.

For different methods of end support Rankine's formula takes the following forms:

(125)	$\frac{P}{A} = \frac{g}{1 + f \left(\frac{l}{t} \right)^2}$	Flat ends (fixed in direction)
(126)	$\frac{P}{A} = \frac{g}{1 + 4f \left(\frac{l}{t} \right)^2}$	Round ends (direction not fixed)
(127)	$\frac{P}{A} = \frac{g}{1 + 2f \left(\frac{l}{t} \right)^2}$	Hinged ends (position fixed, but not direction)
(128)	$\frac{P}{A} = \frac{g}{1 + 1.78f \left(\frac{l}{t} \right)^2}$	One end flat and the other round

56. Values of the empirical constants in Rankine's formula. The values of the empirical constants, f and g , in Rankine's formula have been experimentally determined by Hodgkinson and Christie, with the following results:

For hard steel,	$g = 69,000 \text{ lb./in.}^2,$	$f = \frac{1}{20000}.$
For mild steel,	$g = 48,000 \text{ lb./in.}^2,$	$f = \frac{1}{30000}.$
For wrought iron,	$g = 36,000 \text{ lb./in.}^2,$	$f = \frac{1}{36000}.$
For cast iron,	$g = 80,000 \text{ lb./in.}^2,$	$f = \frac{1}{6400}.$
For timber,	$g = 7,200 \text{ lb./in.}^2,$	$f = \frac{1}{3000}.$

* Rankine's formula can be derived independently of Euler's formula either by assuming that the elastic curve assumed by the center line of the column is a sinusoid or by assuming that the maximum lateral deflection D at the center of the column is given by the expression $D = \mu \frac{l^2}{b}$, where l is the length of the column, b its least width, and μ an empirical constant.

These constants were determined by experiments upon columns for which $20 < \frac{l}{t} < 200$, and therefore can only be relied upon to furnish accurate results when the dimensions of the column lie within these limits.

As a factor of safety to be used in applying the formula, Rankine recommended 10 for timber, 4 for iron under dead load, and 5 for iron under moving load.

57. Johnson's parabolic formula. From the manner in which equation (123) was obtained and afterwards modified by the introduction of the empirical constants f and g , it is clear that Rankine's formula satisfies the requirements for very long or very short columns, while for those of intermediate length it gives the average values of experimental results. A simple formula which fulfills these same requirements has been given by Professor J. B. Johnson, and is called **Johnson's parabolic formula**.

If equation (124) is written

$$\frac{P}{A} = p = \frac{g}{1 + f\left(\frac{l}{t}\right)^2},$$

and then y is written for p , and x for $\frac{l}{t}$, Rankine's formula becomes

$$y = \frac{g}{1 + fx^2}.$$

For this cubic equation Johnson substituted the parabola

$$(129) \quad y = \delta - \epsilon x^2,$$

in which x and y have the same meaning as above, and δ and ϵ are empirical constants. The constants δ and ϵ are then so chosen that the vertex of this parabola is at the elastic limit of the material on the axis of loads (or Y -axis), and the parabola is also tangent to Euler's curve. In this way the formula is made to satisfy the theoretical requirements for very long or very short columns, and for those of intermediate length it is found to agree closely with experiment.

For different materials and methods of end support Johnson's parabolic formulas, obtained as above, are as follows:

KIND OF COLUMN	FORMULA	LIMIT FOR USE
Mild steel		
Hinged ends	$\frac{P}{A} = 42,000 - .97 \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 150$
Flat ends	$\frac{P}{A} = 42,000 - .62 \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 190$
Wrought iron		
Hinged ends	$\frac{P}{A} = 34,000 - .67 \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 170$
Flat ends	$\frac{P}{A} = 34,000 - .43 \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 210$
Cast iron		
Round ends	$\frac{P}{A} = 60,000 - \frac{25}{4} \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 70$
Flat ends	$\frac{P}{A} = 60,000 - \frac{9}{4} \left(\frac{l}{t} \right)^2$	$\frac{l}{t} \leq 120$
Timber (flat ends)		
White pine	$\frac{P}{A} = 2,500 - .6 \left(\frac{l}{t'} \right)^2*$	$\frac{l}{t'} \leq 60$
Short-leaf yellow pine	$\frac{P}{A} = 3,300 - .7 \left(\frac{l}{t'} \right)^2$	$\frac{l}{t'} \leq 60$
Long-leaf yellow pine	$\frac{P}{A} = 4,000 - .8 \left(\frac{l}{t'} \right)^2$	$\frac{l}{t'} \leq 60$
White oak	$\frac{P}{A} = 3,500 - .8 \left(\frac{l}{t'} \right)^2$	$\frac{l}{t'} \leq 60$

The limit for use in each case is the value of $x \left(= \frac{l}{t} \right)$ at the point where Johnson's parabola becomes tangent to Euler's curve. For greater values of $\frac{l}{t}$ Euler's formula should therefore be used.

A graphical representation of the relation between Euler's formula, Rankine's formula, J. B. Johnson's parabolic formula, and T. H. Johnson's straight-line formula (considered in the next article) is given in Fig. 83 for the case of a wrought-iron column with hinged ends.†

* In the formulas for timber t' is the least lateral dimension of the column.

† For a more extensive comparison of these formulas see Johnson's *Framed Structures*, 8th ed., 1905, pp. 159-171; also *Trans. Amer. Soc. Civ. Eng.*, Vol. XV, pp. 518-536.

58. Johnson's straight-line formula. By means of an exhaustive study of experimental data on columns Mr. Thomas H. Johnson has shown that for columns of moderate length a straight line can be made to fit the plotted results of column tests as exactly as a curve. He has therefore proposed the formula

$$(130) \quad \frac{P}{A} = \nu - \sigma \frac{l}{t},$$

FIG. 83. Wrought-Iron Column (Pin Ends)

1, Euler's formula; 2, T. H. Johnson's straight-line formula; 3, J. B. Johnson's parabolic formula; 4, Rankine's formula

or, in the notation of the preceding article,

$$(131) \quad y = \nu - \sigma x,$$

in which ν and σ are empirical constants, this being the equation of a straight line tangent to Euler's curve. This formula has the merit of great simplicity, the only objection to it being that for short columns it gives a value of P in excess of the actual breaking load. The relation of this formula to those which precede is shown in Fig. 83.

The constants ν and σ in formula (130) are connected by the relation

$$(132) \quad \sigma = \frac{\nu}{3} \sqrt{\frac{4\nu}{3n\pi^2 E}},$$

where for fixed ends $n = 1$, for free ends $n = 4$, and for one end fixed and the other free $n = 1.78$.

The following table gives the special forms assumed by Johnson's straight-line formula for various materials and methods of end support: *

KIND OF COLUMN	FORMULA	LIMIT FOR USE
Hard steel		
Flat ends	$\frac{P}{A} = 80,000 - 337 \frac{l}{t}$	$\frac{l}{t} \leq 158.0$
Hinged ends	$\frac{P}{A} = 80,000 - 414 \frac{l}{t}$	$\frac{l}{t} \leq 129.0$
Round ends	$\frac{P}{A} = 80,000 - 534 \frac{l}{t}$	$\frac{l}{t} \leq 99.9$
Mild steel		
Flat ends	$\frac{P}{A} = 52,500 - 179 \frac{l}{t}$	$\frac{l}{t} \leq 195.1$
Hinged ends	$\frac{P}{A} = 52,500 - 220 \frac{l}{t}$	$\frac{l}{t} \leq 159.3$
Round ends	$\frac{P}{A} = 52,500 - 284 \frac{l}{t}$	$\frac{l}{t} \leq 123.3$
Wrought iron		
Flat ends	$\frac{P}{A} = 42,000 - 128 \frac{l}{t}$	$\frac{l}{t} \leq 218.1$
Hinged ends	$\frac{P}{A} = 42,000 - 157 \frac{l}{t}$	$\frac{l}{t} \leq 178.1$
Round ends	$\frac{P}{A} = 42,000 - 203 \frac{l}{t}$	$\frac{l}{t} \leq 138.0$
Cast iron		
Flat ends	$\frac{P}{A} = 80,000 - 438 \frac{l}{t}$	$\frac{l}{t} \leq 121.6$
Hinged ends	$\frac{P}{A} = 80,000 - 537 \frac{l}{t}$	$\frac{l}{t} \leq 99.3$
Round ends	$\frac{P}{A} = 80,000 - 693 \frac{l}{t}$	$\frac{l}{t} \leq 77.0$
Oak		
Flat ends	$\frac{P}{A} = 5,400 - 28 \frac{l}{t}$	$\frac{l}{t} \leq 128.1$

* *Trans. Amer. Soc. Civ. Eng.*, 1886, p. 530.

The limit for use in each case is the value of $x \left(= \frac{l}{t} \right)$ for the point at which Johnson's straight line becomes tangent to Euler's curve.

59. Cooper's modification of Johnson's straight-line formula. In his standard bridge specifications Theodore Cooper has adopted Johnson's straight-line formulas, modifying them by the introduction of a factor of safety. Thus, for medium steel, Cooper specifies that the following formulas shall be used in calculating the *safe* load. For chords

$$(133) \quad \begin{cases} \frac{P}{A} = 8,000 - 30 \frac{l}{t} \text{ for live-load stresses,} \\ \frac{P}{A} = 16,000 - 60 \frac{l}{t} \text{ for dead-load stresses.} \end{cases}$$

For posts

$$(134) \quad \begin{cases} \frac{P}{A} = 7,000 - 40 \frac{l}{t} \text{ for live-load stresses,} \\ \frac{P}{A} = 14,000 - 80 \frac{l}{t} \text{ for dead-load stresses,} \\ \frac{P}{A} = 10,000 - 60 \frac{l}{t} \text{ for wind stresses.} \end{cases}$$

For lateral struts

$$(135) \quad \frac{P}{A} = 9,000 - 50 \frac{l}{t} \text{ for initial stresses.}$$

By *initial stress* in the last formula is meant the stress due to the adjustment of the bridge members during construction.

60. Eccentrically loaded columns. In a column that carries an eccentric load (for example, a column carrying a load on a bracket or the post of a crane) there is a definite amount of bending stress due to the eccentricity of the load in addition to the column stress. As the nature of column stress is such that it is impossible to determine its amount, the simplest method of handling a problem of this kind is to determine its relative security against failure as a column and failure by bending. That is to say, first determine its factor of safety against failure as a column under the given column load. Then consider it as a beam and find the equivalent bending

moment which would give the same factor of safety. Finally, combine this equivalent bending moment with that due to the eccentric load, and calculate the unit stress from the ordinary beam formulas.

To illustrate the method, suppose that a column 18 ft. long is composed of two 12-in. I-beams each weighing 40 lb./ft., and carries a column load of 20 tons at its upper end and also an eccentric load of 10 tons with eccentricity 2 ft., as shown in Fig. 84. Assuming that the column has flat ends, and using Johnson's straight-line formula, $P = A \left(52,500 - 179 \frac{l}{t} \right)$, the factor of safety against column failure is

FIG. 84

$$\frac{A \left(52,500 - 179 \frac{l}{t} \right)}{60,000} = \frac{2(11.76)(52,500 - 179(47.3))}{60,000} = 17.3.$$

Now consider the column as a beam and find the equivalent central load K corresponding to the factor of safety just found, namely, 17.3. The maximum moment in a simple beam bearing a concentrated load K at the center is $M = \frac{Kl}{4}$. Hence, from the beam formula $M = \frac{pI}{e}$ we have $\frac{Kl}{4} = \frac{pI}{e}$; whence $K = \frac{4pI}{le}$. Assuming the ultimate strength of the material to be 60,000 lb./in.², we have

$$p = \frac{60,000}{17.3} \text{ lb./in.}^2, \quad I = 2(245.9) \text{ in.}^4,$$

$$l = 216 \text{ in.}, \quad e = 6 \text{ in.},$$

and, inserting these values, the equivalent load K is found to be

$$K = \frac{4 \times 60,000 \times 491.8}{17.3 \times 216 \times 6} = 5220 \text{ lb.}$$

Now the eccentric load P_2 , acting parallel to the axis of the column, produces the same bending effect as a horizontal reaction H at either end, where $Hl = P_2d$. The bending moment at the center, due to a reaction H perpendicular to the axis of the beam, is, however, $\frac{Hl}{2}$.

Hence the total equivalent moment at the center now becomes

$$M = \frac{Kl}{4} + \frac{Hl}{2} = \frac{Kl}{4} + \frac{P_2 d}{2} = \frac{5220 \times 216}{4} + \frac{20,000 \times 24}{2} \\ = 521,880 \text{ in.-lb.}$$

Consequently, the maximum unit stress in the member becomes

$$p = \frac{M}{S} = \frac{521,880}{81.96} = 6367 \text{ lb./in.}^2,$$

which corresponds to a factor of safety of about 9.

If this factor of safety is larger than desired, assume a smaller I-beam and repeat the calculations.

A method substantially equivalent to the above is to assume that the stress in a column is represented by the empirical factor in the column formula used. Thus, for a short block the actual compressive stress p is given by the relation $P = pA$, whereas in the column formula used above, namely, $P = A \left(52,500 - 179 \frac{l}{t} \right)$, the stress p is replaced by the empirical factor $52,500 - 179 \frac{l}{t}$. Consequently, the fraction

$$\frac{52,500 - 179 \frac{l}{t}}{u_c},$$

where u_c denotes the ultimate compressive strength of the material, represents the reduction in strength of the member due to its slenderness and method of loading; or, what amounts to the same thing, the equivalent unit stress in the column is

$$(136) \quad p_e = \frac{P}{A} \left(\frac{u_c}{52,500 - 179 \frac{l}{t}} \right).$$

Applying this method to the numerical problem given above, we have

$$A = 23.52,$$

$$\frac{l}{t} = 47.3,$$

and

$$\frac{u_c}{52,500 - 179 \frac{l}{t}} = \frac{60,000}{52,500 - 179 \times 47.3} = 1.36.$$

Hence the equivalent stress in the column is

$$p_e = \frac{30 \times 2000}{23.52} \times 1.36 = 3470 \text{ lb./in.}^2$$

Also, the bending stress, produced by the eccentricity of the load, is

$$p = \frac{P_2 d}{S} = \frac{240,000}{81.96} = 2928 \text{ lb./in.}^2$$

Consequently, by this method, the total stress in the column is found to be

$$3470 + 2928 = 6398 \text{ lb./in.}^2$$

If a formula of the Rankine-Gordon type is used, namely,

$$\frac{P}{A} = \frac{g}{1 + f \left(\frac{l}{t} \right)^2},$$

the equivalent stress p_e in the column, due to the given load P , is

$$p_e = \frac{P}{A} \left[\frac{u_c \left(1 + f \left(\frac{l}{t} \right)^2 \right)}{g} \right],$$

where u_c denotes the ultimate compressive strength of the material, as above.

APPLICATIONS

166. A solid, round, cast-iron column with flat ends is 15 ft. long and 6 in. in diameter. What load may be expected to cause rupture?

167. A square wooden post 12 ft. long is required to support a load of 15 tons. With a factor of safety of 10, what must be the size of the post?

168. Two 8-in. steel I-beams, weighing 25.25 lb./ft., are joined by latticework to form a column 25 ft. long. How far apart must the beams be placed, center to center, in order that the column shall be of equal strength to resist buckling in either axial plane?

169. Four medium steel angles, $5 \times 3 \times \frac{3}{8}$ in., have their 3-in. legs riveted to a $\frac{3}{4}$ -in. plate so as to form an I-shaped built column. How wide must the plate be in order that the column shall be of equal strength to resist buckling in either axial plane?

170. A hollow wrought-iron column with flat ends is 20 ft. long, 7 in. internal diameter, and 10 in. external diameter. Calculate its ultimate strength by Rankine's and Johnson's formulas and compare the results.

171. Compute the ultimate strength of the built column in problem 168 by Rankine's and by Johnson's formulas and compare the results.

172. Compute the ultimate strength of the column in problem 169 by Rankine's and by Johnson's straight-line formulas and compare the results.

173. A column 18 ft. long is formed by joining the legs of two 10-in. steel channels, weighing 30 lb./ft., by two plates each 10 in. wide and $\frac{1}{2}$ in. thick, as shown in Fig. 85. Find the safe load for this column by Johnson's straight-line formula, using a factor of safety of 4.

Fig. 85

174. A wrought-iron pipe 10 ft. long, and of internal and external diameter 3 in. and 4 in. respectively, bears a load of 7 tons. What is the factor of safety?

175. What must be the size of a square steel strut 8 ft. long, to transmit a load of 5 tons with safety?

176. Design a column 16 ft. long to be formed of two channels joined by two plates and to support a load of 20 tons with safety.

177. Using Cooper's formula for live load, design the inclined end post of a bridge which is 25 ft. long and bears a load of 30 tons, the end post to be composed of four angles, a top plate, and two side plates.

178. A strut 16 ft. long, fixed rigidly at both ends, is needed to support a load of 80,000 lb. It is to be composed of two pairs of angles united with a single line of $\frac{1}{4}$ -in. lattice bars along the central plane. Determine the size of the angles for a factor of safety of 5. (Note that the angles must be spread $\frac{1}{2}$ in. to admit the latticing.)

179. For short posts or struts, such as are ordinarily used in building construction, it is customary to figure the safe load as $12,000 \text{ lb./in.}^2$ of cross-section area for lengths up to 90 times the radius of gyration; that is, for $\frac{l}{r} \leq 90$. To what factor of safety does this correspond, by Johnson's straight-line formula?

180. The posts used to support a girder in a building are 8 in. $\times$ 8 in. timbers 8 ft. long. Find the diameter of a solid cast-iron column of equal strength.

If a wrought-iron pipe 4 in. in external diameter is used, what must be its thickness to be equally safe?

181. At what ratio of diameter to length would a round mild-steel strut have the same tendency to crush as to buckle?

182. A load of 100 tons is carried jointly by three cast-iron columns 20 ft. long. What saving in material will be effected by using a single column instead of three, the factor of safety to be 15 in both cases?

183. Determine the proper size of a hard-steel piston rod 48 in. long for a piston 18 in. in diameter and a steam pressure of 80 lb./in.² Consult table for proper factor of safety.

184. The side rod of a locomotive is 9 ft. long between centers, 4 in. deep, and 2 in. wide. The estimated thrust in the rod is 12 tons, and the transverse inertia and gravity load 20 lb. per inch of length. Determine the factor of safety.

185. The vertical post of a crane (Fig. 86) is to be made of a single I-beam. The post is pivoted at both ends so as to revolve about its axis. Find the size of I-beam required for factor of safety of 4 and for dimensions and loading, as shown.

Fig. 86

SECTION IX

TORSION

61. Maximum stress in circular shafts. When a uniform circular shaft, such as is shown in Fig. 87, is twisted by the application of moments of opposite signs to its ends, every straight line AB parallel to its axis is deformed into part of a helix, or screw thread, AC . The strain in this case is one of pure shear and is called **torsion**.

FIG. 87

The angle ϕ is called the **angle of shear** and is proportional to the radius BD of the shaft. The angle θ is called the **angle of twist** and is proportional to the length AB of the shaft.

Consider a section of length Δx cut from a circular shaft by planes perpendicular to its axis (Fig. 87). Let $\Delta\theta$ denote the angle of twist for this section. Then, since the angle of twist is proportional to the length of the shaft, $\Delta\theta : \theta = \Delta x : l$; whence

$$\Delta\theta = \theta \frac{\Delta x}{l}.$$

Also, if ϕ and $\Delta\theta$ are expressed in circular measure,

$$BC = \phi \cdot AB = \phi \Delta x, \quad \text{and} \quad BC = \Delta\theta \cdot BD = r \Delta\theta.$$

Therefore $\phi = \frac{r \Delta\theta}{\Delta x} = r \frac{\theta}{l}$. From Hooke's law, $\frac{q}{\phi} = G$. Hence

$$(137) \quad q = G\phi = \frac{Gr\theta}{l}.$$

Therefore q is proportional to r ; that is to say, the unit shear is proportional to its distance from the center, being zero at the center and attaining its maximum value at the circumference.

If q' denotes the intensity of the shear at the circumference, and a denotes the radius of the shaft, then the shear q at a distance r from the center is given by the formula

$$q = \frac{q'r}{a}.$$

Now if q denotes the intensity of the shear on any element of area ΔA , the total force acting on this element is $q\Delta A$, and its moment with respect to the center is $q\Delta Ar$. Therefore the total internal moment of resistance is $\sum q\Delta Ar$, where the summation extends over the entire cross section; and since this must be equal to the external twisting moment M_t , we have

$$M_t = \sum q\Delta Ar.$$

Inserting for q its value in terms of the radius, $q = \frac{q'r}{a}$, this becomes

$$M_t = \frac{q'}{a} \sum r^2 \Delta A,$$

or, since by definition $\sum r^2 \Delta A = I_p$, the polar moment of inertia of the cross section,

$$M_t = \frac{q' I_p}{a}.$$

For a solid circular shaft of diameter D , $I_p = \frac{\pi D^4}{32}$ and $a = \frac{D}{2}$; consequently,

$$(138) \quad q' = \frac{M_t a}{I_p} = \frac{16 M_t}{\pi D^3}.$$

For a hollow circular shaft of external diameter D and internal diameter d , $I_p = \frac{\pi}{32} (D^4 - d^4)$ and $a = \frac{D}{2}$; hence

$$(139) \quad q' = \frac{8 M_t D}{\pi (D^4 - d^4)}.$$

62. Angle of twist in circular shafts. From equation (137),

$$\theta = \frac{ql}{Gr} = \frac{q'l}{Ga}.$$

Therefore, for a solid circular shaft, from equation (138),

$$(140) \quad \theta = \frac{32 M_t l}{\pi D^4 G},$$

and for a hollow circular shaft, from equation (139),

$$(141) \quad \theta = \frac{32 M_t l}{\pi G (D^4 - d^4)}.$$

If M_t is known and θ can be measured, equations (140) and (141) can be used for determining G . If G is known and θ measured, these equations can be used for finding M_t ; in this way the horse power can be determined from the angle of twist.

63. Power transmitted by circular shafts. Let H denote the number of horse power being transmitted by a circular shaft, n its speed in revolutions per minute (R.P.M.), and M_t the torque, or twisting moment, acting on it, expressed in inch-pounds. Then, since the angular displacement of M_t in one minute is $2\pi n$, the work done by the torque in one minute is $2\pi n M_t$. Also, since one horse power = 33,000 ft.-lb./min. = 396,000 in.-lb./min., the total work done by the shaft in one minute is 396,000 H . Therefore

$$2\pi n M_t = 396,000 H;$$

whence

$$(142) \quad M_t = \frac{396,000 H}{2\pi n} = 63,030 \frac{H}{n} \text{ in.-lb.}$$

Therefore, if it is required to find the diameter D of a solid circular shaft which shall transmit a given horse power H with safety, then, from equation (138),

$$q' = \frac{16 M_t}{\pi D^3} = \frac{321,000 H}{n D^3};$$

whence

$$(143) \quad D = 68.5 \sqrt[3]{\frac{H}{n q'}}.$$

As safe values for the maximum unit shear q' , Ewing recommends 9000 lb./in.² for wrought iron, 13,500 lb./in.² for steel, and 4500 lb./in.² for cast iron. Inserting these values of q' in formula (143), it becomes

$$(144) \quad D = \mu \sqrt[3]{\frac{H}{n}},$$

where for steel $\mu = 2.88$, for wrought iron $\mu = 3.29$, and for cast iron $\mu = 4.15$.

Expressed in kilowatts instead of horse power, this formula becomes

$$(145) \quad D = \rho \sqrt[3]{\frac{KW}{n}},$$

where for steel $\rho = 3.175$, for wrought iron $\rho = 3.627$, and for cast iron $\rho = 4.576$.

64. Combined bending and torsion. In many cases shafts are subjected to combined bending and torsion, as, for instance, when a shaft transmits power by means of one or more cranks or pulleys. In this case the bending moment M_b at any point of the shaft produces a normal stress p in accordance with equation (33), article 34, that is,

$$(146) \quad p = \frac{M_b c}{I} = \frac{32 M_b}{\pi D^3},$$

and the torque M_t produces a shearing stress q given by (138), article 61, namely,

$$(147) \quad q = \frac{M_t r}{I_p} = \frac{16 M_t}{\pi D^3}.$$

In more advanced works on the strength of materials, however, it is shown that the maximum and minimum normal and shearing stresses, resulting from any such combination as the above, are given by the relations *

$$(148) \quad p_{\max \atop \min} = \frac{p}{2} \pm \sqrt{4 q^2 + p^2},$$

$$(149) \quad q_{\max \atop \min} = \pm \sqrt{4 q^2 + p^2}.$$

Therefore, inserting in these expressions the values of p and q as given above, the maximum and minimum normal and shearing stresses in terms of the bending and twisting moments are found to be

$$(150) \quad p_{\max \atop \min} = \frac{16}{\pi D^3} (M_b \pm \sqrt{M_b^2 + M_t^2}) \quad (\text{called Rankine's formula}),$$

$$(151) \quad q_{\max \atop \min} = \pm \frac{16}{\pi D^3} \sqrt{M_b^2 + M_t^2} \quad (\text{called Guest's formula}).$$

* Slocum and Hancock, *Strength of Materials*, Revised Edition, pp. 24-25.

Note that Rankine's formula gives the principal normal stresses, that is, tension or compression, whereas Guest's formula gives shear. Since the ultimate strength in tension or compression is usually different from that in shear,* in designing circular shafts carrying combined stress both formulas should be tried with the same working stress (or factor of safety), and the one used which gives the larger dimensions.

65. Resilience of circular shafts. In article 7 the resilience of a body was defined as the internal work of deformation. For a solid circular shaft this internal work is

$$W = \frac{1}{2} M_t \theta,$$

where M_t is the external twisting moment and θ is the angle of twist.

From equation (137), $\theta = \frac{ql}{Gr} = \frac{q'l}{Ga},$

and from equation (138), $M_t = \frac{\pi D^3 q'}{16}.$

Therefore the total resilience of the shaft is

$$(152) \quad W = \frac{1}{2} M_t \theta = \frac{\pi D^3 q'^2 l}{16 G},$$

and consequently the mean resilience per unit of volume is

$$(153) \quad W_1 = \frac{W}{V} = \frac{q'^2}{4 G}.$$

66. Non-circular shafts. The above investigation of the distribution and intensity of torsional stress applies only to shafts of circular section. For other forms of cross section the results are entirely different, each form having its own peculiar distribution of stress.

For any form of cross section whatever, the stress at the boundary must be tangential, for if the stress is not tangential, it can be resolved into two components, one tangential and the other normal to the boundary; but a normal component would necessitate forces parallel to the axis of the shaft, which are excluded by hypothesis.

Since the stress at the boundary must be tangential, the circular section is the only one for which the stress is perpendicular to a radius vector. Therefore the circular section is the only one to

* The shearing strength of ductile materials, both at the elastic limit and at the ultimate stress, is about four fifths of their tensile strength at these points.

which the above development applies, and consequently is the only form of cross section for which Bernoulli's assumption holds true. That is to say, *the circular section is the only form of cross section which remains plane under a torsional strain.*

The subject of the distribution of stress in non-circular shafts has been investigated by St. Venant, and the results of his investigations are summarized below (articles 67-70).

67. Elliptical shaft. For a shaft the cross section of which is an ellipse of semi-axes a and b , the maximum stress occurs at the ends of the *minor* axis instead of at the ends of the major axis, as might be expected. The unit stress at the ends of the minor axis is given by the formula

$$(154) \quad q_{\max} = \frac{2 M_t}{\pi a b^2},$$

and the angle of twist *per unit of length* is

$$(155) \quad \theta_1 = \frac{M_t(a^2 + b^2)}{\pi a^3 b^3 G}.$$

The total angle of twist for an elliptical shaft of length l is therefore

$$(156) \quad \theta = \theta_1 l = \frac{M_t(a^2 + b^2)l}{\pi a^3 b^3 G}.$$

68. Rectangular and square shafts. For a shaft of rectangular cross section the maximum stress occurs at the centers of the longer sides, its value at these points being

$$(157) \quad q_{\max} = \frac{6 M_t}{h b \sqrt{h^2 + b^2}} \left(.68 + .45 \frac{h}{b} \right),$$

in which h is the longer and b the shorter side of the rectangle. The angle of twist per unit of length is, in this case,

$$(158) \quad \theta_1 = 3.57 \frac{M_t(h^2 + b^2)}{G h^3 b^3}.$$

For a square shaft of side b these formulas become

$$(159) \quad q_{\max} = 4.8 \frac{M_t}{b^3}$$

and

$$(160) \quad \theta_1 = 7.14 \frac{M_t}{G b^4}.$$

The value of q for a square shaft found from this equation is about 15 per cent greater than if the formula $q = \frac{Mr}{I_p}$ were used, and the torsional rigidity is about .88 of the torsional rigidity of a circular shaft of equal sectional area.

69. Triangular shafts. For a shaft whose cross section is an equilateral triangle of side c ,

$$(161) \quad q_{\max} = 20 \frac{M_t}{c^3},$$

and the angle of twist per unit of length is

$$(162) \quad \theta_1 = \frac{M_t}{.6 GI_p}.$$

The torsional rigidity of a triangular shaft is therefore .73 of the torsional rigidity of a circular shaft of equal sectional area.

70. Angle of twist for shafts in general. The formula for the angle of twist per unit of length for circular and elliptical shafts can be written

$$(163) \quad \theta_1 = \frac{4 \pi^2 M_t}{G} \cdot \frac{I_p}{A^4},$$

in which I_p is the polar moment of inertia of a cross section about its center, and A is the area of the cross section. This formula is rigorously true for circular and elliptical shafts, and St. Venant has shown that it is approximately true whatever the form of cross section.

APPLICATIONS

186. A steel wire 20 in. long and .182 in. in diameter is twisted by a moment of 20 in.-lb. The angle of twist is then measured and found to be $\theta = 18^\circ 31'$. What is the value of G determined from this experiment?

Solution. From equation (140), article 62,

$$G = \frac{32 M_t l}{\pi D^4 \theta},$$

where, in the present case, using pound and inch units, $M_t = 20$, $l = 20$, $D = .182$, and $\theta = 18^\circ 31' = .3232$ radians. Substituting these numerical values, the result is

$$G = 11,490,000 \text{ lb./in.}^2$$

187. A steel shaft 5 in. in diameter is driven by a crank of 12-in. throw, the maximum thrust on the crank being 10 tons. If the outer edge of the shaft-bearing is 11 in. from the center of the crank pin, what is the stress in the shaft at this point?

Solution. Referring to Fig. 88, the dimensions in the present case are

$$d_1 = 12 \text{ in.}, \quad d_2 = 11 \text{ in.}$$

Consequently,

$$M_t = 10 \cdot 2000 \cdot 12 = 240,000 \text{ in.-lb.}$$

$$M_b = 10 \cdot 2000 \cdot 11 = 220,000 \text{ in.-lb.}$$

Therefore, from equation (150), article 64,

$$\begin{aligned} p_{\max} &= \frac{16}{\pi 5^3} (220,000 + \sqrt{220,000^2 + 240,000^2}) \\ &= 22,200 \text{ lb./in.}^2, \end{aligned}$$

and similarly, from equation (151),

$$q_{\max} = \frac{16}{\pi 5^3} (\sqrt{220,000^2 + 240,000^2}) = 13,275 \text{ lb./in.}^2$$

FIG. 88

188. If P and Q denote the unit stresses at the elastic limits of a material in tension and shear respectively, show that when $\frac{P}{Q} < 1$ the material will fail in tension, whereas when $\frac{P}{Q} > 1$ it will fail in shear, when subjected to combined bending and torsion, irrespective of the relative values of the bending and twisting moments.

Solution. Combining Rankine's and Guest's formulas, we have

$$p' - q' = \frac{16 M_b}{\pi d^3}.$$

Consequently, if the bending moment is zero, $p' = q'$, or $\frac{p'}{q'} = 1$, whereas if it is not zero, $p' > q'$. Similarly, if the twisting moment is zero, $\frac{p'}{q'} = 2$.

Now let F_t and F_s denote the factors of safety in tension and shear respectively. Then

$$\frac{F_t}{F_s} = \frac{\frac{p'}{q'}}{\frac{p'}{q'}} = \frac{Pq'}{Qp'}.$$

Since $p' \equiv q'$, the fraction $\frac{q'}{p'} \equiv 1$. Consequently, if $\frac{P}{Q} < 1$ also, then $\frac{F_t}{F_s} < 1$; that is, $F_t < F_s$, and the material is weaker in tension than in shear. The second part of the theorem is proved in a similar manner.

For a complete discussion of this question see article by A. L. Jenkins, *Engineering* (London, November 12, 1909), pp. 637-639.

189. Three pulleys of radii 8, 4, and 6 in. respectively are keyed on a shaft as shown in Fig. 89. Pulley No. 1 is the driving pulley and transmits 30 H.P. to the shaft, of which amount 10 H.P. is taken off from pulley No. 2 and the remaining 20 H.P. from pulley No. 3. The speed is 50 R.P.M., the belts are all parallel, and the tension in the slack side of each belt is assumed to be one half the tension in the tight side. Find the required size of the shaft for a working stress of 12,000 lb./in.² in tension and 9000 lb./in.² in shear.

Solution. The first step is to find the tensions in the belts. Since power is the rate of doing work, and 1 H.P. = 550 ft.-lb./sec., the formula for power may be written

$$(164) \quad \text{Horse power} = \frac{Fv}{550},$$

FIG. 89

where F denotes the effective force, or difference in tension in the two sides of the belt, expressed in pounds, and v is the belt speed in ft./sec. Hence, for the pulley of 8 in. radius transmitting 30 H.P., we have

$$30 = \frac{F 2 \pi \cdot \frac{3}{4} \cdot 50}{60 \cdot 550};$$

whence $F = 4730$ lb. Since by assumption the tension on the tight side of the belt is twice that on the slack side, their values are

Tension on tight side = 9460 lb.

Tension on slack side = 4730 lb.

The belt tensions for the other pulleys are calculated in a similar manner, the results being indicated on Fig. 89.

Considering the shaft as a beam, the load at each pulley is equal to the sum of the belt tensions for that pulley, as shown in Fig. 90. The reactions of the bearings and the bending-moment diagram are next obtained, the results being given in Fig. 90.

The maximum bending and twisting moments thus occur at pulley No. 1, their numerical values being

$$M_b = 364,230 \text{ in.-lb.},$$

$$M_t = 37,840 \text{ in.-lb.}$$

Therefore, substituting in equation (150), we have

$$12,000 = \frac{16}{\pi D^3} (364,230 + \sqrt{364,230^2 + 37,840^2});$$

whence

$$D = 6.725 \text{ in.}$$

and similarly, from equation (151),

$$9000 = \frac{16}{\pi D^3} \sqrt{364,230^2 + 37,840^2};$$

whence

$$D = 5.842 \text{ in.}$$

The proper diameter for the shaft is then the larger of these two values, say $6\frac{3}{4}$ in.

190. If the angle of twist for the wire in problem 186 is $\theta = 40^\circ$, how great is the torsional moment acting on the wire?

191. Compare the angle of twist given by St. Venant's general formula with the values given by the special formulas in articles 67, 68, and 69.

192. A steel shaft is required to transmit 300 H.P. at a speed of 200 revolutions per minute, the maximum moment being 40 per cent greater than the average. Find the diameter of the shaft.

193. Under the same conditions as in problem 192, find the inside diameter of a hollow circular shaft whose outside diameter is 6 in. Also compare the amount of metal in the solid and hollow shafts.

194. The semi-axes of the cross section of an elliptical shaft are 3 in. and 5 in. respectively. What is the diameter of a circular shaft of equal strength?

195. An oak beam 6 in. square projects 4 ft. from a wall and is acted upon at the free end by a twisting moment of 25,000 ft.-lb. How great is the angle of twist?

196. A steel shaft 10 ft. long between centers of bearings and 4 in. in diameter carries a pulley 14 in. in diameter at its center. If the driving tension in the belt is 250 lb., and the following side runs slack, what is the maximum stress in the shaft, and how many H.P. is it transmitting when running at 80 R.P.M.?

197. Find the required diameter of a solid wrought-iron circular shaft which is required to transmit 150 H.P. at a speed of 60 R.P.M.

198. Find the angle of twist in problem 192.

199. Find the angle of twist in problem 193 and compare it with the angle of twist for the solid shaft in problem 198.

200. How many H.P. can a hollow circular steel shaft of 15 in. external diameter and 11 in. internal diameter transmit at a speed of 50 R.P.M. if the maximum allowable unit stress is not to exceed 12,000 lb./in.²?

201. Find the diameter of a structural-steel engine shaft to transmit 900 H.P. at 75 R.P.M. with a factor of safety of 10.

202. Find the factor of safety for a wrought-iron shaft 5 in. in diameter which is transmitting 60 H.P. at 125 R.P.M.

203. A structural-steel shaft is 60 ft. long and is required to transmit 500 H.P. at 90 R.P.M. with a factor of safety of 8, and to be of sufficient stiffness so that the angle of torsion shall not exceed .5° per foot of length. Find its diameter.

204. Under the same conditions as in problem 203, find the size of a hollow shaft if the external diameter is twice the internal.

205. A hollow wrought-iron shaft 9 in. in external diameter and 2 in. thick is required to transmit 600 H.P. with a factor of safety of 10. At what speed should it be run?

FIG. 91

206. A horizontal steel shaft 4 in. in diameter and 10 ft. long between centers of bearings carries a 300-lb. pulley 14 in. in diameter at its center. The belt on the pulley has a tension of 50 lb. on the slack side and 175 lb. on the driving side. Find the maximum stress in the shaft, assuming that the belt exerts a horizontal pull on the shaft.

207. An overhung steel crank, like that shown in Fig. 88, carries a maximum thrust on the crank pin of 2 tons. Length of crank, 9 in.; distance from center of pin to center of bearing, 5 in. Determine the size of crank and shaft for a factor of safety of 5.

208. A propeller shaft 9 in. in diameter transmits 1000 H.P. at 90 R.P.M. If the thrust on the screw is 12 tons, determine the maximum stress in the shaft.

209. A round steel bar 2 in. in diameter, supported at points 4 ft. apart, deflects .029 in. under a central load of 300 lb. and twists 1.62° in a length of $2\frac{1}{2}$ ft. under a twisting moment of 1500 ft.-lb. Find E and G for the material.

FIG. 92

210. A steel shaft subjected to combined bending and torsion has an elastic limit in tension of 64,600 lb./in.² and an elastic limit in shear of 29,170 lb./in.² Show that Guest's formula, rather than Rankine's, applies to this material.

FIG. 93

211. A shaft subjected to combined bending and twisting is made of steel for which the elastic limit in tension is 28,800 lb./in.² and the elastic limit in shear is 16,000 lb./in.² Show that if the bending moment is one half the twisting moment, the shaft will be weakest in shear, whereas if the bending moment is

twice the twisting moment, it will be weakest in tension.

212. A cast-iron flanged shaft coupling is connected by eight $1\frac{1}{4}$ in. bolts, the axis of each bolt being 6 in. from the axis of the shaft. The diameter of the

shaft is 5 in. Find the shear on each bolt when the maximum shearing stress in the shaft is 9000 lb./in.²

213. A crank shaft revolves in bearings at *A* and *B*, as indicated in Fig. 91. The cranks are in the same plane, and the crank-pin pressures *P* and *Q* are assumed to act at right angles to the cranks. If $P = 2500$ lb., find *Q* and the reactions of the bearings at *A* and *B*. Find also the maximum stress in the crank pin *C* and draw the bending moment and shear diagrams.

214. A crank shaft revolves in bearings at *B* and *F* (Fig. 92) and carries two cranks *C* and *E* in the same plane. The shaft transmits a pure torque at the left end, and the crank-pin pressures are assumed to act perpendicular to the plane of the cranks. Find the stresses in the cranks *C* and *E*, and in the shaft at *B* and *D*, and draw the bending moment and shear diagrams.

215. A crank shaft revolves in bearings at *B* and *D* (Fig. 93), the planes of the two cranks being 90° apart. Taking the dimensions given in the figure, assume $P = 3000$ lb. and find *Q*, the reactions of the bearings, and the stress in the crank pin *C*.

SECTION X

SPHERES AND CYLINDERS UNDER UNIFORM PRESSURE

71. Hoop stress. When a hollow sphere or cylinder is subjected to uniform pressure, as in the case of steam boilers, standpipes, gas, water, and steam pipes, fire tubes, etc., the effect of the radial pressure is to produce stress in a circumferential direction, called **hoop stress**. In the case of a cylinder closed at the ends, the pressure on the ends produces longitudinal stress in the side walls in addition to the hoop stress.

If the thickness of a cylinder or sphere is small compared with its diameter, it is called a **shell**. In analyzing the stress in a thin shell subjected to uniform pressure, such as that due to water, steam, or gas, it may be assumed that the hoop stress is distributed uniformly over any cross section of the shell. This assumption will be made in what follows.

FIG. 94

72. Hoop tension in hollow sphere. Consider a spherical shell subjected to uniform internal pressure, and suppose that the shell is cut into hemispheres by a diametral plane (Fig. 94). Then, if w denotes the pressure per unit of area within the shell, the resultant force acting on either hemisphere is $P = \frac{\pi d^2 w}{4}$, where

d is the radius of the sphere. If p denotes the unit tensile stress on the circular cross section of the shell, the total stress on this cross section is $\pi d h p$, approximately, where h is the thickness of the shell. Consequently,

$$\frac{\pi d^2 w}{4} = \pi d h p;$$

whence

$$(165) \quad p = \frac{w d}{4 h},$$

which gives the hoop tension in terms of the radial pressure.

73. Hoop tension in hollow circular cylinder. In the case of a cylindrical shell, its ends hold the cylindrical part together in such a way as to relieve the hoop tension at either extremity. Suppose, then, that the portion of the cylinder considered is so far removed from either end that the influence of the end constraint can be assumed to be zero.

FIG. 95

Suppose the cylinder cut in two by a plane through its axis, and consider a section cut out of either half cylinder by two planes perpendicular to the axis, at a distance apart equal to c (Fig. 95). Then the resultant internal pressure P on the strip under consideration is $P = cdw$, and the resultant hoop tension is $2chp$, where the letters have the same meaning as in the preceding article. Consequently, $cdw = 2chp$; whence

$$(166) \quad p = \frac{dw}{2h}.$$

This result is applicable to shells under both inner and outer pressure, if p is taken to be the excess of the internal over the external pressure.

74. Longitudinal stress in hollow circular cylinder. If the ends of a cylinder are fastened to the cylindrical part, the internal pressure against the ends produces longitudinal stresses in the side walls. In this case the cylindrical part is subjected both to hoop tension and to longitudinal tension.

FIG. 96

To find the amount of the longitudinal tension, consider a cross section of the cylinder near its center, where the influence of the end restraints can be assumed to be zero (Fig. 96). Then the resultant pressure on either end is $P = \frac{\pi d^2 w}{4}$, and the resultant longitudinal stress on the cross section is $\pi d h p$. Therefore $\frac{\pi d^2 w}{4} = \pi d h p$; whence

$$(167) \quad p = \frac{wd}{4h}.$$

This is the same formula as for the sphere, which was to be expected, since the cross section is the same in both cases.

75. Thick cylinders. Lamé's formulas. Consider a thick circular cylinder of external radius a and internal radius b , subjected to either

FIG. 97

internal or external uniform pressure, or to both simultaneously, and suppose that a section is cut out of the cylinder by two planes perpendicular to the axis at a unit distance apart (Fig. 97).

Now consider a thin ring of the material anywhere in the given section, of external radius r_e and internal radius r_i . Then, under the strain, r_e will become

$$r_e + \Delta r_e = r_e \left(1 + \frac{\Delta r_e}{r_e} \right) = r_e (1 + s'_e),$$

where s_e denotes the unit deformation of the fiber, which never exceeds $\frac{1}{1000}$ for safe working stresses. Similarly, r_i will become

$$r_i + \Delta r_i = r_i \left(1 + \frac{\Delta r_i}{r_i} \right) = r_i (1 + s_i),$$

where the unit deformation s_i is also very small. Since any safe strain produces no appreciable change in the sectional area of the thin ring here considered, by equating its sectional areas before and after strain we have

$$\pi (r_e^2 - r_i^2) = \pi [r_e^2 (1 + s_e)^2 - r_i^2 (1 + s_i)^2].$$

Canceling out the common factor π and reducing, this becomes

$$r_e^2 (s_e^2 + 2 s_e) = r_i^2 (s_i^2 + 2 s_i);$$

or, since the unit deformations s_e and s_i are very small, their squares may be neglected in comparison with their first powers, and consequently this expression further reduces to

$$\frac{r_e^2}{r_i^2} = \frac{s_i}{s_e}.$$

Since by Hooke's law the unit stress is proportional to the unit deformation within the elastic limit, if p_e denotes the unit stress on the outside fiber of the thin ring, and p_i on the inside fiber, then

$$\frac{p_i}{p_e} = \frac{s_i}{s_e} = \frac{r_e^2}{r_i^2},$$

or

$$p_i r_i^2 = p_e r_e^2.$$

Hence, if p_h denotes the hoop stress on any element of the thin ring and r the radius of this element, then

$$(168) \quad p_h r^2 = \text{constant, say } C.$$

It should be noted that this relation applies only to a thin ring and not to the thick cylinder as a whole. It may be used, however, to find the change in the hoop stress corresponding to a small change in the radius, that is to say, the difference in the hoop stress on two adjacent fibers, as explained in what follows.

Now again consider a thin ring of the material and let its internal radius be r and its thickness Δr . Also, let p_h denote the hoop stress in this thin ring, p_r the radial stress acting on its inner surface, and $p_r + \Delta p_r$ the radial stress acting on its outer surface. Then the difference in pressure on the inside and outside of the ring must be equal to the total force holding the ring together; that is,

$$(p_r + \Delta p_r) 2(r + \Delta r) - 2rp_r = 2p_h \Delta r;$$

but, since $\Delta p_r \Delta r$ is infinitesimal in comparison with the other terms, this reduces to

$$(169) \quad p_h = p_r + r \frac{\Delta p_r}{\Delta r}.$$

If the ends of the cylinder are free from restraint, or if the cylinder is subjected to a uniform longitudinal stress, the longitudinal deformation must be constant throughout the cylinder. But the lateral action of p_r and p_h produce longitudinal deformation in accordance with Poisson's law (article 8). Thus, if $\frac{1}{m}$ denotes Poisson's ratio, the longitudinal deformation due to the action of p_r and p_h is $\frac{p_r}{mE} + \frac{p_h}{mE}$, or $\frac{1}{mE}(p_r + p_h)$. Therefore, in order that

this expression may be constant, $p_r + p_h$ must be constant. Denoting this constant by k , we have

$$(170) \quad p_r + p_h = k.$$

Now eliminating p_h between equations (168) and (170), we have

$$(k - p_r)r^2 = C.$$

As the radius r increases, the stress p_r increases or decreases according to whether the constant C is positive or negative; that is, whether the internal pressure is greater or less than the external. Since the sign of C has no effect on the result, we may say that for a point at a distance $r + \Delta r$ from the axis the radial stress is of amount $p_r - \Delta p_r$, such that

$$[k - (p_r + \Delta p_r)](r + \Delta r)^2 = C.$$

Simplifying this expression, it becomes

$$(k - p_r)r^2 + 2r\Delta r(k - p_r) - r^2\Delta p_r = C,$$

and subtracting from it the original relation, namely,

$$(k - p_r)r^2 = C,$$

we have

$$2r\Delta r(k - p_r) - r^2\Delta p_r = 0,$$

whence

$$\frac{\Delta p_r}{\Delta r} = \frac{2r(k - p_r)}{r^2} = \frac{2p_h}{r};$$

and, since $p_h = \frac{C}{r^2}$, this becomes

$$(171) \quad \frac{\Delta p_r}{\Delta r} = \frac{2C}{r^3}.$$

Substituting equation (171) in equation (169) and making use of equation (170), we have

$$p_h = p_r + \frac{2C}{r^2} = k - p_h + \frac{2C}{r^2};$$

whence

$$(172) \quad p_h = \frac{k}{2} + \frac{C}{r^2},$$

and therefore, from (170),

$$(173) \quad p_r = \frac{k}{2} - \frac{C}{r^2}.$$

If, therefore, the cylinder is subjected to a uniform internal pressure of amount w_i per unit of area, and also to a uniform external pressure of amount w_e per unit of area, then $p_r = w_e$ when $r = a$, and $p_r = w_i$ when $r = b$. Substituting these simultaneous values in equation (173),

$$w_e = \frac{k}{2} - \frac{C}{a^2}, \quad w_i = \frac{k}{2} - \frac{C}{b^2};$$

whence
$$C = \frac{a^2 b^2 (w_e - w_i)}{a^2 - b^2}, \quad \frac{k}{2} = \frac{w_e a^2 - w_i b^2}{a^2 - b^2}.$$

Hence, substituting these values of C and $\frac{k}{2}$ in equations (172) and (173), they become

$$(174) \quad \begin{cases} p_r = \frac{w_e a^2 - w_i b^2}{a^2 - b^2} - \frac{a^2 b^2 (w_e - w_i)}{(a^2 - b^2) r^2}, \\ p_h = \frac{w_e a^2 - w_i b^2}{a^2 - b^2} + \frac{a^2 b^2 (w_e - w_i)}{(a^2 - b^2) r^2}, \end{cases}$$

which give the radial and hoop stresses in a thick cylinder subjected to internal and external pressure. Equations (174) are known as Lamé's formulas.

76. Maximum stress in thick cylinder under uniform internal pressure. Consider a thick circular cylinder which is subjected only to internal pressure. Then $w_e = 0$, and equations (174) become

$$(175) \quad p_r = \frac{w_i b^2}{a^2 - b^2} \left(\frac{a^2}{r^2} - 1 \right), \quad p_h = -\frac{w_i b^2}{a^2 - b^2} \left(\frac{a^2}{r^2} + 1 \right).$$

Since p_h is negative, the hoop stress in this case is tension.

Since p_r and p_h both increase as r decreases, the maximum stress occurs on the inner surface of the cylinder, where $r = b$ and $p_r = w_i$. Hence

$$(176) \quad p_h = -\frac{w_i (a^2 + b^2)}{a^2 - b^2}.$$

Clearing the latter of fractions, we have $\frac{a^2}{b^2} = \frac{p_h + w_i}{p_h - w_i}$; whence the thickness of the tube, $h = a - b$, is given by

$$(177) \quad h = b \left(\sqrt{\frac{p_h + w_i}{p_h - w_i}} - 1 \right).$$

77. Bursting pressure for thick cylinder. Let u_t denote the ultimate tensile strength of the material of which the cylinder is composed. Then, from equation (176), the maximum allowable internal pressure w_i is obtained from the equation

$$u_t = -\frac{w_i(a^2 + b^2)}{(a^2 - b^2)};$$

whence

$$(178) \quad w_i = \frac{u_t(a^2 - b^2)}{a^2 + b^2}.$$

Equation (178) gives the maximum internal pressure w_i which the cylinder can stand without bursting.

78. Maximum stress in thick cylinder under uniform external pressure. Consider a thick circular cylinder subjected only to external pressure. In this case $w_i = 0$ and equations (174) become

$$p_r = \frac{w_e a^2}{a^2 - b^2} \left(1 - \frac{b^2}{r^2}\right),$$

$$p_h = \frac{w_e a^2}{a^2 - b^2} \left(1 + \frac{b^2}{r^2}\right).$$

Since p_h is positive, the hoop stress in this case is compression.

For a point on the inner surface of the cylinder $r = b$, $p_r = 0$, and

$$(179) \quad p_h = \frac{2 w_e a^2}{a^2 - b^2}.$$

79. Comparison of formulas for the strength of tubes under uniform internal pressure.

I. *Thin Cylinder* $\left(\frac{h}{a} \leq .023\right)$. From article 73 the formula for the hoop (or circumferential) stress in a thin circular cylinder is

$$(180) \quad p_h = \frac{w d}{2 h},$$

and from article 74, when the ends of the cylinder are closed, the longitudinal stress is

$$(181) \quad p_l = \frac{w d}{4 h}.$$

The actual stress in a thin cylinder, due to the combination of these two stresses and based on a value of Poisson's ratio = .3, is then found to be*

$$(182) \quad p = .425 \frac{wd}{h}.$$

II. *Thick Cylinder. Lamé's Formula.* In article 76 the maximum stress in a thick cylinder under uniform internal pressure is given by equation (176) in terms of the radii a and b . If the internal and external diameters of the tube are denoted by d and D respectively, then $d = 2b$, $D = 2a$, and the formula becomes

$$(183) \quad p = \frac{w(D^2 + d^2)}{D^2 - d^2}.$$

III. *Barlow's Formula.* This formula, which is widely used because of its simplicity, assumes that the area of cross section of the tube remains constant under the strain, and that the length of the tube also remains unaltered. As neither of these assumptions is correct, the formula can give only approximate results. In the notation previously used Barlow's formula is

$$(184) \quad p = \frac{wD}{2h}.$$

It is therefore of the same form as the formula for the hoop stress in a thin cylinder, except that it is expressed in terms of the *outside diameter* D inside of the inside diameter.

From the results of their experience in the manufacture and testing of tubes, the National Tube Company asserts that for any ratio of $\frac{h}{D} < .3$ Barlow's formula "is best suited for all ordinary calculations pertaining to the bursting strength of commercial tubes, pipes, and cylinders."

For certain classes of seamless tubes and cylinders, however, and for critical examination of welded pipe, where the least thickness of wall, yield point of the material, etc. are known with accuracy, and close results are desired, they recommend that the following formulas, due to Clavarino and Birnie, be used rather than Barlow's.

* Slocum and Hancock, *Strength of Materials*, Revised Edition, p. 156.

IV. *Clavarino's formula.* In this formula each particle of the tube is assumed to be subjected to radial stress, hoop stress, and longitudinal stress, due to a uniform internal pressure acting jointly on the tube wall and its closed ends. The formula also involves Poisson's ratio of lateral contraction, and is theoretically correct, provided the maximum stress does not exceed the elastic limit of the material. Assuming a value of Poisson's ratio = .3 and using the same notation as above, Clavarino's formula is

$$(185) \quad p = \frac{w(13 D^2 + 4 d^2)}{10(D^2 - d^2)};$$

whence

$$(186) \quad d = D \sqrt{\frac{10 p - 13 w}{10 p + 4 w}}.$$

V. *Birnie's formula.* This formula is based upon the same assumptions as Clavarino's, except that the longitudinal stress is assumed to be zero. Using the same notation as before and assuming Poisson's ratio for steel to be .3, Birnie's formula is

$$(187) \quad p = \frac{w(13 D^2 + 7 d^2)}{10(D^2 - d^2)};$$

whence

$$(188) \quad d = D \sqrt{\frac{10 p - 13 w}{10 p + 7 w}}.$$

80. **Thick cylinders built up of concentric tubes.** From equations (174) it is evident that in a thick cylinder subjected to internal pressure the stress is greatest on the inside of the cylinder and decreases toward the outside. In order to equalize the stress throughout the cylinder and thus obtain a more economical use of material, the device used consists in forming the cylinder of several concentric tubes and producing an initial compressive stress on the inner ones. For instance, in constructing the barrel of a cannon or the cylinder of a hydraulic press the cylinder is built up of two or more tubes. The outer tubes in this case are made of somewhat smaller diameter than the inner tubes, and each is heated until it has expanded sufficiently to be slipped over the one next smaller. In cooling, the metal of the outer tube contracts, thus producing a compressive stress in the inner tube and a tensile

stress in the outer tube. If, then, this composite tube is subjected to internal pressure, the first effect of the hoop tension thus produced is to relieve the initial compressive stress in the inner tube and increase that in the outer tube. Thus the resultant stress in the inner tube is equal to the difference between the initial stress and that due to the internal pressure, whereas the resultant stress in the outer tube is equal to the sum of these two. In this way the strain is distributed more equally throughout the cylinder. It is evident that the greater the number of tubes used in building up the cylinder, the more nearly can the strain be equalized.

The preceding discussion of the stress in thick tubes can also be applied to the calculation of the stress in a rotating disk. For example, a grindstone is strained in precisely the same way as a thick tube under internal pressure, the load in this case being due to centrifugal force instead of to the pressure of a fluid or gas.

81. Practical formulas for the collapse of tubes under external pressure. A rigorous analysis of the stress in thin tubes, due to external pressure, using Poisson's ratio $\frac{1}{m}$ of transverse to longitudinal deformation, gives the formula *

$$w = \frac{E}{4 \left(1 - \frac{1}{m^2}\right)} \left(\frac{h}{a}\right)^3,$$

or, in terms of the diameter $D = 2a$,

$$w = \frac{2E}{1 - \frac{1}{m^2}} \left(\frac{h}{D}\right)^3.$$

This formula, however, is based on the assumptions that the tube is perfectly symmetrical, of uniform thickness, and of homogeneous material — conditions which are never fully realized in commercial tubes. From recent experiments on the collapse of tubes,† however, it is now possible to determine the practical limitations of this formula and to so modify it, by a method similar to that by

* Love, *Mathematical Theory of Elasticity*, Vol. II, pp. 308-316.

† Carman, "Resist. of Tubes to Collapse," *Univ., Ill. Bull.*, Vol. III, No. 17; Stewart, "Collap. Press. Lap-Welded Steel Tubes," *Trans. A.S.M.E.*, 1906, pp. 730-820.

which the Gordon-Rankine column formula was deduced from Euler's formula (articles 54, 55), as to obtain a rational formula which shall, nevertheless, conform closely to experimental results. By determining the ellipticity, or deviation from roundness, and the variation in thickness of the various types of tubes covered by the tests mentioned above, it is found that by introducing empirical constants the rational formulas can be made to fit experimental results as closely as any empirical formulas, with the advantage of being unlimited in their range of application.* The formula so obtained is

$$(189) \quad w = \frac{2EC}{1 - \frac{1}{m^2}} \left(\frac{h}{D} \right)^3, \quad \left\{ \begin{array}{l} \text{for thin tubes} \\ \frac{h}{D} \cong .023 \end{array} \right.$$

where h = average thickness of tube in inches,
 D = maximum outside diameter in inches,

$\frac{1}{m}$ = Poisson's ratio $\cong .3$ for steel,

C = .69 for lap-welded steel boiler flues,
 = .76 for cold-drawn seamless steel flues,
 = .78 for drawn seamless brass tubes.

By a similar procedure for thick tubes $\left(\frac{h}{D} > .023 \right)$ a practical rational formula has been obtained from Lamé's formula (article 75) for this case also, namely,

$$(190) \quad w = \frac{2Khu_c}{D} \left(1 - K \frac{h}{D} \right), \quad \left\{ \begin{array}{l} \text{for thick tubes} \\ \frac{h}{D} > .023 \end{array} \right.$$

where u_c = ultimate compressive strength of the material,
 K = .89 for lap-welded steel boiler flues.

Only one value of K is given, as the experiments cited were all made on one type of tube.

The correction constants C and K include corrections both for ellipticity, or flattening of the tube, and for variation in thickness.

* Slocum, "The Collapse of Tubes under External Pressure," *Engineering*. London, January 8, 1909. Also abstract of same article in Kent, 8th ed., 1910, pp. 320-322.

Thus, if the correction for ellipticity is denoted by C_1 and the correction for variation in thickness by C_2 , we have

$$C_1 = \frac{\text{minimum outside diameter}}{\text{maximum outside diameter}},$$

$$C_2 = \frac{\text{minimum thickness}}{\text{average thickness}},$$

and the correction constants C and K are therefore defined as

$$C = C_1^3 C_2^3,$$

$$K = C_1 C_2.$$

By an experimental determination of C_1 and C_2 the formulas can therefore be applied to any given type of tube.

82. Shrinkage and forced fits. In machine construction shrinkage and forced, or pressed, fits are frequently employed for connecting certain parts, such as crank disk and shaft, wheel and axle, etc. To make such a connection the shaft is finished slightly larger than the hole in the disk or ring in which it belongs. The shaft is then either tapered slightly at the end and pressed into the ring cold, or the ring is enlarged by heating until it will slip over the shaft, in which case the shrinkage due to cooling causes it to grip the shaft.

To analyze the stresses arising from shrinkage and forced fits, let D_1 denote the diameter of the hole in the ring or disk, and D_2 the diameter of the

FIG. 98

shaft (Fig. 98). When shrunk or forced together, D_1 must increase slightly and D_2 decrease slightly; that is, D_1 and D_2 must of necessity take the same value D . Consequently, the circumference of the hole changes from πD_1 to πD , and hence the unit deformation s_1 of a fiber on the inner surface of the hole is

$$s_1 = \frac{\pi D - \pi D_1}{\pi D_1} = \frac{D - D_1}{D_1}.$$

Similarly, the unit deformation s_2 of a fiber on the surface of the shaft is

$$s_2 = \frac{\pi D_2 - \pi D}{\pi D_2} = \frac{D_2 - D}{D_2}.$$

From Hooke's law, $\frac{p}{s} = E$, we have, therefore, for the unit stress p_1 on the inside of the disk

$$\frac{p_1}{E_1} = s_1 = \frac{D - D_1}{D_1},$$

and for the unit stress p_2 on the surface of the shaft

$$\frac{p_2}{E_2} = s_2 = \frac{D_2 - D}{D_2}.$$

Adding these two equations to eliminate the unknown quantity D , the result is

$$\frac{p_1 D_1}{E_1} + \frac{p_2 D_2}{E_2} = D_2 - D_1 = K,$$

where K denotes the **allowance**, or difference in diameter of shaft and hole. For a thick disk or heavy ring this allowance K may be determined from the nominal diameter D of the shaft by means of the following empirical formulas: *

$$\text{For shrinkage fits,} \quad K = \frac{\frac{17}{16} D + \frac{1}{2}}{1000}.$$

$$\text{For pressed fits,} \quad K = \frac{2 D + \frac{1}{2}}{1000}.$$

$$\text{For driven fits,} \quad K = \frac{\frac{1}{2} D + \frac{1}{2}}{1000}.$$

For thin rings, however, the allowance given by these formulas will be found to produce stresses in the ring entirely too large for safety. In deciding on the allowance for any given class of work the working stresses in shaft and ring may first be assigned and the allowance then determined from the formulas given below, so that the actual stresses shall not exceed these values.

* S. H. Moore, *Trans. Am. Soc. Mech. Eng.*, Vol. XXIV.

From Lamé's formulas the stresses p_1 and p_2 may be obtained in terms of the unit pressure between the surfaces in contact. Thus, from formula (183), the stress on the inside of the hole is

$$(191) \quad p_1 = \frac{w(D_1^2 + D_3^2)}{D_3^2 - D_1^2},$$

where D_3 denotes the outside diameter of the ring, while, by substituting $r = a$ and $b = 0$ in the equations of article 78, the stresses on the outer surface of the shaft are found to be

$$p_h = w, \quad p_r = w,$$

and consequently

$$p_2 = w.$$

Eliminating w between these expressions for p_1 and p_2 , we have

$$(192) \quad p_1 = p_2 \left(\frac{D_1^2 + D_3^2}{D_3^2 - D_1^2} \right).$$

Now, to simplify the solution, let the coefficient of p_2 be denoted by H ; that is, let

$$H = \frac{D_1^2 + D_3^2}{D_3^2 - D_1^2},$$

in which case

$$p_1 = Hp_2.$$

Eliminating p_1 between this relation and the above expression for the allowance K , we have finally

$$(193) \quad \begin{cases} p_2 = \frac{K}{\frac{HD_1}{E_1} + \frac{D_2}{E_2}}, \\ p_1 = Hp_2. \end{cases}$$

In applying these formulas the constant H is first computed from the given dimensions of the parts. If the allowance K is given, the unit stresses p_1 and p_2 in ring and shaft are then found from the above. If K is to be determined, a safe value for the stress in the ring p_1 is assigned, and p_2 is calculated from the second equation. This value is then substituted in the first equation, and K is calculated.

APPLICATIONS

216. The outside diameter of a pipe is 4 in., and thickness of wall $\frac{1}{2}$ in. Find the safe internal fluid pressure by Clavarino's formula for a working stress in the steel of 10,000 lb./in.²

Solution. The thickness ratio in this case is $\frac{h}{D} = \frac{\frac{1}{2}}{4} = 0.125$ in. Also, $D = 4$ in., $d = 3$ in., $p = 10,000$ lb./in.², and consequently

$$w = \frac{10(16 - 9)}{13 \times 16 + 4 \times 9} 10,000 = 2869 \text{ lb./in.}^2$$

217. A cast-iron gear, 8 in. external diameter, 3 in. wide, and $1\frac{3}{4}$ in. internal diameter, is to be forced on a steel shaft. Find the stresses developed, the pressure required to force the gear on the shaft, and the tangential thrust required to shear the fit, that is, to produce relative motion between gear and shaft.

Solution. From the formula $K = \frac{2D + \frac{1}{2}}{1000}$ the allowance is found to be .004 in., making the diameter of the shaft $D_2 = 1.754$ in. Also, since $D_1 = 1.75$ in. and $D_3 = 8$ in., we have $H = 1.1005$. Hence, assuming $E_1 = 15,000,000$ lb./in.² and $E_2 = 30,000,000$ lb./in.², we have

$$p_1 = 23,550 \text{ lb./in.}^2, \quad p_2 = 21,400 \text{ lb./in.}^2$$

To find the pressure required to force the gear on the shaft it is first necessary to calculate the pressure between the surfaces in contact. From the relation $p_2 = w$ this amounts to

$$w = 21,400 \text{ lb./in.}^2$$

The coefficient of friction depends on the nature of the surfaces in contact. Assuming it to be $\mu = .15$ as an average value, and with a nominal area of contact of $\pi \times 1\frac{3}{4} \times 3 = 16.497$ in.², the total pressure P required is

$$P = 16.497 \times 21,400 \times .15 = 52,955 \text{ lb.} = 26.5 \text{ tons.}$$

To find the torsional resistance of the fit, we have, as above,

$$\text{Bearing area} = 16.497 \text{ in.}^2, \quad \text{Unit pressure} = 21,400 \text{ lb./in.}^2,$$

$$\mu = .15, \text{ radius of shaft} = .875 \text{ in.}$$

Hence the torsional resistance is

$$M_t = 16.497 \times 21,400 \times .15 \times .875 = 46,336 \text{ in.-lb.}$$

Consequently the tangential thrust on the teeth of the gear necessary to shear the fit is

$$\frac{46,336}{4} = 11,584 \text{ lb.} = 5.8 \text{ tons.}$$

218. The outside diameter of a steel pipe is $5\frac{1}{2}$ in., thickness of wall $\frac{1}{4}$ in., and internal fluid pressure 1500 lb./in.² Find by Clavarino's formula the maximum fiber stress in the wall of the pipe.

219. The outside diameter of a steel pipe is 8 in., the internal fluid pressure is 2000 lb./in.², and the allowable stress in the steel is 15,000 lb./in.² Find the required thickness of pipe wall.

220. Solve problem 216 by the other four formulas listed in article 79 and compare the results.

221. Find the thickness necessary to give to a steel locomotive cylinder of 22 in. internal diameter if it is required to withstand a maximum steam pressure of 150 lb./in.² with a factor of safety of 10, using both Lamé's and Clavarino's formulas.

FIG. 99

222. In a four-cycle gas engine the cylinder is of steel with an internal diameter of 6 in., and the initial internal pressure is 200 lb./in.² absolute. With a factor of safety of 15, how thick should the walls of the cylinder be made, according to Lamé's formula?

223. The steel cylinder of a hydraulic press has an internal diameter of 5 in. and an external diameter of 7 in. With a factor of safety of 3, how great an internal pressure can the cylinder withstand, according to Lamé's formula?

224. In a fire-tube boiler the tubes are of drawn steel, 2 in. internal diameter and $\frac{1}{8}$ in. thick. What is the factor of safety for a working gauge pressure of 200 lb./in.²?

225. How great is the stress in a copper sphere 2 ft. in diameter and .25 in. thick under an internal pressure of 175 lb./in.²?

226. A cast-iron water pipe is 24 in. in diameter and 2 in. thick. What is the greatest internal pressure which it can withstand, according to the formula for thin cylinders?

227. A wrought-iron cylinder is 8 in. in external diameter and $1\frac{1}{2}$ in. thick. How great an external pressure can it withstand?

228. An elevated water tank is cylindrical in form, with a hemispherical bottom (Fig. 99). The diameter of the tank is 20 ft. and its height 52 ft. (exclusive of the bottom). If the tank is to be built of wrought iron and the factor of safety is taken to be 6, what should be the thickness of the bottom plates and also of those in the body of the tank near its bottom?

NOTE. Formulas (165) and (166) give the required thickness of the plates, provided the tank is without joints. The bearing power of the rivets at the joints, however, is, in general, the consideration which determines the thickness of the plates (article 90).

229. A marine boiler shell is 16 ft. long, 8 ft. in diameter, and 1 in. thick. What is the stress in the shell for a working gauge pressure of 160 lb./in.²?

230. The air chamber of a pump is made of cast iron of the form shown in Fig. 100. If the diameter of the air chamber is 10 in. and its height 24 in., how thick must the walls of the air chamber be made in order to stand a pressure of 500 lb./in.², with a factor of safety of 4?

231. The end plates of a boiler shell are curved out to a radius of 5 ft. If the plates are $\frac{5}{8}$ in. thick, find the tensile stress due to a steam pressure of 175 lb./in.²

232. If the thickness of the end plates in problem 231 is changed to $\frac{1}{2}$ in., the steam pressure being the same, to what radius should they be curved in order that the tensile stress in them shall remain the same?

FIG. 100

233. The cylinder of an hydraulic press is 12 in. inside diam. How thick must it be in order to stand a pressure of 1500 lb./in.² if it is made of cast steel and the factor of safety is 10?

234. A high-pressure cast-iron water main is 4 in. inside diameter and carries a pressure of 800 lb./in.² Find its thickness for a factor of safety of 15.

235. The water chamber of a fire engine has a spherical top 18 in. in diameter and carries a pressure of 250 lb./in.² It is made of No. 7 B. and S. gauge copper, which is reduced in manufacture to a thickness of about .1 in. Determine the factor of safety.

236. A cast-iron ring 3 in. thick and 8 in. wide is forced onto a steel shaft 10 in. in diameter. Find the stresses in ring and shaft, the pressure required to force the ring onto the shaft, and the torsional resistance of the fit.

NOTE. Since the ring in this case is relatively thin, assume an allowance of about half the amount given by Moore's formula. Then, having given $D_2 = 10$ in., $D_3 = 16$ in., and having computed the allowance K , we have also $D_1 = D_2 - K$, and, inserting these values in the formulas of article 82, the required quantities may be found, as explained in problem 217.

237. The following data are taken from Stewart's experiments on the collapse of thin tubes under external pressure, the tubes used for experiment being lap-welded steel boiler flues. Compute the collapsing pressure from the rational formula for thin tubes, given in article 81, for both the average thickness and least thickness, and note that these two results lie on opposite sides of the value obtained directly by experiment.

OUTSIDE DIAMETER IN INCHES			THICKNESS h IN INCHES			ACTUAL COLLAPS- ING PRESSURE lb./in. ²
Average	At place of collapse		Average	At place of collapse		
	Greatest = D	Least = d		Greatest	Least	
8.604	8.610	8.580	0.219	0.230	0.210	870
8.664	8.670	8.625	0.226	0.227	0.204	840
8.665	8.670	8.660	0.212	0.240	0.211	880
8.653	8.665	8.590	0.208	0.220	0.203	970
8.688	8.715	8.605	0.274	0.280	0.266	1430
8.664	8.695	8.635	0.258	0.261	0.248	1320
8.645	8.665	8.635	0.263	0.268	0.259	1590
8.674	8.675	8.675	0.273	0.282	0.270	2030
8.638	8.645	8.615	0.289	0.298	0.280	2200
10.055	10.180	9.950	0.157	0.182	0.150	210

238. The following data are taken from Stewart's experiments on the collapse of thick tubes under external pressure. The ultimate compressive strength of the material was not given by the experimenter, but from the other elastic properties given it is here assumed to be $u_c = 38,500$ lb./in.² Compute the collapsing pressure from the rational formula for thick tubes, given in article 81, for both average and least thickness, and compare these results with the actual collapsing pressure obtained by experiment.

OUTSIDE DIAMETER IN INCHES			THICKNESS h IN INCHES			ACTUAL COLLAPS- ING PRESSURE lb./in. ²
Average	At place of collapse		Average	At place of collapse		
	Greatest = D	Least = d		Greatest	Least	
4.010	4.020	3.980	0.173	0.203	0.140	2050
4.014	4.050	3.990	0.178	0.277	0.158	2225
4.012	4.050	3.960	0.173	0.200	0.170	2425
4.018	4.050	4.010	0.184	0.192	0.165	2540
2.997	3.010	2.980	0.147	0.151	0.138	3350
2.987	3.010	2.970	0.139	0.139	0.125	2575
2.990	3.010	2.970	0.190	0.218	0.166	4200
2.996	3.020	2.980	0.191	0.216	0.176	4200
2.997	3.020	2.960	0.190	0.215	0.161	4175
3.000	3.020	2.960	0.182	0.192	0.165	3700

239. What is the maximum external pressure which a cast-iron pipe 18 in. in diameter and $\frac{1}{2}$ in. thick can stand without crushing?

240. Solve problem 226 by Birnie's and Barlow's formulas.

SECTION XI

FLAT PLATES

83. Theory of flat plates. The analysis of stress in flat plates is at present the most unsatisfactory part of the strength of materials. Although flat plates are of frequent occurrence in engineering constructions (as, for example, in manhole covers, cylinder ends, floor panels, etc.), no general theory of such plates has as yet been given. Each form of plate is treated by a special method, which in most cases is based upon an arbitrary assumption either as to the dangerous section or as to the reactions of the supports, and therefore leads to questionable results.

Although the present theory of flat plates is plainly inadequate, it is nevertheless of value in pointing out the conditions to which such plates are subject, and in furnishing a rational basis for the estimation of their strength. The formulas derived in the following paragraphs, if used in this way, with a clear understanding of their approximate nature, will be found to be invaluable in designing, or in determining the strength of flat plates.

The following has come to be the standard method of treatment and is chiefly due to Bach.*

84. Maximum stress in homogeneous circular plate under uniform load. Consider a flat, circular plate of homogeneous material, which bears a uniform load of amount w per unit of area, and suppose that the edge of the plate rests freely on a circular rim slightly smaller than the plate, every point of the rim being maintained at the same level. The strain in this case is greater than it would be if the plate was fixed at the edges, and consequently the formula deduced will give the maximum stress in all cases.

* For an approximate method of solution see article by S. E. Slocum entitled "The Strength of Flat Plates, with an Application to Concrete-Steel Floor Panels," *Engineering News*, July 7, 1904.

Now suppose a diametral section of the plate taken, and regard either half of the plate as a cantilever (Fig. 101). Then if r is the radius of the plate, the total load on this semicircle is $\frac{\pi r^2}{2}w$, and its resultant is applied at the center of gravity of the semicircle, which is at a distance of $\frac{4r}{3\pi}$ from AB . The moment of this resultant about the support AB is therefore $\frac{\pi r^2}{2}w \cdot \frac{4r}{3\pi}$, or $\frac{2r^3w}{3}$. Similarly, the resultant of the supporting forces at the edge of the plate is of amount $\frac{\pi r^2}{2} \cdot w$ and is applied at the center of gravity of the semi-circumference, which is at a distance of $\frac{2r}{\pi}$ from AB . The moment of this resultant about AB is therefore $\frac{\pi r^2 w}{2} \cdot \frac{2r}{\pi}$, or r^3w . Hence the total external moment M at the support is

$$M = r^3w - \frac{2r^3w}{3} = \frac{r^3w}{3}.$$

FIG. 101

Now assume that the stress at any point of the plate is independent of the distance of this point from the center. Under this arbitrary assumption the stress in the plate is given by the fundamental formula in the theory of beams, namely,

$$p = \frac{Me}{I}.$$

If the thickness of the plate is denoted by h , then, since the breadth of the section is $b = 2r$,

$$I = \frac{bh^3}{12} = \frac{rh^3}{6}, \text{ and } e = \frac{h}{2}.$$

Consequently,

$$p = \frac{Me}{I} = \frac{\frac{r^3w}{3} \cdot \frac{h}{2}}{\frac{rh^3}{6}};$$

whence

$$(194) \quad p = w \left(\frac{r}{h} \right)^2.$$

Föppl has shown that the arbitrary assumption made in deriving this formula can be avoided, and the same result obtained, by a more rigorous analysis than the preceding, and Bach has verified the formula experimentally. Formula (194) is therefore well established both theoretically and practically.

85. Maximum stress in homogeneous circular plate under concentrated load. Consider a flat, circular plate of homogeneous material, and suppose that it bears a single concentrated load P which is distributed over a small circle of radius r_0 concentric with the plate. Taking a section through the center of the plate and regarding either half as a cantilever, as in the preceding article, the total rim pressure is $\frac{P}{2}$, and it is applied at a distance of $\frac{2r}{\pi}$ from the center. The total load on the semicircle of radius r_0 is $\frac{P}{2}$, and it is applied at a distance of $\frac{4r_0}{3\pi}$ from the section. Therefore the total external moment M at the section is

$$M = \frac{Pr}{\pi} - \frac{2Pr_0}{3\pi} = \frac{Pr}{\pi} \left(1 - \frac{2r_0}{3r}\right).$$

Assuming that the stress is uniformly distributed throughout the plate, the stress due to the external moment M is given by the formula

$$p = \frac{Me}{I}.$$

If the thickness of the plate is denoted by h , then

$$I = \frac{r h^3}{6} \quad \text{and} \quad e = \frac{h}{2}.$$

Therefore

$$p = \frac{Me}{I} = \frac{\frac{Pr}{\pi} \left(1 - \frac{2r_0}{3r}\right) \frac{h}{2}}{\frac{r h^3}{6}};$$

whence

$$(195) \quad p = \frac{3P}{\pi h^2} \left(1 - \frac{2r_0}{3r}\right).$$

If $r_0 = 0$, that is to say, if the load is assumed to be concentrated at a single point at the center of the plate, formula (195) becomes

$$(196) \quad p = \frac{3P}{\pi h^2}.$$

If the load is uniformly distributed over the entire plate, then $r_0 = r$ and $P = \pi r^2 w$, where w is the load per unit of area. In this case formula (195) becomes

$$p = \frac{3 \pi r^2 w}{\pi h^2} \left(1 - \frac{2}{3}\right) = w \left(\frac{r}{h}\right)^2,$$

which agrees with the result of the preceding article.

86. Dangerous section of elliptical plate. Consider a homogeneous elliptical plate of semi-axes a and b and thickness h , and suppose that an axial cross is cut out of the plate, composed of two strips AB and CD , each of unit width, intersecting in the center of the plate, as shown in Fig. 102.

FIG. 102

Now suppose that a single concentrated load acts at the intersection of the cross and is distributed to the support in such a way that the two beams AB and CD each deflect the same amount at the center. Since AB is of length $2a$, from article 40, equation (54), the deflection at the center of AB is $D_1 = \frac{P(2a)^3}{48 EI}$. From symmetry, the reactions at A and B are equal. Therefore, if each of these reactions is denoted by R_1 , $2 R_1 = P$ and, consequently,

$$D_1 = \frac{R_1 a^3}{3 EI}.$$

Similarly, if R_2 denotes the equal reactions at C and D , the deflection D_2 of CD at its center is

$$D_2 = \frac{R_2 b^3}{3 EI}.$$

If the plate remains intact, the two strips AB and CD must deflect the same amount at the center. Therefore $D_1 = D_2$, and hence

$$(197) \quad \frac{R_1}{R_2} = \frac{b^3}{a^3}.$$

For the beam AB of length $2a$ the maximum external moment is $R_1 a$. Also, since AB is assumed to be of unit width, $I = \frac{h^3}{12}$ and $e = \frac{h}{2}$. Hence the maximum stress p' in AB is

$$p' = \frac{Me}{I} = 6R_1 \frac{a}{h^2}.$$

Similarly, the maximum stress p'' in CD is

$$p'' = 6R_2 \frac{b}{h^2}.$$

Consequently,

$$\frac{p'}{p''} = \frac{R_1 a}{R_2 b},$$

or, since from equation (197) $\frac{R_1}{R_2} = \frac{b^3}{a^3}$,

$$\frac{p'}{p''} = \frac{b^2}{a^2}.$$

By hypothesis $a > b$. Therefore $p'' > p'$; that is to say, the maximum stress occurs in the strip CD (that is, in the direction of the shorter axis of the ellipse). In an elliptical plate, therefore, rupture may be expected to occur along a line parallel to the major axis — a result which has been confirmed by experiment.

87. Maximum stress in homogeneous elliptical plate under uniform load. The method of finding the maximum stress in an elliptical plate is to consider the two limiting forms of an ellipse, namely, a circle and a strip of infinite length, and express a continuous relation between the stresses for these two limiting forms. The method is therefore similar to that used in Article 54 in obtaining the modified form of Euler's column formula.

Consider first an indefinitely long strip with parallel sides, supported at the edges and bearing a uniform load of amount w per unit of area. Let the width of the strip be denoted by $2b$ and its thickness by h . Then, if this strip is cut into cross strips of unit width, each of these cross strips can be regarded as an independent beam, the load on one of these unit cross strips being $2bw$ and the maximum moment at the center being $\frac{(2b)^2 w}{8}$.

Consequently, the maximum stress in the cross strips, and therefore in the original strip, is

$$(198) \quad p = \frac{Me}{I} = \frac{\frac{4b^2w}{8} \cdot \frac{h}{2}}{\frac{h^3}{12}} = 3 \frac{b^2w}{h^2}.$$

In the preceding article it was shown that the maximum stress in an elliptical plate occurs in the direction of the minor axis. Therefore equation (198) gives the limiting value which the stress in an elliptical plate approaches as the ellipse becomes more and more elongated.

For a circular plate of radius b and thickness h the maximum stress was found to be

$$(199) \quad p = \frac{b^2w}{h^2}.$$

Comparing equations (198) and (199), it is evident that the maximum stress in an elliptical plate is given, in general, by the formula

$$p = k \frac{b^2w}{h^2},$$

where k is a constant which lies between 1 and 3. Thus, for $\frac{b}{a} = 1$ (that is, for a circle) $k = 1$; whereas, if $\frac{b}{a} = 0$ (that is, for an infinitely long ellipse), $k = 3$. The constant k may therefore be assumed to have the value

$$k = 3 - 2 \frac{b}{a},$$

which reduces to the values 1 and 3 for the limiting cases, and in other cases has an intermediate value depending on the form of the plate. Consequently,

$$(200) \quad p = \left(3 - 2 \frac{b}{a}\right) \frac{b^2w}{h^2} = \frac{(3a - 2b)b^2w}{ah^2},$$

which is the required formula for the maximum stress p in a homogeneous elliptical plate of thickness h and semi-axes a and b .

88. Maximum stress in homogeneous square plate under uniform load. In investigating the strength of square plates the method of taking a section through the center of the plate and regarding the

portion of the plate on one side of this section as a cantilever is used, but experiment is relied upon to determine the position of the dangerous section. From numerous experiments on flat plates Bach has found that homogeneous square plates under uniform load always break along a diagonal.*

FIG. 103

Consider a homogeneous square plate of thickness h and side $2a$, which bears a uniform load w per unit of area. Suppose that a diagonal section of this plate is taken, and consider either half as a cantilever, as shown in Fig. 103. Then the total load on the plate is $4wa^2$, and the reaction of the support under each edge is wa^2 .

If d denotes the length of the diagonal AC , the resultant pressure on each edge of the plate is applied at a distance $\frac{d}{4}$ from AC , and therefore the moment of these resultants about AC is $2(wa^2)\frac{d}{4}$, or $\frac{wa^2d}{2}$. The total load on the triangle ABC is $2wa^2$, and its resultant is applied at the center of gravity of the triangle, which is at a distance of $\frac{d}{6}$ from AC . Therefore the moment of the load about AC is $(2wa^2)\frac{d}{6}$, or $\frac{wa^2d}{3}$. Therefore the total external moment M at the section AC is

$$M = \frac{wa^2d}{2} - \frac{wa^2d}{3} = \frac{wa^2d}{6}.$$

Hence the maximum stress in the plate is

$$p = \frac{Me}{I} = \frac{\frac{wa^2d}{6} \cdot \frac{h}{2}}{\frac{dh^3}{12}},$$

from which

$$(201) \quad p = w \left(\frac{a}{h} \right)^2.$$

The maximum stress in a square plate of side $2a$ is therefore the same as in a circular plate of diameter $2a$.

* Bach, *Elasticität und Festigkeitslehre*, 3d. ed., p. 561.

89. Maximum stress in homogeneous rectangular plate under uniform load. In the case of rectangular plates experiment does not indicate so clearly the position of the dangerous section as it does for square plates. It will be assumed in what follows, however, that the maximum stress occurs along a diagonal of the rectangle. This assumption is at least approximately correct if the length of the rectangle does not exceed two or three times its breadth.

Let the sides of the rectangle be denoted by $2a$ and $2b$, and the thickness of the plate by h (Fig. 104). Also let d denote the length of the diagonal AC , and c the altitude of the triangle ABC . Now suppose that a diagonal section AC of the plate is taken, and consider the half plate ABC as a cantilever, as shown in Fig. 104. If w denotes the unit load, the total load on the plate is $4abw$, and consequently the resultant of the reactions of the supports along AB and BC is of amount $2abw$ and is applied at a distance $\frac{c}{2}$ from AC .

Therefore the moment of the supporting force about AC is $abwc$.

Also, the total load on the triangle ABC is $2abw$, and it is applied at the center of gravity of the triangle, which is at a distance of $\frac{c}{3}$ from AC . Consequently, the total moment of the load about AC is $\frac{2abwc}{3}$. Therefore the total external moment M at the section AC is

$$M = abwc - \frac{2abwc}{3} = \frac{abwc}{3},$$

and the maximum stress in the plate is

$$p = \frac{Me}{I} = \frac{\frac{abwc}{3} \cdot \frac{h}{2}}{\frac{dh^3}{12}} = \frac{2wabc}{dh^2},$$

FIG. 104

or, since $cd = 4ab$,

$$(202) \quad p = w \frac{c^2}{2h^2},$$

which gives the required maximum stress.

For a square plate $a = b$ and $c = a\sqrt{2}$, and formula (202) reduces to formula (201) for square plates, obtained in the preceding article.

APPLICATIONS

241. The cylinder of a locomotive is 20 in. internal diameter. What must be the thickness of the steel end plate if it is required to withstand a pressure of 160 lb./in.² with a factor of safety of 6?

242. A circular cast-iron valve gate $\frac{1}{2}$ in. thick closes an opening 6 in. in diameter. If the pressure against the gate is due to a water head of 150 ft., what is the maximum stress in the gate?

243. Show that the maximum concentrated load which can be borne by a circular plate is independent of the radius of the plate.

244. A cast-iron manhole cover 1 in. thick is elliptical in form and covers an elliptical opening 3 ft. long and 18 in. wide. How great a uniform pressure will it stand?

245. What must be the thickness of a wrought-iron plate covering an opening 4 ft. square in order to carry a load of 200 lb./ft.² with a factor of safety of 5?

246. A wrought-iron trap door is 5 ft. long, 3 ft. wide, and $\frac{3}{8}$ in. thick. How great a uniform load will it bear?

247. The steel diaphragm separating two expansion chambers of a steam turbine is subjected to a pressure of 150 lb./in.² on one side and 80 lb./in.² on the other. Find the required thickness for a factor of safety of 10.

248. The cylinder of a hydraulic press is made of cast steel, 10 in. inside diameter, with a flat end of the same thickness as the walls of the cylinder. Find the required thickness for a factor of safety of 20. Also find how much larger the factor of safety would be if the end was made hemispherical. Assume $w = 1200$ lb./in.²

249. The cylinder of a steam engine is 16 in. inside diameter and carries a steam pressure of 125 lb./in.² If the cylinder head is mild steel, find its thickness for a factor of safety of 10.

250. A cast-iron valve gate 10 in. in diameter is under a pressure head of 200 ft. Find its thickness for a factor of safety of 15.

251. A cast-iron elliptical manhole cover is 18 in. $\times$ 24 in. in size and is designed to carry a concentrated load of 1000 lb. If the cover is ribbed, how thick must it be for a factor of safety of 20, assuming that the ribs double its strength?

252. Thurston's rule for the thickness of cylinder heads for steam engines is

$$h = .00035 wD,$$

where

h = thickness of head in inches,

D = inside diameter of cylinder in inches,

w = pressure in lb./in.²

Compare this formula with Bach's, assuming the material to be wrought iron and using the data of problem 249.

253. Show that Thurston's rule for thickness of cylinder head, given in problem 252, makes thickness of head = $1\frac{1}{4}$ times thickness of walls.

254. Nichols's rule for the proper thickness of unbraced flat wrought-iron boiler heads is

$$h = \frac{Aw}{10p},$$

where h = thickness of head in inches,
 A = area of head in square inches,
 w = pressure per square inch,
 p = working stress = $\frac{50,000}{10} = \frac{\text{ultimate strength in tension}}{\text{factor of safety}}.$

Compare this empirical rule with Bach's formula, using the data of problem 249 and assuming the material to be wrought iron.

255. Nichols's rule for the collapsing pressure of unbraced flat wrought-iron boiler heads is

$$w = \frac{10 hu_t}{A},$$

where w = collapsing pressure in lb./in.², h = thickness of head in inches,
 u_t = ultimate tensile strength in lb./in.², A = area of head in square inches.

Show that Nichols's two formulas are identical and that therefore they cannot be rational.

256. The following data are taken from Nichols's experiments on flat wrought-iron circular plates.

DIAMETER IN INCHES	THICKNESS IN INCHES	ACTUAL BURSTING PRESSURE LB./IN. ²
34.5	$\frac{9}{16}$	280
34.5	$\frac{3}{8}$	200
28.5	$\frac{3}{8}$	300
26.5	$\frac{3}{8}$	370

Using these data, compare Bach's and Grashof's rational formulas with Nichols's and Thurston's empirical formulas, as given below:

Circular plate, supported at edge and uniformly loaded.

Bach, $h = r \sqrt{\frac{w}{p}} = .5 D \sqrt{\frac{w}{p}},$

Grashof, $h = \sqrt{\frac{5 r^2 w}{6 p}} = .4564 D \sqrt{\frac{w}{p}},$

Nichols, $h = \frac{Aw}{10p} = .0785 \frac{w D^2}{p},$

Thurston, $h = .00035 w D,$

where h = thickness of head in inches, D = diameter of head in inches = $2r$,
 w = pressure in lb./in.², p = working stress in lb./in.²,

$$A = \text{area of head in square inches} = \frac{\pi D^2}{4}.$$

Note that the Nichols and Thurston formulas apply only to wrought iron.

SECTION XII

RIVETED JOINTS AND CONNECTIONS

90. Efficiency of riveted joint. In structural work such as plate girders, trusses, etc., and also in steam boilers, standpipes, and similar constructions, the connections between the various members are made by riveting the parts together. Since the holes for the rivets weaken the members so joined, the strength of the structure is determined by the strength of the joint.

Failure of a riveted joint may occur in various ways; namely, by shearing across the rivet, by crushing the rivet, by crushing the plate in front of the rivet, by shearing the plate (that is, pulling out the rivets), or by tearing the plate along the line of rivet holes. Experience has shown, however, that failure usually occurs either by shearing across the rivet or by tearing the plate along the line of rivet holes.

The strength of any given type of riveted joint is expressed by what is called its **efficiency**, defined as

$$\text{Efficiency of riveted joint} = \frac{\text{strength of joint}}{\text{strength of unriveted member}}.$$

Thus, in Fig. 105, if d denotes the diameter of a rivet and c the distance between rivet holes, or **pitch** of the rivets, as it is called, the efficiency, e , of the joint against tearing of the plate along the line of rivet holes is

$$e = \frac{c - d}{c}.$$

To determine the efficiency of the joint against shearing across the rivets, let q denote the ultimate shearing strength of the rivet and p the ultimate tensile strength of the plate. Then, for a single-riveted lap joint (Fig. 105), if h denotes the thickness of the plate, the area corresponding to one rivet is hc , and the area in shear for

each rivet is $\frac{\pi d^2}{4}$. Consequently the efficiency of this type of joint against rivet shearing is

$$e = \frac{\pi d^2 q}{4 c h p}$$

For an economical design these two efficiencies should be equal. For practical reasons, however, it is not generally possible to make these

SINGLE-RIVETED LAP JOINT
EFFICIENCY 50-60 PER CENT

SINGLE-RIVETED BUTT JOINT
EFFICIENCY 76-78 PER CENT

DOUBLE-RIVETED LAP JOINT
EFFICIENCY 70-72 PER CENT

DOUBLE-RIVETED BUTT JOINT
EFFICIENCY 82-83 PER CENT

FIG. 105

exactly equal, and in this case the smaller of the two determines the strength of the joint.

For a double-riveted lap joint the efficiency against tearing of the plate is

$$e = \frac{c - d}{c},$$

as above, but since in this case there are two rivets for each strip of length c , the efficiency against rivet shear is

$$e = \frac{\pi d^2 q}{2 chp}.$$

Similarly, for a single-riveted butt joint with two cover plates the efficiency of the joint against tearing of the plate is

$$e = \frac{c-d}{c},$$

and against rivet shear is

$$e = \frac{\pi d^2 q}{2 chp}.$$

For a double-riveted butt joint with two cover plates the efficiency against tearing of the plate is

$$e = \frac{c-d}{c},$$

and against rivet shear is

$$e = \frac{\pi d^2 q}{chp}.$$

The average efficiencies of various types of riveted joints as used in steam boilers are given in Fig. 105.

91. Boiler shells. In designing steam-boiler shells it is customary in this country to determine first the thickness of shell plates, by the following rule:

To find the thickness of shell plates, multiply the maximum steam pressure to be carried (safe working pressure in lb./in.²) by half the diameter of the boiler in inches. This gives the hoop stress in the shell per unit of length. Divide this result by the safe working stress (working stress = ultimate strength, usually about 60,000 lb./in.², divided by the factor of safety, say 4 or 5), and divide the quotient by the average efficiency of the style of joint to be used, expressed as a decimal. The result will be the thickness of the shell plates expressed in decimal fractions of an inch.

Having determined the thickness of shell plates by this method, the diameter of the rivets is next found from the empirical formula

$$d = k\sqrt{t},$$

where $k = 1.5$ for lap joints and $k = 1.3$ for butt joints with two cover plates.

The pitch of the rivets is next determined by equating the strength of the plate along a section through the rivet holes to the strength of the rivets in shear and solving the resulting equation for c .

To illustrate the application of these rules, let it be required to design a boiler shell 48 in. in diameter to carry a steam pressure of 125 lb./in.² with a double-riveted, double-strapped butt joint.

By the above rule for thickness of shell plates we have

$$h = \frac{125 \times \frac{4.8}{2}}{\frac{60000}{5} \times .82} = .3, \text{ say } \frac{5}{16} \text{ in.}$$

The diameter of rivets is then

$$d = 1.3 \sqrt{\frac{5}{16}} = .73, \text{ say } \frac{3}{4} \text{ in.}$$

To determine the pitch of the rivets, the strength of the plate for a section of width c on a line through the rivet holes is

$$(c - d)hp = (c - \frac{3}{4})\frac{5}{16} \times 60,000,$$

and the strength of the rivets in shear for a strip of this width is

$$4 \times \frac{\pi d^2}{4} q = \pi \frac{9}{16} \times 40,000.$$

Equating these two results and solving for c , we have

$$(c - \frac{3}{4})\frac{5}{16} \times 60,000 = \pi \frac{9}{16} \times 40,000,$$

whence

$$c = 4.5 \text{ in.}$$

As a check on the correctness of our assumptions the efficiency of the joint is found to be

$$e = \frac{c - d}{c} = \frac{4.5 - .75}{4.5} = .83.$$

92. Structural steel. For bridge and structural work the following empirical rules are representative of American practice: *

The pitch (or distance from center to center) of rivets should not be less than 3 diameters of the rivet. In bridge work the pitch should not exceed 6 in. or 16 times the thickness of the thinnest

* Given by Cambria Steel Co.

outside plates except in special cases hereafter noted. In the flanges of beams and girders, where plates more than 12 in. wide are used, an extra line of rivets with a pitch not greater than 9 in. should be driven along each edge to draw the plates together.

At the ends of compression members the pitch should not exceed 4 diameters of the rivet for a length equal to twice the width or diameter of the member.

In the flanges of girders and chords carrying floors the pitch should not exceed 4 in.

For plates in compression the pitch in the direction of the line of stress should not exceed 16 times the thickness of the plate, and the pitch in a direction at right angles to the line of stress should not exceed 32 times the thickness, except for cover plates of top chords and end posts, in which the pitch should not exceed 40 times their thickness.

The distance between the edge of any piece and the center of the rivet hole should not be less than $1\frac{1}{4}$ in. for $\frac{3}{4}$ -in. and $\frac{7}{8}$ -in. rivets, except in bars less than $2\frac{1}{2}$ in. wide; when practicable it should be at least 2 diameters of the rivet for all sizes, and should not exceed 8 times the thickness of the plate.

Typical illustrations of riveted connections in structural steel work are shown in Figs. 106 and 107.

93. Unit stresses. In structural-steel work it is customary to proportion the various members on the basis of certain specified unit stresses. The following specifications for the greatest allowable unit stresses represent the best American practice:

Axial tension , net section	18,000 lb./in. ²
Axial compression , gross section, but not to exceed 17,000 lb./in. ²	$16,000 - 17\frac{l}{r}$
l = length of member in inches,	
r = least radius of gyration of member in inches.	
Compression in flanges of deck plate girders and built or rolled beams, but not to exceed 17,000 lb./in. ²	$18,000 - 70\frac{l}{r}$
Compression in flanges of through plate girders, but not to exceed 17,000 lb./in. ²	$18,000 - 70\frac{l}{r}$
l = unsupported length of flange in inches,	
r = least radius of gyration of flange section laterally in inches,	
	$-\frac{17DW}{d^2}$

D = depth from top of girder to bottom of floor beams,	
d = depth of floor beams back to back of angles or flanges of beams,	
W = width center to center of girders.	
Shear in webs of plate girders, net section	13,500 lb./in. ²
Shear in pins and shop-driven rivets	13,500 lb./in. ²
Shear in field-driven rivets	10,800 lb./in. ²
Tension in extreme fiber of flanges of beams proportioned by moment of inertia, net section	18,000 lb./in. ²
Tension or compression in the extreme fiber of pins, assuming the stresses to be applied in the centers of bearings . . .	27,000 lb./in. ²
Bearing on pins in members not subject to reversal of stress,	24,000 lb./in. ²
Bearing on pins in members subject to reversal of stress, using the greater of the two stresses	12,000 lb./in. ²
Bearing on shop-driven rivets and stiffeners of girders, and other parts in contact	27,000 lb./in. ²
Bearing on concrete masonry	500 lb./in. ²
Bearing on sandstone and limestone masonry	400 lb./in. ²
Bearing on expansion rollers in pounds per lineal inch, where d = diameter of roller in inches	600 d .

APPLICATIONS

257. In a single-riveted lap joint calculate the pitch of the rivets and the distance from the center of the rivets to the edge of the plate under the assumption that the diameter of the rivets is twice as great as the thickness of the plate.

Solution. Consider a strip of width equal to the rivet pitch, that is, a strip containing one rivet. Let q denote the unit shearing strength of the rivet and p the unit tensile strength of the plate. Then if h denotes the thickness of the plate, in order that the shearing strength of the rivet may be equal to the tensile strength of the plate along the line of rivet holes, we must have

$$\frac{\pi d^2}{4} q = (c - d) h p.$$

Since the rivet is usually of better material than the plate, we may assume that the ultimate shearing strength of the rivet is equal to the ultimate tensile strength of the plate; that is, assume that $p = q$. Under this assumption the above relation becomes

$$\frac{\pi d^2}{4} = (c - d) h = (c - d) \frac{d}{2};$$

whence

$$c = 2.5 d, \text{ approximately.}$$

Similarly, in order that the joint may be equally secure against shearing off the rivet and pulling it out of the plate, that is, shearing the plate in front of the rivet, the condition is

$$\frac{\pi d^2}{4} q = 2 \left(a - \frac{d}{2} \right) h q',$$

FIG. 106. Detail of column riveting with Bethlehem I-beams and H columns

FIG. 107. Riveted joints in shop-building construction with Bethlehem wide-flange beams used for columns and crane girders

where a denotes the *margin* or distance from center of rivets to edge of plate, and q' denotes the ultimate shearing strength of the plate. Assuming that $q' = \frac{4}{3}q$ and $h = \frac{d}{2}$, and solving the resulting expression for a , we have

$$a = 1.5d.$$

258. Find the required rivet pitch for a single-riveted lap joint with $\frac{1}{2}$ -in. steel plates and $\frac{3}{4}$ -in. steel rivets, in order that the joint shall be of equal strength in shear and tension.

Solution. For a strip of length c the strength in shear is $\frac{\pi d^2}{4}q$, and in tension is $(c-d)hp$. Hence, to satisfy the given condition,

$$(c-d)hp = \frac{\pi d^2}{4}q,$$

whence
$$c = \frac{\pi d^2 q}{4hp} + d = \frac{\pi (\frac{3}{4})^2 50,000}{4(\frac{1}{2}) 60,000} + \frac{3}{4} = 1.486, \text{ say } 1\frac{1}{2} \text{ in.}$$

Also, efficiency of joint is then

$$e = \frac{\text{sectional area through rivet holes}}{\text{sectional area unriveted plate}} = \frac{c-d}{c} = \frac{.75}{1.50} = 50 \text{ per cent.}$$

259. Determine the maximum diameter of rivet in terms of thickness of plate so that the crushing strength of the joint shall not be less than its shearing strength.

Solution. It is customary to assume the ultimate crushing strength of the material as about twice its ultimate shearing strength, or say 100,000 lb./in.² Let

p_c = ultimate crushing strength,

p_t = ultimate tensile strength,

q = ultimate shearing strength.

Then, if the shearing strength of the rivets is to equal their crushing strength, we have for **lap joints**

$$\frac{\pi d^2}{4}q = dhp_c \quad \text{and} \quad p_c = 2q;$$

whence

$$d = 2.54h.$$

A larger rivet than this will crush before it shears, whereas a smaller one will shear before it crushes. Therefore the crushing strength of a lap joint need not be considered when $d < 2.54h$, as it usually is.

For a rivet in double shear, assuming that the strength of the rivet in double shear is twice that of the same rivet in single shear,* we have for **butt joints**

$$2 \cdot \frac{\pi d^2}{4}q = dhp_c \quad \text{and} \quad p_c = 2q;$$

whence

$$d = 1.273h.$$

If the diameter exceeds this value, the rivet will fail by crushing, whereas if it is smaller, it will fail by shear. Consequently the crushing strength of a butt joint need not be considered when $d < 1.273h$.

* This seems to be substantiated by experiment, although the English Board of Trade specifies that a rivet in double shear shall be assumed to be only 1.75 times as strong as if in single shear.

260. Design a double-riveted butt joint to have an efficiency of 75 per cent, using 1-in. steel plates and steel rivets.

Solution. For any strip along the joint of length equal to the pitch c , two rivets are in double shear. Hence for equal strength in tension and shear we have

$$(c - d)hp = 4 \frac{\pi d^2}{4} q.$$

Also, $\frac{c - d}{c} = e$, whence $d = c(1 - e)$ and $c - d = ce$. Hence, substituting these values of $c - d$ and d in the first equation, the result is

$$cehp = \pi c^2 (1 - e)^2 q;$$

whence
$$c = \frac{ehp}{\pi (1 - e)^2 q} = 4.586 \text{ in.},$$

or say
$$c = 4 \frac{5}{8} \text{ in.}$$

Then
$$d = c(1 - e) = 1.156, \text{ or say } d = 1 \frac{3}{16} \text{ in.}$$

From the results of problem 259 it is apparent that the crushing strength of the joint need not be considered, since here $d < 1.273 h$.

The butt straps should apparently each be half as thick as the plate, but when so designed they are found to be the weakest part of the joint. It is therefore customary to make the thickness of the butt straps about $\frac{5}{8} h$, or, in the present case, $\frac{5}{8}$ in.

261. In a double-riveted lap joint the plates are $\frac{1}{2}$ in. thick, rivets $\frac{7}{8}$ in. in diameter, and pitch 3 in. Calculate the efficiencies of the joint and determine how it will fail.

262. A boiler shell is to be 4 ft. in diameter, with double-riveted lap joints, and is to carry a steam pressure of 90 lb./in.² with a factor of safety of 5. Determine the thickness of shell plates and diameter and pitch of rivets. Also calculate the efficiency of the joint.

263. A cylindrical standpipe is 75 ft. high and 25 ft. inside diameter, with double-riveted, two-strap butt joints. Determine the required thickness of plates near the bottom for a factor of safety of 5, and also the diameter and pitch of rivets.

264. A boiler shell $\frac{3}{8}$ in. thick and 5 ft. in diameter has longitudinal, single-riveted lap joints, with 1-in. rivets and $2\frac{1}{4}$ -in. rivet pitch. Calculate the maximum steam pressure which can be used with a factor of safety of 5.

265. A cylindrical standpipe 80 ft. high and 20 ft. inside diameter is made of $\frac{1}{2}$ -in. plates at the base, with longitudinal, single-riveted, two-strap butt joints, connected by 1-in. rivets with a pitch of $3\frac{1}{2}$ in. Compute the factor of safety when the pipe is full of water.

266. Determine the diameter and pitch of rivets required to give the strongest single-riveted lap joint, using $\frac{7}{8}$ -in. steel plates and steel rivets, and calculate the efficiency of the joint.

SECTION XIII

REËNFORCED CONCRETE

94. Physical properties. The use of concrete dates from the time of the Romans, who obtained a good artificial stone from a mixture of slaked lime, volcanic dust, sand, and broken stone. The modern use of concrete, however, is of comparatively recent development, its universal use being a matter of only the last quarter of a century, while reënforced concrete is of still more recent origin.

Concrete is made by mixing broken stone, varying in size from a walnut to a hen's egg, with clean, coarse sand and Portland cement, using enough water to make a mixture of the consistency of heavy cream. The proportion of these three materials depends on their relative size; in general, enough sand being needed to fill the voids in the broken stone and enough cement to fill the voids in the sand. The cement and water cause the mass to begin to stiffen in about half an hour, and in from ten to twenty-four hours it becomes hard enough to resist pressure with the thumb. In a month the mixture becomes thoroughly hard, although the hardness continues gradually to increase for some time.

Portland cement was invented by Joseph Aspdin of Leeds, England, who took out a patent for its manufacture in 1824, the name Portland being due to its resemblance to a popular limestone quarried in the Isle of Portland. Its manufacture was begun in 1825, but its use did not become general until 1850, when the French and the Germans became active in its scientific production and succeeded in greatly improving both the method of manufacture and the quality of the finished product. Portland cement was first brought to the United States in 1865, but not until 1896 did its annual domestic production reach a million barrels.

When ordinary limestone (calcium carbonate) is heated to about 800° F., carbon dioxide is driven off, leaving an oxide of calcium called **quicklime**. This has a great affinity for water, and when

combined with it is said to be **slaked**. Slaked lime when dry falls into a fine powder.

Lime mortar is formed by mixing slaked lime with a large proportion of sand. Upon exposure to the air this mortar becomes hard by reason of the lime combining with carbon dioxide and forming again calcium carbonate, the product being a sandy limestone. Lime mortar is used in laying brick walls and in structures where the mortar will not be exposed to water, since it will not set, that is, combine with carbon dioxide, under water.

When limestone contains a considerable amount of clay, the lime produced is called **hydraulic lime**, for the reason that mortar made by using it will harden under water. If the limestone contains about 30 per cent of clay and is heated to 1000° F., the carbon dioxide is driven off, and the resulting product, when finely ground, is called **natural cement**. When about 25 per cent of water is added, this cement hardens because of the formation of crystals of calcium and aluminium compounds.

If limestone and clay are mixed in the proper proportions, usually about three parts of lime carbonate to one of clay, and the mixture roasted to a clinker by raising it to a temperature approaching 3000 F., the product, when ground to a fine powder, is known as **Portland cement**. The proper proportion of limestone and clay is determined by finding the proportions of the particular clay and stone that will make perfect crystallization possible. In the case of natural cement the lime and clay are not present in such proportions as to form perfect crystals, and consequently it is not as strong as Portland cement.

The artificial mixing of the limestone and clay in the manufacture of Portland cement is accomplished in different ways. Throughout the north central portion of the United States large beds of marl are found, and also in the same localities beds of suitable clay. This marl is nearly pure limestone and is mixed with the clay when wet. (These materials are also mixed dry.) Both the marl and clay are pumped to the mixer, where they are mixed in the proper proportions. The product is then dried, roasted, and ground. Most American Portland cements, however, are made by grinding a clay-bearing limestone with sufficient pure limestone to give the

proper proportions. After being thoroughly mixed, the product is roasted and ground to a powder.

Slag cement (puzzolan) is made by thoroughly mixing with slaked lime the granulated slag from an iron blast furnace and then grinding the mixture to a fine powder. Slag cements are usually lighter in color than the Portland cements and have a lower specific gravity, the latter ranging from 2.7 to 2.8. They are also somewhat slower in setting than the Portland cements and have a slightly lower tensile strength. They are not adapted to resist mechanical wear, such as would be necessary in pavements and floors, but are suitable for foundations or any work not exposed to dry air or great strain.

True Portland cement may be made from a mixture of blast-furnace slag and finely powdered limestone, the mixture being burned in a kiln and the resultant clinker ground to powder. Both the Portland and the puzzolan cements will set under water; that is, they are hydraulic.

Gravel or broken stone forms the largest part of the mass of a good concrete and is called the **coarse aggregate**. Its particles may be from $\frac{1}{4}$ in. to $\frac{3}{4}$ in. in diameter for thin walls or where reinforcement is used, or up to $2\frac{1}{2}$ in. for heavy foundations or walls over a foot thick. The coarse aggregate should always be *clean* and *hard*.

The sand, or **fine aggregate**, should be *clean* and *coarse*; that is, a large proportion of the grains should measure $\frac{1}{32}$ to $\frac{1}{8}$ in. in diameter. All should pass through a screen of $\frac{1}{4}$ -in. mesh. Too fine a sand weakens the mixture and requires a larger proportion of cement.

The following standard proportions may be taken as a guide to the proper mixture for various classes of work:*

1. **A rich mixture** for columns and other structural parts subjected to high stresses or required to be exceptionally water-tight. Proportions 1 : $1\frac{1}{2}$: 3; that is, one barrel (4 bags) of packed Portland cement to one and one half barrels (5.7 cu. ft.) of loose sand to three barrels (11.4 cu. ft.) of loose gravel or broken stone.

* Taylor and Thompson, "Concrete Plain and Reinforced"; also, Atlas Portland Cement Co., "Concrete Construction," and Turneaure and Maurer, "Principles of Reinforced Concrete Construction," p. 10.

2. **A standard mixture** for reinforced floors, beams, and columns, for arches, for reinforced engine or machine foundations subject to vibrations, and for tanks, sewers, conduits, and other water-tight work. Proportions 1 : 2 : 4 ; that is, one barrel (4 bags) of packed Portland cement to two barrels (7.6 cu. ft.) of loose sand to four barrels (15.2 cu. ft.) of loose gravel or broken stone.

3. **A medium mixture** for ordinary machine foundations, retaining walls, abutments, piers, thin foundation walls, building walls, ordinary floors, sidewalks, and sewers with heavy walls. Proportions 1 : $2\frac{1}{2}$: 5 ; that is, one barrel (4 bags) of packed Portland cement to two and one half barrels (9.5 cu. ft.) of loose sand to five barrels (19 cu. ft.) of loose gravel or broken stone.

4. **A lean mixture** for unimportant work in masses, for heavy walls, for large foundations supporting a stationary load, and for backing for stone masonry. Proportions 1 : 3 : 6 ; that is, one barrel (4 bags) of packed Portland cement to three barrels (11.4 cu. ft.) of loose sand to six barrels (22.8 cu. ft.) of loose gravel or broken stone.

95. **Design of reinforced concrete beams.** Since concrete is a material which does not conform to Hooke's law and moreover does not obey the same elastic law for tension as for compression, the exact analysis of stress in a plain or reinforced concrete beam would be much more complicated than that obtained under the assumptions of the common theory of flexure. The physical properties of concrete, however, depend so largely on the quality of material and workmanship, that for practical purposes the conditions do not warrant a rigorous analysis. The following simple formulas, although based on approximate assumptions, give results which agree closely with experiment and practice.

Consider first a plain concrete beam, that is, one without reinforcement. The elastic law for tension is in this case (see Fig. 108)

$$\frac{p_t^{m_1}}{s_t} = E_t$$

and for compression

$$\frac{p_c^{m_2}}{s_c} = E_c$$

To simplify the solution, however, assume the straight-line law of distribution of stress ; that is, assume $m_1 = m_2 = 1$. Note, however,

that this does not make the moduli equal. Assume also that cross sections which were plane before flexure remain plane after flexure (Bernoulli's assumption), which leads to the relation

$$\frac{s_c}{s_t} = \frac{e_c}{e_t},$$

where e_c and e_t denote the distances of the extreme fibers from the neutral axis (Fig. 108).

Now let the ratio of the two moduli be denoted by n ; that is, let

$$\frac{E_c}{E_t} = n.$$

Then

$$\frac{p_c}{p_t} = \frac{s_c E_c}{s_t E_t} = n \frac{e_c}{e_t}.$$

FIG. 108

For a section of unit width the resultant compressive stress R_c on the section is $R_c = \frac{1}{2} p_c e_c$, and similarly the resultant tensile stress R_t is $R_t = \frac{1}{2} p_t e_t$. Also, since R_c and R_t form a couple, $R_c = R_t$. Hence

$$p_c e_c = p_t e_t \text{ or } \frac{p_c}{p_t} = \frac{e_t}{e_c}, \text{ and, equating this}$$

to the value of the ratio $\frac{p_c}{p_t}$ obtained above, we have

$$e_t = e_c \sqrt{n}.$$

Since the total depth of the beam h is $h = e_c + e_t$, we have, therefore, $e_c = h - e_c \sqrt{n}$, whence

$$e_c = \frac{h}{1 + \sqrt{n}};$$

and similarly $e_t = h - \frac{e_t}{\sqrt{n}}$, whence

$$e_t = \frac{h}{1 + \frac{1}{\sqrt{n}}}.$$

Now, by equating the external moment M to the moment of the stress couple, we have

$$M = \left(\frac{1}{2} p_c e_c\right) \frac{2}{3} h \quad \text{or} \quad M = \left(\frac{1}{2} p_t e_t\right) \frac{2}{3} h;$$

whence, by solving for the unit stresses p_c and p_s ,

$$p_c = \frac{3M}{h^2}(1 + \sqrt{n}), \quad p_s = \frac{3M}{h^2}\left(1 + \frac{1}{\sqrt{n}}\right);$$

or, solving one of these two relations for h , say the first, we have

$$h = \sqrt{\frac{3M}{p_c}(1 + \sqrt{n})}.$$

For ordinary concrete n may be taken as 25. Also, using a factor of safety of 8, the working stress p_c becomes $p_c = 300 \text{ lb./in.}^2$. Substituting these numerical values in the above, the formula for the depth of the beam in terms of the external moment takes the simple form

$$h = \frac{\sqrt{M}}{4},$$

h being expressed in inches, and M in inch-pounds per inch of width of beam.

For a reinforced concrete beam the tensile strength of the concrete may be neglected. Let E_c and E_s denote the moduli of elasticity for concrete and steel respectively, and let $\frac{E_s}{E_c} = n$. Then, if x denotes the distance of the neutral axis from the top fiber (Fig. 109), the assumptions in this case are expressed by the relations

$$\frac{s_c}{s_s} = \frac{x}{h-x}, \quad \frac{p_c}{s_c} = E_c, \quad \text{and} \quad \frac{p_s}{s_s} = E_s,$$

whence

$$\frac{s_c}{s_s} = \frac{p_c E_s}{p_s E_c} = n \frac{p_c}{p_s} = \frac{x}{h-x};$$

or, solving for x ,

$$x = h \frac{np_c}{p_s + np_c}.$$

Now if A denotes the area of steel reinforcement per unit width of beam, then

$$R_s = p_s A \quad \text{and} \quad R_c = \frac{1}{2} p_c x;$$

and consequently, since $R_c = R_s$,

$$\frac{1}{2} p_c x = p_s A.$$

FIG. 109

Moreover, equating the external moment M to the moment of the stress couple, we have

$$M = \frac{1}{2} p_c x \left(h - \frac{x}{3} \right) \quad \text{or} \quad M = p_s A \left(h - \frac{x}{3} \right).$$

Substituting the value of x in either one of these expressions, say the first, we have

$$M = \frac{1}{2} p_c h \frac{n p_c}{p_s + n p_c} \left(h - \frac{h}{3} \frac{n p_c}{p_s + n p_c} \right);$$

whence, solving for h ,

$$h = \frac{p_s + n p_c}{p_c} \sqrt{\frac{6 M}{n (3 p_s + 2 n p_c)}}.$$

For practical work assume $n = 15$, $p_c = 500$ lb./in.² (factor of safety of 5), and $p_s = 15,000$ lb./in.² (factor of safety of 4). Substituting these numerical values in the above, the results take the simple form

$$\begin{aligned} h &= .116 \sqrt{M}, & h &= 3 x, \\ A &= \frac{h}{180}, & x &= 60 A, \\ H &= h + \frac{d}{2} + \frac{1}{2}, \end{aligned}$$

where H denotes the total depth of the beam in inches, d is the diameter of the reënforcement in inches, and M is the external moment in inch-pounds per inch of width.

In designing beams by these formulas first find h , then A , and finally H .

96. Calculation of stirrups, or web reënforcement. For a beam reënforced with horizontal rods only, that is, having no vertical or web reënforcement, the ultimate shearing strength is found to be about 100 lb./in.², calculated as the average shearing stress on the cross section. The working stress in shear for the concrete is therefore assumed to be 25 or 30 lb./in.², equivalent to a factor of safety of 3 or 4.

If the average shear on any cross section exceeds 30 lb./in.², vertical, or web, reënforcement is required, usually supplied in the

form of stirrups, or loops (Fig. 110). It can be shown that the maximum shear in a beam is inclined at an angle of 45° to the axis of the beam.* Therefore to be effective, vertical stirrups cannot be spaced farther apart than the depth of the beam. In actual practice it is customary to make the distance apart about one half this amount, or $\frac{h}{2}$, where h denotes the depth of the beam.

Since the maximum shear is inclined at an angle of 45° to the vertical, the effective area of the stirrups is $\sqrt{2}$ times their cross-sectional area; but since the maximum shear is also approximately equal to $\sqrt{2}$ times the average shear,† it is usual simply to design the stirrups to carry the average shear.

FIG. 110

For instance, suppose that the maximum shear on any cross section of a beam has an average value of 100 lb./in.^2 , which, as shown by the results of tests, is about the maximum limit for good work. Then, assuming that the concrete carries 30 lb./in.^2 of this shear, the vertical stirrups must be designed to carry the remainder, or 70 lb./in.^2 . Therefore if the beam is of breadth b and depth h , the total shear to be carried by the stirrups is $70bh$, and consequently for a working stress of $15,000 \text{ lb./in.}^2$ in the steel, the required area is

$$A_s = \frac{70bh}{15,000} = .0047bh,$$

or .47 per cent of the total area of the cross section. Since the stirrups are usually in the form of a double loop, the required

* Slocum and Hancock, *Strength of Materials*, revised edition, article 28, p. 25, Ginn and Company.
† Ibid., article 55, pp. 59 and 60.

cross-sectional area of each stirrup rod is .23 per cent of the total area of the cross section.

Inclined reënforcing rods, formed by bending up part of the horizontal bottom rods at an angle of 45° , are usually too large and too far apart to form an effective web reënforcement, but a combination of the two, as shown in Fig. 110, constitutes the most effective design.

97. Reënforced concrete columns. It is seldom necessary to design reënforced concrete columns by the formulas for long columns. In ordinary construction the ratio of length to least width seldom exceeds 12 or 15, while actual tests show that they may be practically considered as short blocks for ratios up to 20 or 25. The strength of a reënforced concrete column considered as a short block will therefore first be determined, and in exceptional cases this result may then be corrected by applying a general column formula.

The method of reënforcing concrete columns is either

1. by means of longitudinal rods extending the full length of the column;

2. by means of hoops or spiral bands;

3. by a combination of longitudinal rods and hoops or spirals.

Let A denote the total cross-sectional area of the column; A_c the area of the concrete; A_s the area of the steel; and p_c, p_s the safe unit stresses in the concrete and steel respectively. Then the safe load P for the column is given by

$$P = p_c A_c + p_s A_s.$$

The unit deformations of the concrete and steel corresponding to these stresses are

$$s_c = \frac{p_c}{E_c}, \quad s_s = \frac{p_s}{E_s},$$

where E_c and E_s denote Young's moduli for the concrete and steel respectively. Since the concrete and steel must deform the same amount, $s_c = s_s$, and, consequently,

$$\frac{p_s}{p_c} = \frac{E_s}{E_c} = n,$$

or

$$p_s = np_c,$$

where n denotes the ratio of the two moduli, ordinarily assumed to be 15.

It is desirable to express the load P in terms of the total area of the cross section A . For this purpose let k denote the percentage of reinforcement, or the ratio of the area of the steel to the total area; that is, let

$$k = \frac{A_s}{A}.$$

Then $A_c = A - A_s = A - kA = A(1 - k)$.

Therefore $P = p_c A_c + p_s A_s = p_c A(1 - k) + np_s kA$;

whence $P = p_c A [1 + (n - 1)k]$.

If the column was plain concrete without reinforcement, its safe load would be $P' = p_c A$. The relative strength of a plain concrete column as compared with one reinforced is therefore

$$\frac{P}{P'} = 1 + (n - 1)k.$$

Thus, if $k = 1$ per cent and $n = 15$, we have

$$1 + (n - 1)k = 1.14;$$

that is, a reinforcement of 1 per cent of metal increases the strength 14 per cent.

In the case of reinforcement in the form of hoops or spirals, the increase in strength depends on the effect of the hoops or coils in preventing lateral deformation. The results of tests show that this effect is very slight for loads up to the ultimate strength of plain concrete, but beyond this point there is a notable increase in the ultimate strength of the column. Tests on hooped columns made under the direction of Professor A. N. Talbot at the University of Illinois showed that the ultimate strength of the column in terms of the percentage of steel reinforcement may be calculated by the formulas

$$\text{for mild steel,} \quad p = 1600 + 65,000 k,$$

$$\text{for high steel,} \quad p = 1600 + 100,000 k;$$

where p denotes the stress per square inch, and k is the percentage of steel with reference to the concrete core inside the hoops. The compressive strength of plain concrete is here assumed to be 1600 lb./in.² As regards ultimate strength, the effect of the

hoop reënforcement was found to be from 2 to 4 times as great as for the same amount of metal in the form of longitudinal rods.

From extensive investigations and experiments on hooped columns, Considère has derived the formula

$$P = p_c A + 2.4 p_s k A,$$

where P = ultimate strength of the column,
 p_c = ultimate strength of the concrete,
 p_s = elastic limit of the steel.

This formula indicates that hoops or spirals are 2.4 times as effective as the same amount of metal in the form of longitudinal rods.

When longitudinal rods are used without hoops, it is necessary to tie them together at intervals to prevent them from buckling and pulling away from the concrete. The distance apart for these horizontal ties may be determined by considering the longitudinal reënforcing rods as long columns and applying Euler's formula; namely,

$$P = \frac{\pi^2 EI}{l^2}.$$

Assuming a factor of safety of 5, and taking $E = 25,000,000$ lb./in.² for wrought iron, we have

$$l^2 = \frac{\pi^2 EI}{5P} = \frac{10 \times 25,000,000 \times \frac{\pi d^4}{64}}{5 p_s \frac{\pi d^2}{4}}$$

where d denotes the diameter of the reënforcing rods, from which the unsupported length l of the rods, or distance between ties, is found to be

$$l = \frac{1750 d}{\sqrt{p_s}}.$$

For instance, suppose that a concrete column 12 in. square, carrying a load of 80,000 lb., is reënforced with four rods $\frac{7}{8}$ in. in diameter placed in the four corners, 1 in. from the outer faces. Then

$$A = 144 \text{ in.}^2, \quad A_s = 4(.6013) = 2.4052 \text{ in.}^2,$$

$$k = \frac{A_s}{A} = 1.67 \text{ per cent, } p_c = \frac{80,000}{144 [1 + 14(1.67)]} = 450 \text{ lb./in.}^2,$$

$$p_s = n p_c = 6750 \text{ lb./in.}^2$$

Therefore the distance between ties should be

$$l = \frac{1750 \cdot \frac{7}{8}}{\sqrt{6750}} = 18 \text{ in.}$$

98. Radially reinforced flat slabs. A system of floor construction without the use of beams or ribs, called the "mushroom system," has been devised by Mr. C. A. P. Turner (Fig. 111). The essential features of this system are that the floor slab is of uniform thickness throughout, the reinforcement is radial, and the column top is enlarged and reinforced with hoops. This type of construction is best adapted to large areas with few large openings. The following is a simple analysis of the chief features of the design.

FIG. 111

99. Diameter of top. In the case of a continuous floor slab supported by several columns, it is obvious that the slab will be concave downwards over the columns and concave upward in the center of each panel. Between these two extremes there must be a boundary at which there is no curvature, that is, a line of inflection. In a restrained beam of length l , bearing a uniform load, the two points of inflection occur at a distance of $0.212l$ from each support (article 45). For a continuous slab, therefore, the line of inflection for square panels may be assumed to be approximately a circle of radius between $\frac{1}{5}l$ and $\frac{1}{4}l$, where l denotes the span, or distance

center to center of columns. For practical purposes the diameter D of the column top may therefore be assumed as

$$D = \frac{7}{16} l,$$

which is approximately the mean of the above values.

There is another condition, however, which also affects the diameter of the top, namely, the distribution of slab reinforcement. If the column top is too small, there will be portions of the slab which contain no reinforcement, as shown by the triangular areas a, b, c, d (Fig. 112).

FIG. 112

FIG. 113

The arrangement shown in Fig. 113, however, has no such gaps; it requires that

$$D^2 + D^2 = (l - D)^2,$$

$$\text{or } D = \frac{l}{1 + \sqrt{2}} = 0.414 l,$$

which determines the minimum diameter of column top. If, then, D is assumed as

$$D = \frac{7}{16} l = 0.4375 l,$$

a slight overlap of the reinforcing rods is assured.

100. Efficiency of the spider hoops. The efficiency of tensile reinforcement in the form of hoops as compared with direct reinforcement may be obtained approximately as follows:

In the case of column tops as here considered, take a diametral section of the top and consider half of one hoop and the portion of the material reinforced by this segment, as shown in Fig. 114. Let w denote the radial stress, or pressure on the inside of the hoop per unit of length, expressed in pounds per linear inch of hoop. Also let r denote the radius of the hoop, A its cross-sectional area, and p the unit stress in the metal. Then the radial force acting on any portion of the hoop of length Δs is $w\Delta s$, and the component of this force perpendicular to the plane of the section is $w\Delta s \sin \alpha$. Or, if Δx denotes the projection of Δs on the diameter, then $\Delta s \sin \alpha = \Delta x$, and this component of the force therefore becomes

$w\Delta x$. Therefore, equating the tension in the hoop to the sum of the components of the radial stress perpendicular to the plane of the section, we have

$$2pA = \sum w\Delta x = w \sum \Delta x = w 2r;$$

whence

$$A = \frac{wr}{p}.$$

Now let A' denote the cross-sectional area of an equivalent amount of radial reënforcement. Then, since the length of the arc considered is πr and the radial stress is of amount w per unit of length, the total amount of radial reënforcement required would be given by the equation

$$pA' = \pi rw;$$

whence $A' = \frac{\pi rw}{p}.$

Comparing these expressions for A and A' , it is found

that $A' = \pi A = 3.14 A.$

FIG. 114

Consequently, the theoretical efficiency of tensile reënforcement in the form of hoops is 3.14 times as great as the same cross-sectional area of direct, or radial, reënforcement. The amount of metal in a hoop of radius r , however, is $2\pi rA$, whereas that in the radial reënforcement is $A' 2r$, and since $A' = \pi A$, these volumes are equal. Consequently, there is no saving in material effected by making the reënforcement in the form of hoops. But when there is such a complex system of reënforcement as that shown in Fig. 111, some of the metal may be used to better advantage in the form of hoops, as this lessens somewhat the congestion of metal at the columns.

101. Maximum moment. For a continuous beam of span l , carrying a total uniform load of amount W , the moment at the supports is $\frac{Wl}{12}$; whereas the moment at midspan is one half this

amount, or $\frac{Wl}{24}$. Assuming that each of the four sets of slab rods carries one fourth the total load, the bending moment from which to determine the thickness of the slab and the amount of reinforcement in the head becomes

$$M = \frac{\frac{W}{4}l}{12} = \frac{Wl}{48},$$

where W denotes the total load on the panel, and l is the distance center to center of columns. The formula determined by experiment and used in practice is $M = \frac{Wl}{50}$. For a top of diameter $D = \frac{7}{16}l$,

the moment per foot of width, say M_1 , is therefore $M_1 = \frac{Wl}{50} \div \frac{7}{16}l$,

or
$$M_1 = \frac{Wl}{50} \div \frac{7}{16}l = \frac{W}{22} \text{ ft.-lb.}$$

102. Thickness of slab. Let

p_s = unit working tensile stress in reinforcement,

p_c = unit working compressive stress in concrete,

$n = \frac{E_s}{E_c}$ = ratio of elastic moduli,

M_1 = bending moment in foot-pounds per foot of width.

Then the thickness of slab h from the outer fiber in compression to the center of the reinforcement is given by the formula (article 95)

$$h = \frac{p_s + np_c}{p_c} \sqrt{\frac{6 M_1}{n(3 p_s + 2 np_c)}}.$$

For working values of $p_s = 16,000 \text{ lb./in.}^2$, $p_c = 600 \text{ lb./in.}^2$, and $n = 15$ this becomes

$$h = 0.1026 \sqrt{M_1}.$$

Since $M_1 = \frac{W}{22}$, as explained above, this may also be written

$$h = 0.02187 \sqrt{W},$$

where h is expressed in inches and W denotes the total load on the panel in pounds.

The total thickness of slab is then found by adding to this value of h the amount needed for the reinforcing rods plus a small amount for bond below the bottom rods.

The amount of reinforcement for the unit stresses assumed above is then found from the relation (article 95)

$$A = 0.081 h,$$

where h is expressed in inches and A in square inches per foot of width.

103. Area of slab rods. As stated above, the moment at the center of the slab is half as great as at the supports. The effect of this on the required dimensions of the slab and reinforcement would be to divide both h and A by $\sqrt{2}$. But since the slab is necessarily of the same thickness throughout and hence is thicker at the center than necessary, the cross-sectional area of the reinforcement in the slab may be lessened, so as to make the moment of the stress in the reinforcement equal to the moment required for the thinner slab. If, then, p denotes the unit stress in the metal, where p is assumed to be the same in both cases, and A' denotes the cross-sectional area of metal actually required, the condition that the moment shall be constant is

$$kpA'h = kp \frac{A}{\sqrt{2}} \cdot \frac{h}{\sqrt{2}};$$

whence

$$A' = \frac{A}{2}.$$

Consequently, the design is made up by placing half the required cross-sectional area, obtained from the formula $A = 0.081 h$, in the slab rods and the other half in the hoops and spider in the column top.

104. Application of formulas. To illustrate the use of these formulas, consider a floor system with panels 20 ft. square, carrying a live load of 200 lb./ft.² By a preliminary calculation it is found that the floor slab will be about 9 in. thick, giving a dead load of 115 lb./ft.² The total live and dead load is therefore 315 lb./ft.² Consequently, $W = 20^2 \times 315 = 126,000$ lb., and

$$h = 0.02187 \sqrt{126,000} = 7.76 \text{ in.}$$

The total area of reënforcement in the head for each radial system is then $A = 0.081 h = 0.628$ sq. in. per foot.

Since the diameter of the top is assumed as $D = \frac{7}{16} l = 8.75$ ft., the total area required for one radial system is $8.75 A = 5.5$ sq. in.

The required area of slab rods in one system is then

$$\frac{5.5}{2} = 2.75 \text{ sq. in.,}$$

equivalent to 18 rods $\frac{7}{16}$ in. in diameter, spaced 5 in. apart.

Since the four sets of rods overlap where they cross the column top, and since h denotes the distance from the extreme fiber in compression to the center of the reënforcement, the total thickness of slab becomes $h + 2d + \frac{1}{2}$ in. = 9 in.

Since the hoops around the column head are assumed to be π times as effective as the same cross-sectional area of radial reënforcement, the total area of the hoops, neglecting the spider, is $\frac{2.75}{3.14} = 0.88$ sq. in. If two hoops are used, the diameter of each hoop rod may therefore be assumed as $\frac{3}{4}$ in., giving a total cross-sectional area of 0.88 sq. in.

It should be noted, however, that the effectiveness of hoop reënforcement depends on the hoop being placed where it can carry the tensile stress in the column top. Since the outermost hoop of the top is placed as nearly as possible on the line of inflection, there is practically no stress in the slab at this point, except shear, and hence the cross-sectional area of the outer hoop should be neglected in dimensioning the top hoops.

105. Dimension table. By the use of the above formulas as just explained, the accompanying table has been calculated, giving the required thickness of slab and also the size and number of slab rods for various spans and loads.

In this table the thickness of concrete below the bottom of the rods has been assumed as about $\frac{1}{2}$ in., which has been proved by experiment to be sufficient for fireproofing purposes. Some fireproofing specifications require more, however, in which case it will be necessary to increase the thickness of slab given in the table to the required amount.

TABLE FOR FLAT-SLAB SYSTEM

(r. in table indicates round rods)

$$\left\{ \begin{array}{l} h = 0.02187 \sqrt{W} \\ A = 0.081 h \end{array} \right.$$

Calculated from formulas $\left\{ \begin{array}{l} \text{Column top, } D = \frac{7}{16} l \end{array} \right.$

LIVE LOAD	80 lb./ft. ²	100 lb./ft. ²	125 lb./ft. ²	150 lb./ft. ²	200 lb./ft. ²	250 lb./ft. ²	300 lb./ft. ²
Panel 14 × 14 ft. Column top 6 ft. 2 in.	5-in. slab 65 lb. D.L. 13- $\frac{5}{16}$ -in. r.	5-in. slab 65 lb. D.L. 14- $\frac{5}{16}$ -in. r.	5 $\frac{1}{4}$ -in. slab 70 lb. D.L. 15- $\frac{5}{16}$ -in. r.	5 $\frac{3}{4}$ -in. slab 75 lb. D.L. 11- $\frac{3}{8}$ -in. r.	6 $\frac{1}{4}$ -in. slab 80 lb. D.L. 12- $\frac{3}{8}$ -in. r.	6 $\frac{3}{4}$ -in. slab 85 lb. D.L. 13- $\frac{3}{8}$ -in. r.	7 $\frac{1}{4}$ -in. slab 90 lb. D.L. 14- $\frac{3}{8}$ -in. r.
Panel 16 × 16 ft. Column top 7 ft.	5 $\frac{3}{4}$ -in. slab 75 lb. D.L. 12- $\frac{3}{8}$ -in. r.	6-in. slab 75 lb. D.L. 13- $\frac{3}{8}$ -in. r.	6 $\frac{1}{4}$ -in. slab 80 lb. D.L. 14- $\frac{3}{8}$ -in. r.	6 $\frac{1}{2}$ -in. slab 85 lb. D.L. 15- $\frac{3}{8}$ -in. r.	7-in. slab 90 lb. D.L. 16- $\frac{3}{8}$ -in. r.	7 $\frac{1}{2}$ -in. slab 95 lb. D.L. 17- $\frac{3}{8}$ -in. r.	8-in. slab 100 lb. D.L. 14- $\frac{7}{16}$ -in. r.
Panel 18 × 18 ft. Column top 8 ft.	6 $\frac{1}{4}$ -in. slab 80 lb. D.L. 15- $\frac{3}{8}$ -in. r.	6 $\frac{1}{2}$ -in. slab 85 lb. D.L. 16- $\frac{3}{8}$ -in. r.	7-in. slab 90 lb. D.L. 18- $\frac{3}{8}$ -in. r.	7 $\frac{1}{2}$ -in. slab 95 lb. D.L. 14- $\frac{7}{16}$ -in. r.	8-in. slab 100 lb. D.L. 15- $\frac{7}{16}$ -in. r.	8 $\frac{3}{4}$ -in. slab 110 lb. D.L. 17- $\frac{7}{16}$ -in. r.	9 $\frac{1}{2}$ -in. slab 120 lb. D.L. 14- $\frac{1}{2}$ -in. r.
Panel 20 × 20 ft. Column top 8 ft. 9 in.	7-in. slab 90 lb. D.L. 14- $\frac{7}{16}$ -in. r.	7 $\frac{1}{4}$ -in. slab 90 lb. D.L. 15- $\frac{7}{16}$ -in. r.	7 $\frac{3}{4}$ -in. slab 95 lb. D.L. 16- $\frac{7}{16}$ -in. r.	8 $\frac{1}{4}$ -in. slab 105 lb. D.L. 17- $\frac{7}{16}$ -in. r.	9-in. slab 115 lb. D.L. 18- $\frac{7}{16}$ -in. r.	9 $\frac{1}{2}$ -in. slab 120 lb. D.L. 16- $\frac{1}{2}$ -in. r.	10 $\frac{1}{2}$ -in. slab 130 lb. D.L. 17- $\frac{1}{2}$ -in. r.

106. Dimensions of spider. The size and number of rods in the spider are determined from the condition that the total area shall be sufficient to carry the shear, as will now be shown.

It is customary to place a fillet at the top of each column, as shown in Fig. 111, the depth of the fillet being not less than the thickness of the slab, and its diameter about twice the diameter of the column. The area of concrete in shear at the face of the column is then $\pi d(2t)$, where d denotes the diameter of the column and t the thickness of the slab, and at the outside of the fillet is $\pi(2d)t$, which is the same amount. Consequently, a fillet of these dimensions doubles the area of concrete in shear. The chief purpose of the fillet, however, is to avoid the weakening effect of an angle and thus enable the slab to develop its full strength at its junction with the column.

Since there are more slab rods to carry the shear at the outside of the fillet than at the face of the column, the latter section is weakest in shear. To illustrate the method of dimensioning for shear, consider the same numerical problem as above; namely, a panel 20×20 ft. with a total live and dead load of 320 lb./ft.² The total load on each column, or the total shear, is then $320 \times 400 = 128,000$ lb. Assuming that the columns supporting the floor are 1 ft. in diameter, and taking a cylindrical section at the face of the column, the total area of concrete in shear, including the fillet, will be $\pi \times 12 \times 2 \times 9.5 = 716$ sq. in. For a working stress in shear of 30 lb./in.², the shearing strength of the concrete alone is therefore $716 \times 30 = 21,480$ lb.

The amount of metal in the slab rods was determined previously as 0.628 in.²/ft. Since the column is 1 ft. in diameter, and since each set of rods is in double shear and there are four sets of rods, the total area of metal in shear at the surface of the column is $8 \times 0.628 = 5$ sq. in. Assuming the working stress in shear of the metal as 10,000 lb./in.², the shearing strength developed by the slab rods alone is $5 \times 10,000 = 50,000$ lb. Since the total load is 126,000 lb., there still remains to be taken care of $126,000 - (21,480 + 50,000) = 54,520$ lb. of shear.

To design the spider to carry this shear, assume that it is made up of 8 rods, as shown in Fig. 111. Then the amount of shear on

each rod will be $\frac{54,520}{8} = 6815$ lb., and hence the required area of

each rod will be $\frac{6815}{10,000} = 0.68$ sq. in., giving a rod slightly less than 1 in. in diameter. If the fillet under the head is neglected in computing the shearing strength of the concrete, a spider made up of eight 1-in. rods will still give sufficient area to develop the required shearing strength.

APPLICATIONS

267. A plain concrete beam 6 in. $\times$ 6 in. in cross section, and with a 68-in. span, is supported at both ends and loaded in the middle. The load at failure is 1008 lb. Find the maximum fiber stress.

268. A concrete building block 24 in. in length and having an effective cross section of 8 in. $\times$ 10 in. minus 4 in. $\times$ 6 in. is tested by being supported at points 2 in. from each end and loaded in the middle. The load at failure is found to be 5000 lb. Find the maximum fiber stress, the height of the block being 10 in.

269. A reinforced concrete beam 10 in. wide and 22 in. deep has four $1\frac{1}{4}$ -in. round bars with centers 2 in. above the lower face. The span is 16 ft. The beam is simply supported at the ends. Find the safe load per linear foot for a working stress in the concrete of 500 lb./in.², and also find the tensile stress in the reinforcement.

270. A reinforced-concrete beam of 16 ft. span is 12 in. wide and has to support a uniform load of 1000 lb. per linear foot. Determine the total depth and amount of steel reinforcement required, bars to have centers 2 in. above the lower face of beam.

271. A reinforced concrete beam 8 in. $\times$ 10 in. in cross section, and 15 ft. long, is reinforced on the tension side by six $\frac{1}{2}$ -in. plain steel rounds. The steel has a modulus of elasticity of 30,000,000 lb./in.² and the center of the reinforcement is placed 2 in. from the bottom of the beam. Assuming that $E_c = 3,000,000$ lb./in.², and $p_c = 600$ lb./in.², find the position of the neutral axis and the moment M .

272. For a stress $p_c = 2700$ lb./in.² on the outer fiber of concrete in the beam given in problem 271, find the stress p_s in the steel reinforcement.

273. A concrete beam is 10 $\times$ 16 in. in cross section and 20 ft. long. It is reinforced with four $\frac{3}{4}$ -in. steel rods with centers 2 in. above the lower face of the beam. The safe compressive strength of the concrete is 600 lb./in.², and the steel used has an elastic limit of 40,000 lb./in.² What single concentrated load will the beam carry at its middle? What tension will be developed in the steel? What shearing stress along the reinforcement?

274. Find what load, uniformly distributed, the beam in the preceding problem will carry and find the tension in the steel and bond for this case.

275. A reinforced concrete floor is to carry a load of 200 lb./ft.² over panels 14 ft. square. Find the required thickness of the slab and the area of the reinforcement for working stresses of 500 lb./in.² in the concrete and 15,000 lb./in.² in the reinforcement.

276. Design a floor panel 14 ft. square, to be made of reinforced concrete and to sustain a total uniform load of 120 lb./ft.², with a factor of safety of 4.

SECTION XIV

SIMPLE STRUCTURES

107. Composition and resolution of forces. It will now be necessary to recall some of the results previously obtained concerning the composition and resolution of forces.

It was shown in article 11 that any number of concurrent forces may be combined by means of a vector triangle or vector polygon into a single resultant. Also that, conversely, any force may be resolved into components forming with the given force a closed triangle or polygon.

FIG. 115

Thus, in Fig. 115, if F_1 , F_2 , denote two forces and R their resultant, resolve F_1 into rectangular components x_1 , y_1 , and F_2 into components x_2 , y_2 . Then, if x , y , denote the components of the resultant R , we have

$$x = x_1 + x_2, \quad y = y_1 + y_2,$$

and, consequently,

$$R = \sqrt{x^2 + y^2} = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}.$$

In article 10 the moment of a force with respect to any point was defined as the product of the force by its perpendicular distance from the point in question; that is,

$$\text{Moment} = \text{force} \times \text{lever arm.}$$

It was also proved that the sum of the moments of any number of forces lying in the same plane with respect to a point in this plane is equal to the moment of their resultant with respect to this point.

It remains to consider the case when the system of forces lie in the same plane but are not concurrent, that is, do not all meet in a point. This involves the properties of a **force couple**, defined as two equal and opposite parallel forces F , F , not acting in the same line (Fig. 116).

For any couple F , F , let x denote the distance of any point O in its plane from the nearest force of the couple, and d the lever arm of the couple (Fig. 116). Then the moment M of the couple with respect to the point O is

$$M = F(d + x) - Fx = Fd.$$

Therefore the moment of the couple is constant and equal to Fd with respect to any point in its plane. Moreover, since the moment of the couple involves only the magnitude of the forces

FIG. 116

and their distance apart, it is evident that the couple can be revolved through any angle without altering its value. A couple may, therefore, be moved about anywhere in its plane without altering its numerical value or changing its effect in any way.

It is also obvious that the forces of a couple may be altered in amount, provided that the lever arm is at the same time changed so as to keep their product constant. Two or more couples may therefore be combined by first reducing them to equivalent couples having the same lever arm and then taking the algebraic sum of the forces, thus giving a single resultant couple with this same lever arm.

Now consider any number of forces $F_1, F_2, F_3 \dots$, lying in the same plane but not concurrent. At any arbitrary point O (Fig. 117), introduce two forces F_1', F_1'' , opposite in direction, but each equal in amount to F_1 . Since F_1' and F_1'' are equal and opposite they will not disturb the equilibrium of the system. But F_1 and F_1'' together form a couple of moment $F_1 d_1$, leaving the single force F_1' , equal

to F_1 , acting at O . Similarly, each of the other forces is equivalent to a couple plus a single force (equal and parallel to the given force) acting at O . The given force system is therefore equivalent

FIG. 117

to a system of equal but *concurrent* forces acting at O , and an equal number of couples, the moment of each couple being equal to the moment of the corresponding given force with respect to the point O .

This concurrent force system, however, may now be combined into a single resultant force, and the couples also combined into

a single resultant couple, as just explained. Consequently, we have the following general theorem:

Any system of forces lying in the same plane is equivalent to a single force acting at any assigned point in this plane plus a couple whose moment is equal to the sum of the moments of the given forces with respect to this point.

108. Conditions of equilibrium of a system of coplanar forces.

When a body acted upon by two or more forces is at rest or in uniform motion relative to any system of coördinate axes, it is said to be in **equilibrium**, and the forces acting on it are said to **equilibrate**. The conditions for equilibrium are, therefore, that the resultant force acting on the body must be zero, and that the resultant moment or couple acting on it must also be zero. That is to say, the algebraic sum of all the forces acting on the body must be zero, and the algebraic sum of the moments of these forces with respect to any point must also be zero. Expressed symbolically the conditions of equilibrium are

$$\sum F = 0, \quad \sum M = 0.$$

In general it is convenient in applying these conditions to resolve each force F into rectangular components X , Y , and replace the single condition $\sum F = 0$ by the two independent conditions $\sum X = 0$, $\sum Y = 0$.

These three conditions, $\sum X=0$, $\sum Y=0$, $\sum M=0$, are obviously both necessary and sufficient to assure equilibrium. For if the first two are satisfied, the system will be in equilibrium as regards translation, and if $\sum M=0$, it will also be in equilibrium as regards rotation; and, furthermore, it will not be in equilibrium unless all three are satisfied.

The conditions for equilibrium of a system of forces lying in the same plane may then be reduced to the following convenient form:

1. *For equilibrium against translation,*

$$\begin{cases} \sum \text{horizontal components} = 0, \\ \sum \text{vertical components} = 0. \end{cases}$$

2. *For equilibrium against rotation,*

$$\sum \text{moments about any point} = 0.$$

When a body is acted on by only three forces, lying in the same plane, the conditions for equilibrium are that these three forces shall meet in a point, and that one of them shall be equal and opposite to the resultant of the other two.

109. Equilibrium polygon. As explained above, the resultant of any system of forces lying in the same plane may be found by means of a vector force polygon, the resultant being the closing side of the polygon formed on the given system of forces as adjacent sides. Although this construction gives the magnitude and direction of the resultant, it does not determine its position or its line of action. The most convenient way to determine the line of action of the resultant is to introduce into the given system two equal and opposite forces of arbitrary amount and direction, such as P' and P'' (Fig. 118). Since P' and P'' balance one another, they will not affect the equilibrium of the given system. To find the line of action of the resultant R , combine P' and P_1 into a resultant R_1 acting along $B'A'$, parallel to the corresponding ray OB of the force polygon. Prolong $A'B'$ until it intersects P_2 and then combine R_1 and P_2 into a resultant R_2 acting along $C'B'$, parallel to the corresponding ray OC of the force polygon, etc. Proceed in this way until the last partial resultant R_4 is obtained. Then the resultant

of P' and R_4 will give the line of action, as well as the magnitude, of the resultant of the original system P_1, P_2, P_3, P_4 . The closed figure $A'B'C'D'E'F'$ obtained in this way is called an **equilibrium polygon**.

FIG. 118

For a system of parallel forces the equilibrium polygon is constructed in the same manner as above, the only difference being that in this case the force polygon becomes a straight line (Fig. 119).

FIG. 119

Since P' and P'' are entirely arbitrary in both magnitude and direction, the point O , called the **pole**, may be chosen anywhere in the plane. Therefore, in constructing an equilibrium polygon

corresponding to any given system of forces, the force polygon $ABCDE$ (Figs. 118 and 119) is first drawn, then any convenient point O is chosen and joined to the vertices A, B, C, D, E , of the force polygon, and finally the equilibrium polygon is constructed by drawing its sides parallel to the rays OA, OB, OC , etc. of the force diagram. Since the position of the pole O is entirely arbitrary, there is an infinite number of equilibrium polygons corresponding to any given set of forces. The position and magnitude of the resultant R , however, is independent of the choice of the pole, and will be the same no matter where O is placed.

For a system of concurrent forces (that is, forces which all pass through the same point) the closing of the force polygon is the necessary and sufficient condition for equilibrium. If, however, the forces are not concurrent, or if they are parallel, this condition is necessary but not sufficient, for in this case the given system of forces may be equivalent to a couple, the effect of which would be to produce rotation. To assure equilibrium against rotation, therefore, it is also necessary that the equilibrium polygon shall close.

The graphical and analytical conditions for equilibrium are then as follows:

CONDITIONS OF EQUILIBRIUM

	Analytical	Graphical
Translation	$\sum F = 0$	Force polygon closes
Rotation	$\sum M = 0$	Equilibrium polygon closes

110. Application of equilibrium polygon to determining reactions.

One of the principal applications of the equilibrium polygon is in determining the unknown reactions of a beam or truss. To illustrate its use for this purpose, consider a simple beam placed horizontally and bearing a number of vertical loads P_1, P_2 , etc. (Fig. 120). To determine the reactions R_1 and R_2 , the force diagram is first constructed by laying off the loads P_1, P_2 , etc. to scale on a line AF , choosing any convenient point O as pole and drawing the rays OA, OB , etc. The equilibrium polygon corresponding to this force diagram is then constructed, starting from any point, say A' , in R_1 .

Now the closing side $A'G'$ of the equilibrium polygon determines the line of action of the resultants P' and P'' at A' and G' respectively. For a simple beam, however, the reactions are vertical. Therefore, in order to find these reactions, each of the forces P' and P'' must be resolved into two components, one of which shall be vertical. To accomplish this, suppose that a line OH is drawn from the pole O in the force diagram parallel to the closing side $G'A'$ of the equilibrium polygon. Then HO (or P') may be replaced by its components HA and AO , parallel to R_1 and $A'B'$ respectively;

FIG. 120

and similarly, OH may be replaced by its components FH and OF , parallel to R_2 and $F'G'$ respectively. HA and FH are therefore the required reactions.

111. Equilibrium polygon through two given points. Let it be required to pass an equilibrium polygon through two given points, say M and N (Fig. 121).

To solve this problem a trial force diagram is first drawn with any arbitrary point O as pole, and the corresponding equilibrium polygon $MA'B'C'D'E'$ constructed, starting from one of the given points, say M . The reactions are then determined by drawing a line OH parallel to the closing side ME' of the equilibrium polygon, as explained in the preceding article.

The reactions, however, are independent of the choice of the pole in the force diagram, and, consequently, they must be of amount AH and HE , no matter where O is placed. Moreover, if the equilibrium

polygon is to pass through both M and N , its closing side must coincide with the line MN , and therefore the pole of the force diagram must lie somewhere on a line through H parallel to MN .

FIG. 121

Let O' be any point on this line. Then, if a new force diagram is drawn with O' as pole, the corresponding equilibrium polygon starting at M will pass through N .

112. Equilibrium polygon through three given points. Let it be required to pass an equilibrium polygon through three given points, say M , N , and L (Fig. 122).

Fig. 122

As in the preceding article, a trial force diagram is first drawn with any point O as pole, and the corresponding equilibrium polygon constructed, thus determining the reactions R_1 and R_2 as HA and EH respectively.

Now, if the equilibrium polygon is to pass through N , the pole of the force diagram must lie somewhere on a line HK drawn through H parallel to MN , as explained in the preceding article. The next step, therefore, is to determine the position of the pole on this line HK , so that the equilibrium polygon through M and N shall also pass through L . This is done by drawing a vertical LS through L and treating the points M and L exactly as M and N were treated. Thus $OABCD$ is the force diagram for this portion of the original figure, and $MA'B'C'S$ is the corresponding equilibrium polygon, the reactions for this partial figure being $H'A$ and DH' . If, then, the equilibrium polygon is to pass through L , its closing side must be the line ML , and consequently the pole of the force diagram must lie on a line $H'K'$ drawn through H' parallel to ML . The pole is therefore completely determined as the intersection O' of the lines HK and $H'K'$. If, then, a new force diagram is drawn with O' as pole, the corresponding equilibrium polygon starting from the point M will pass through both the points L and N .

Since there is but one position of the pole O' , only one equilibrium polygon can be drawn through three given points. In other words, an equilibrium polygon is completely determined by three conditions.

113. Application of equilibrium polygon to calculation of stresses. Consider any structure, such as an arch or arched rib, supporting a system of vertical loads, and suppose that the force diagram and equilibrium polygon are drawn as shown in Fig. 123. Then each ray of the force diagram is the resultant of all the forces which precede it and acts along the segment of the equilibrium polygon parallel to this ray. For instance, OC is the resultant of all the forces on the left of P_3 and acts along $C'D'$. Consequently, the stresses acting on any section of the structure, say mn , are the same as would result from a single force OC acting along $C'D'$.

Let θ denote the angle between the segment $C'D'$ of the equilibrium polygon and the tangent to the arch at the point S . Then the stresses acting on the section mn at S are due to a tangential thrust of amount $OC \cos \theta$; a shear at right angles to this, of amount $OC \sin \theta$; and a moment of amount $OC \cdot d$, where d is the perpendicular distance of $C'D'$ from S .

From Fig. 123 it is evident that the horizontal component of any ray of the force diagram is equal to the pole distance OH . Therefore, if OC is resolved into its vertical and horizontal components, the moment of the vertical component about S is zero, since it passes through this point; and hence the moment $OC \cdot d = OH \cdot z$, where z is the vertical intercept from the equilibrium polygon to the center of moments S . Having determined the moment at any given point, the stresses at this point can easily be calculated.

FIG. 123

114. Relation of equilibrium polygon to bending moment diagram.

In the preceding article it was proved that the moment acting at any point of a structure is equal to the pole distance of the force diagram multiplied by the vertical intercept on the equilibrium polygon from the center of moments. For a system of vertical loads, however, the pole distance is a constant. Consequently, the moment acting on any section is proportional to the vertical intercept on the equilibrium polygon from the center of moments. Therefore, if the equilibrium polygon is drawn to such a scale as to make this factor of proportionality equal to unity, the equilibrium polygon will be identical with the bending moment diagram for the given system of loads.

115. Structures: external forces. The external forces acting upon any stationary structure must be in equilibrium. Hence they may be found, in general, by applying the conditions of equilibrium given in article 109. The conditions of equilibrium may be applied either analytically or graphically. The former method has the advantage of being available under all circumstances; whereas the

latter method requires the accurate use of instruments, and is therefore confined chiefly to office work. Both methods are illustrated in what follows.

1. *Analytical method.* Consider first the analytical determination of the external forces acting on a simple structure, such as the loaded jib crane, shown in Fig. 124. This consists of a vertical

FIG. 124

mast ED , supported by a collar B and footstep C , and carrying a jib AD , supported by the guy AEF . The external forces acting on the crane are the load W , the counterweight W_1 (including hoisting engine and machinery), and the reactions at B and C . The reaction of the collar B can have no vertical component, as the collar is made a loose fit so that the crane may be free to

swivel. For convenience, the reaction of the footstep C may be replaced by its horizontal and vertical components H and V .

Applying the conditions of equilibrium to the structure as a whole, we have, therefore,

$$\sum \text{vertical forces} = 0, \quad W + W_1 + \text{weight of crane} - V = 0,$$

$$\sum \text{horizontal forces} = 0, \quad H_1 + H_2 = 0,$$

$$\sum \text{moments} = 0 \text{ (taken about } B), \quad Wl_2 - W_1l_1 + H_1c = 0.$$

From the first condition the vertical reaction of the footstep is found to be equal to the entire weight of the structure and its loads. In applying the last condition, moments are taken about B , since one unknown H_2 is thus eliminated, leaving the resulting moment equation with only one unknown H_1 . The other unknown H_2 is then found from the second condition, $H_2 = -H_1$.

The moment of the counterweight W_1l_1 should, when possible, be made equal to $\frac{Wl_2}{2}$, where W is the maximum load the crane is designed to lift. The mast will then never be subjected to a bending moment of more than one half that due to the lifted load; that is to

say, the horizontal reactions H_1 and H_2 will never have more than one half the value they would have if the crane was not counterweighted.

2. *Graphical method.* To illustrate this method, consider the Pratt truss, shown in Fig. 125. Assume the loads in this case to

be the weight of the truss W , a uniform load of amount W_1 , assumed for present purposes to be concentrated at its center of gravity, and two concentrated loads P_1 , P_2 . Since the only other external forces acting on the truss are the reactions R_1 , R_2 , they must hold the loads in equilibrium, and hence the force polygon must close. The force polygon, however, consists in the present case simply of a straight line 1 2 3 4 5, and therefore does not suffice to deter-

mine the values of R_1 and R_2 . For this purpose an equilibrium polygon must be drawn. Thus, choose any pole O on the force diagram and draw the rays $O1$, $O2$, $O3$, etc.; then construct the corresponding equilibrium polygon by starting from any point a in R_1 and drawing ab parallel to $O1$, from b drawing bc parallel to $O2$, etc. Having found the closing side af of the equilibrium polygon, draw through O the ray $O6$ parallel to af , thereby determining R_1 as 56 and R_2 as 61.

If, for any reason, it is desired to draw the equilibrium polygon through two fixed points, say a and f' in the figure, the reactions are first determined as above. Then a line is drawn through 6 parallel to af' , and a pole O' is chosen somewhere on this line. The closing side of the equilibrium polygon will then be parallel to $O'6$ (or af'), and hence if the polygon starts at a , it must end at f' .

FIG. 125

116. Structures: joint reactions. Since all parts of a structure at rest are in equilibrium, the conditions of equilibrium may evidently be applied to the forces acting upon any portion of the structure. This portion may be a single joint, a single member or part of a member, or it may include several joints and members. The forces acting upon the part considered may be partly external forces and partly internal forces, or stresses, or they may be wholly stresses.

As in finding external reactions, the conditions of equilibrium may be applied either analytically or graphically.

FIG. 126

1. *Analytical method.* To illustrate this method, as applied to the joints of a structure, let it be required to find the stresses in the members of the shear legs, shown in Fig. 126.

Starting with the joint A , the forces acting at this point are the weight W , the tension P in the guy AC , and the reaction of the legs of the A frame. To simplify the solution the latter may be assumed for the present equivalent to a single force R acting along the center line $A'C$ between the legs of the A frame. The conditions of equilibrium applied to this joint are then

$$\sum \text{vertical forces} = 0, \quad W + P \cos(\theta + \phi) - R \cos \theta = 0,$$

$$\sum \text{horizontal forces} = 0, \quad P \sin(\theta + \phi) - R \sin \theta = 0,$$

giving two simultaneous equations for R and P .

Since R is by assumption equivalent to the combined action of the two shear legs, the thrust T in each may be found by resolving forces along R . Thus $T \cos \alpha = \frac{1}{2} R$, which determines T , since R has already been found.

Similarly, the force at the bottom of the shear legs tending to make them spread is $T \sin \alpha$.

At the point C the forces acting are the upward pull V on the anchorage, the horizontal pull H on it, and the tension P in the guy. Hence, applying the conditions of equilibrium, we have

$$H = P \cos \beta, \quad V = P \sin \beta.$$

2. Graphical method.

To illustrate the graphical calculation of stresses from joint reactions, consider the roof truss shown in Fig. 127.

Since the loading in this case is symmetrical, the reactions of the supports will each be equal to half the weight on the truss.

The most convenient notation is to letter the spaces between the various lines of the diagram. Each member of the truss and each external force will then be designated by the adjoining letters on opposite sides of it, as the member AH , the load BC , etc.

Starting with the left support, we have three forces meeting at a point. The magnitude of one, namely R_1 , or AB , is known, and the directions of all three are known. Hence the other two can be determined by means of a triangle of forces. Thus, if ab is laid off to scale to represent R_1 , and aj , bj , are drawn from a and b parallel

FIG. 127

to AJ and BJ , they will represent the stresses in these members to the same scale as that to which R_1 was laid off (Fig. 127, *I*).

Proceeding to the next joint, $BJIC$, we have four forces meeting at a point, one of which, BJ , has just been determined, and another, BC , is known. Hence the other two are found by drawing a force polygon, $bjic$, giving the stresses in CI and IJ (Fig. 127, *II*).

Similarly, passing to the next joint, $AJII$, the stresses in AJ and JI having been found, those in IH and AI may be determined from the force polygon $ajih$ (Fig. 127, *III*), and finally for the joint HCD the remaining stresses are determined from the force polygon $ghcd$ (Fig. 127, *IV*).

Since each force polygon contains one side of each of the others, by placing these sides together they may all be combined into

FIG. 128

one figure, as shown in Fig. 127, *V*. In the present case separate diagrams were drawn for each joint to illustrate the method. In practice, however, but one diagram, the combined one, is drawn, as it affords a saving in time and space and produces a neater and more

compact appearance. Such a figure is called a **Maxwell diagram**.

117. Structures: method of sections. If a section is passed through a structure, cutting not more than two members whose stresses are unknown, the single condition that the force polygon, drawn for the forces acting upon the portion of the structure on one side of the section, must close, will enable the stresses in these members to be found. Commencing at one end of a structure and passing a section cutting but two members, the stresses in these can thus be determined. Then, passing a section cutting three members, one of which has already been treated, the stresses in the other two can be found, etc. Thus, by means of successive sections, all of the stresses can be determined by simple force polygons.

1. Analytical method. To illustrate the analytical application of this method, consider a Warren truss used as a deck bridge, as

shown in Fig. 128. Let the depth of truss and panel length be each 15 ft., and the loads carried at the joints of the upper chord be 7, 10, 9, and 15 tons respectively. The reactions at B and J are found by taking moments about J and B to be $17\frac{5}{8}$ tons and $23\frac{3}{8}$ tons respectively.

Since this form of truss has parallel chords and a single web system, it is not necessary to begin at any particular point, but a section may be taken anywhere, provided it cuts both chords and a single web member. Taking any section xy , and considering only the portion of the structure on one side of the section, the external forces acting on this portion will be in equilibrium with the stresses P , Q , R , in the members cut (Fig. 129). Since Q is the only stress having a vertical component, it must equilibrate the external forces at B and C . That is to say, from the condition of equilibrium

FIG. 129

$\sum$ vertical forces = 0, we have $Q \sin 63^\circ 26' = 17\frac{5}{8} - 7$; whence $Q = 11.854$ tons and is compressive.

To find P take moments about D . Then since Q and R both pass through D , their moments about this point are zero; therefore

$$P \times 15 = 17\frac{5}{8} \times 15 - 7 \times 7.5;$$

whence $P = 14\frac{1}{8}$ tons. By observing the signs of the moments of the external forces at B and C about D , P is found to act in the direction shown by the arrow, that is, in compression.

Similarly, to find the stress R in DE , take the section xy just to the left of E , then take moments about E . Since P and Q pass through E , their moments about this point are zero, and hence

$$R \times 15 = 17\frac{5}{8} \times 22.5 - 7 \times 15;$$

whence

$$R = 19.44 \text{ tons.}$$

Since the loads are vertical, R might also have been found from P and Q by the condition $\sum$ horizontal forces = 0; that is, $P + Q \cos 63^\circ 26' = R$; whence $R = 19.427$ tons.

2. *Graphical method.* Before proceeding with the explanation of the graphical method it will be necessary to show how the moment of any number of forces with respect to a given point may be obtained from the equilibrium polygon.

Let P_1, P_2, P_3, P_4 denote any set of forces and B the given point about which their moment is required (Fig. 130). First draw the force polygon for these forces, choose any pole O , and construct the corresponding equilibrium polygon $abcde$. Now in the force diagram, drop a perpendicular oh from the pole O on the resultant R . This is called the pole distance

of R and will be denoted by H . Also, in the equilibrium diagram draw through the given point B a line parallel to R , making the intercept xy on the equilibrium polygon. Then the triangle OAE in the force diagram is similar to the triangle xye in the equilibrium diagram, and hence

$$r : xy = H : AE,$$

or

$$Rr = H \times xy.$$

But Rr is the moment of the resultant R about B and is equal to the sum of the moments of all the given forces about this point. The following moment theorem may therefore be stated:

The moment of any system of forces about a given point is equal to the pole distance of their resultant multiplied by the intercept made by the equilibrium polygon on a line drawn through the given point parallel to the resultant.

The moment of a part of the given set of forces about any point may also be found by this theorem. For example, let it be required to find the moment of P_1 and P_2 about B . The resultant of P_1, P_2 , is given in amount by AC and acts through the point f , as shown. Hence draw through B a line parallel to this partial resultant,

FIG. 130

making the intercept mn on the equilibrium polygon. Then, since the triangles fmn and OAC are similar, we have

$$r' : mn = H' : R',$$

or

$$R'r' = H' \times mn,$$

which is the expression required by the theorem.

For a system of parallel forces the pole distance H is constant and hence the equilibrium polygon is similar to the moment diagram for the forces on either side of any given point. Therefore the moment of all the forces on one side of a given point, taken with respect to this point, is equal to the constant pole distance H multiplied by the intercept made by the equilibrium polygon on a vertical through the point in question.

To apply this method to the roof truss shown in Fig. 131, for example, draw the force polygon and the corresponding equilibrium polygon, as shown in the figure. Now take any section of the truss, such as xy in the figure, and take moments of the stresses in the members cut about one of the joints, say B . Then the condition of equilibrium

$$\sum \text{moments about } B = 0$$

may be written

$$\text{Moment of stress in } AF + \sum \text{moments of } P_1, P_2, R_1 = 0.$$

But by the above theorem

$$\sum \text{moments of } P_1, P_2, R_1, \text{ about } B = bb' \times Oh,$$

Hence

$$\text{Stress in } AF = \frac{bb' \times Oh}{BG}.$$

FIG. 131

Similarly, by taking moments about A the stress in BF is found to be

$$\text{Stress in } BF = \frac{aa' \times Oh}{AT},$$

and the stress in BC , with center of moments at F , is

$$\text{Stress in } BC = \frac{cc' \times Oh}{FS}.$$

By observing the signs of the moments the stresses in AB , BC , and BF are found to be compressive and that in AF tensile.

In the present case, from symmetry, the stresses in the remaining members of the truss are the same as in those already found. For unsymmetrical loading it would be necessary to apply the above method to each individual member.

APPLICATIONS

277. Two equal weights of 50 lb. each are joined by a cord which passes over two pulleys in the same horizontal line, distant 12 ft. between centers. A weight of 5 lb. is attached to the string midway between the pulleys. Find the sag.

278. The rule used by the makers of cableways for finding the stress in the cable is to calculate a factor = $\frac{\text{one half the span}}{\text{twice the sag}}$, and multiply the load, assumed to be at the middle, by this factor. Show how this formula is obtained.

FIG. 132

FIG. 133

279. It is usual to allow a sag in a cable equal to one twentieth of the span. What does the numerical factor in the preceding problem become in this case, and how does the tension in the cable compare with the load?

280. Find the relation between F and W and the total

pull on the upper support in the systems of pulleys shown in Fig. 132.

281. In a Weston differential pulley two sheaves, of radii a and b , are fastened together, and by means of a continuous cord passing around both and also around a movable pulley, support a weight W . Find the relation between F and W , neglecting friction (Fig. 133).

282. In a Weston differential pulley the diameters of the sheaves in the upper block are 8 in. and 9 in. Find the theoretical advantage.

283. In the differential axle shown in Fig. 134 the rope is wound in opposite directions around the two axles so that it unwinds from one and winds up on the other at the same time. Find its mechanical advantage if the radius of the large drum is R , of the small drum is r , and of the crank is c .

284. A differential screw consists of two screws, one inside the other. The outer screw works through a fixed block, and is turned by means of a lever. This screw is cored out and tapped for a smaller screw of less pitch which works through another block, free to move along the axis of the screw, but prevented from rotating. Find the mechanical advantage of such a differential screw if the lever arm is 3 ft. long, the outer screw has 8 threads to the inch, and the inner screw has 10 threads to the inch

FIG. 134

FIG. 135

285. The bed of a straight river makes an angle α with the horizontal. Taking a cross section perpendicular to the course of the river, the sides of the valley are inclined at an angle β to the horizontal. Find the angle which the tributaries of the river make with it.

286. Find the least horizontal force necessary to pull a wheel 30 in. in diameter, carrying a load of 500 lb., over an obstacle 4 in. high.

287. A steelyard weighs 6 lb. and has its center of gravity in the short arm at a distance of 1 in. from the fulcrum, and the center of suspension is 3 in. from the fulcrum. The movable weight weighs 4 lb. Find the zero graduation, and the distance between successive pound graduations.

288. A ladder 50 ft. long, weighing 75 lb., rests with its upper end against a smooth vertical wall and its lower end on rough horizontal ground. Find the reactions of the supports when the ladder is inclined 20° to the vertical.

289. A circular, three-legged table, 4 ft. in diameter, weighs 50 lb. and carries a load of 100 lb. 10 inches from the center and in a line joining the center and one leg. Find the pressure between each foot and the floor. Find also the smallest load which when hung from the edge of the table will cause it to tip over.

290. The average turning moment exerted on the handle of a screw driver is 120 in.-lb. The screw has a square slot, but the point of the screw driver is beveled to an angle of 10° (Fig. 135). If the point of the screw driver is $\frac{1}{2}$ in. wide, find the vertical force tending to raise the screw driver out of the slot.

291. Three smooth cylindrical water mains, each weighing 500 lb., are placed in a wagon box, two of them just filling the box from side to side and the third being placed on top of these two. Find the pressure between the pipes and also against the bottom and sides of the wagon.

292. An engine is part way across a bridge, the weights and distances being as shown in Fig. 136. Find the reactions of the abutments.

FIG. 136

FIG. 137

293. In the letter scales shown in Fig. 137 the length of the parallel links is 3 in., and the distance of the center of gravity of the moving parts below the pivot O is 2 in. If the radius of the scale is 8 in. and the weight of the moving parts is 12 oz., find the distance between successive ounce graduations on the scale.

294. A scale is arranged as shown in Fig. 138. Determine the relation between the load and the weight P . (Quintenz scales, Strassburg, 1821.)

FIG. 138

FIG. 139

Solution. With the given dimensions we have, by the principle of moments,

$$R_1 = W \frac{x}{l}, \quad F_1 = W \frac{l-x}{l}, \quad F_2 = R_1 \frac{d}{e+d},$$

and

$$Pa = F_1 b + F_2 c;$$

whence

$$Pa = Wb - W \frac{x}{l} \left(b - \frac{d}{e+d} c \right).$$

Since this is independent of x , the position of the load on the platform does not affect the result.

Let the dimensions be so proportioned that $\frac{b}{c} = \frac{d}{e+d}$, and also $a = 10b$. Then $P = \frac{W}{10}$. A scale so arranged is called a decimal scale.

295. In the toggle-joint press shown in Fig. 139, the length of the hand lever is $l_1 = 3\frac{1}{2}$ ft., and $l_2 = 4$ in. If the pull $P = 100$ lb., find the pressure between the

jaws of the press when the toggle is inclined at 10° to the vertical.

FIG. 140

296. In the crab hook shown in Fig. 140, assume that the load $Q = 300$ lb. and the coefficient of friction $\mu = .5$, and determine the kind and amount of strain in the members ED , AD , and AB for $\alpha = 30^\circ$, $\beta = 90^\circ$, $a = 6$ in., $b = 3$ in., and $l = 18$ in. Show also that in order to hold the weight without slipping, the condition which must be satisfied is

$$\mu \left(\frac{l}{2a \sin \alpha} - \frac{b}{2a} \right) \geq \frac{1}{2}.$$

297. A steam cylinder is 20 in. in diameter, the steam pressure is 150 lb./in.², the crank is 18 in. long, and the connecting rod is 5 cranks long. Find the stress in the connecting rod, pressure on cross-head guides, and tangential pressure on crank pin when the crank makes an angle of 45° with the horizontal on the "in end" of the stroke. Find also the maximum tangential pressure on the crank pin.

298. Calculate the stresses in all the members of the dockyard crane shown in Fig. 141 when carrying a load of 40 tons.

299. Calculate analytically the stresses in the members of the jib crane shown in Fig. 142 when lifting a load of 28 tons., the dimensions being as given in the figure.

FIG. 141

300. In the locomotive crane shown in Fig. 143, calculate the stresses in boom, mast, back stays, hoisting line, and boom line when the boom is in its lowest position, the dimensions for this case being as given in the figure.

301. Calculate the maximum stresses in all the members of the stiff-leg derrick shown in Fig. 144 when lifting a load of 10 tons, the dimensions being as follows: mast = 25 ft., boom = 38 ft., each leg = 40 ft. It is customary to load the sills with stone to give the necessary stability. Find the amount of stone required on each sill when lifting the 10-ton load,

FIG. 142

FIG. 143

FIG. 144

FIG. 145

302. The testing machine shown in Fig. 145 is designed for a maximum load on the platform of 100,000 lb. The dimensions of the various levers are as shown in Fig. 146, the lever *C* being V-shaped, with the lever *D* hung inside. Neglecting the

FIG. 146

weights of the arms, compute the weight of the slider when in its extreme position required to balance a load of 100,000 lb. on the platform.

303. Determine analytically the stresses in the members *CD*, *DE*, and *EF* of the curved-chord Pratt truss shown in Fig. 147, assuming the load at each panel point to be 50,000 lb.

FIG. 147

FIG. 148

304. The roof truss shown in Fig. 148 is anchored at one end *A*, and rests on rollers at the other end *B*. The span $l = 80$ ft., rise $h = 30$ ft., distance between trusses $b = 18$ ft. The weight of the truss is given approximately by the formula $W = \frac{1}{2}bl^2$; the wind load, assumed to be from the left, is taken as 45 lb./ft.² of roof surface, and the snow load is 30 lb./ft.² of horizontal projection. Calculate analytically the reactions of the supports due to all loads acting on the truss.

305. In the saw-tooth type of roof truss shown in Fig. 149, determine analytically the stress in *FH*.

306. In the Pratt truss shown in Fig. 150, the dimensions and loads are as follows: span = 150 ft., height = 30 ft., number of panels = 6. The dead load per linear foot in pounds for single-track bridge of this type is given by the formula $w = 5l + 350$, where l denotes the span in feet; the weight of single track

may be taken as 400 lb. per linear foot, and live load as 3500 lb. per linear foot. Calculate analytically the stresses in all the members.

NOTE. Each truss carries one half the total load. In the present case, therefore, the total load per linear foot per truss is $\frac{1100 + 400 + 3500}{2} = 2500$ lb.

307. In the saw-tooth type of roof truss shown in Fig. 149, obtain graphically the stresses in all the members, the dimensions being as follows: span = 25 ft., distance apart of trusses = 15 ft., and pitch of roof = $\frac{2}{3}$, making the inclination of the longer leg to the horizontal = $21^\circ 48'$.

FIG. 149

As the span is short and the roof comparatively flat, it is sufficiently accurate to assume that the combined action of wind and snow is equivalent to a uniform vertical load, which in the present case may be assumed as 25 lb./ft.² of roof. The weight of this type of truss will be taken as 1.5 lb./ft.² of roof, and the weight of roof covering as 7 lb./ft.² of roof. As the top-chord panel length is 8 ft., each panel load will be $8 \times 15 \times 33.5 = 4020$ lb.

308. Analyze graphically for both dead and snow loads the French type of roof truss shown in Fig. 151, the dimensions being as follows: span $l = 100$ ft., rise $h = 30$ ft., $d = 5$ ft., and distance apart of trusses $b = 20$ ft.

The weight of truss in pounds for this type is given by the formula

$W = \frac{1}{24} b l^2$, where b and l are expressed in feet. The weight of the roof covering may be assumed as 15 lb./ft.² of roof surface, and the snow load as 20 lb./ft.² of horizontal projection.

First calculate the dead load carried at each joint, due to weight of truss and roof covering, and draw the diagram for this system of loads. The diagram for

FIG. 150

snow load will be a similar figure, and the stresses in the two cases will be proportional to the corresponding loads. Hence the snow-load stresses may be obtained by multiplying each dead-load stress by a constant factor equal to the ratio of the loads.

FIG. 151

In drawing the diagram start at one abutment, say the left, and take the joints in order, thus determining the stresses in OA , AM , AB , PB , BC , and CM . At the middle of the rafter, where the load PQ is applied, there will be three unknowns, and since these cannot all be determined simultaneously, one of the three must be obtained by some other means before the construction can proceed. For this purpose,

proceed to the load QR and determine the stress in EF by the auxiliary construction shown by the dotted lines in the diagram. Then determine the stress in ED by the same auxiliary construction. Having found the stress in ED , we may then go back to the load PQ and complete the diagram.

FIG. 152

309. Analyze graphically for both dead and wind loads the Howe roof truss shown in Fig. 152 for a span of 50 ft. and of one-half pitch, that is, with a rise of $2\frac{1}{2} = 12.5$ ft. The trusses are spaced 16 ft. apart; the weight of each truss may be taken as 2.5 lb./ft.² of horizontal area, the roof covering as 10 lb./ft.² of roof, and the snow load as 20 lb./ft.² of roof. The wind load, based on a pressure of

30 lb./ft.² of vertical projection, gives for a roof of one-half pitch an equivalent load of 22.4 lb./ft.² of roof surface.

Assume the left end of the truss to be on rollers and the right end fixed. The total horizontal thrust due to wind load is then carried by the right abutment.

In drawing the wind-load diagram, first calculate the reactions R_1 and R_2 by the method of moments. Having thus determined the point M , the remainder of the diagram is easily drawn.

310. Solve problem 304 graphically.

ANSWERS TO PROBLEMS

The following list comprises answers to about two thirds of the problems given at the close of each section as applications of the text. This is ample to enable the student to verify the correctness of his numerical applications, while enough remain unanswered to cultivate self-reliance and independence of thought.

1. Answer given. 2. Answer given. 7. 16,500,000 lb./in.² approximately. 8. .0044 in. approximately; $s = .00002037$. 9. 27,708 lb. 10. .000307 in. 11. .0005076 in. 12. $s = .0018$. 13. 5.4 in. 14. .0000104. 15. 1005 lb. 16. .245 in. square. 17. 5 in. for $p = 16,000$ lb./in.² 18. 113 tons at failure. 19. 20.0086 ft. 20. 2363 lb./in.² 21. .13 in. 22. 10,980 lb./in.² 23. 75,280 lb./in.² for root area. 24. $10\frac{1}{2}$. 25. 5.8. 26. 17,440 lb. 27. 1,645,300 lb./in.² 28. $100 + 554$ lb. 29. $\frac{1}{4}$-in. pin. 30. 2036 lb. tension in each.	3. Answer given. 4. 17.7 lb./in.² 31. $t = 1\frac{5}{8} d$. 32. Copper, 5.73 cm.; alloy, 4.27 cm. 34. Brass, 791 lb.; steel, 409 lb. 35. Eight $\frac{7}{8}$-in. bolts. 36. 2 in. 37. $\frac{\text{crushing}}{\text{shearing}} = \frac{3}{\pi}$. 38. 1.7 approximately. 39. $18\frac{1}{2}$. 40. $\frac{3}{4}$-in. steel-wire rope. 54. 95.45 in.⁴ 55. 441 in.⁴ 56. 337 in.⁴ 57. 111 in.⁴ 58. $1\frac{1}{4}$ in. above base. 59. $83\frac{1}{3}$ in.⁴ 60. 3.36 in. from top. 61. $17\frac{1}{4}$ in. 62. $576\frac{2}{3}$; $364\frac{1}{8}$. 63. 3.28 in. 64. 51 in.⁴ 68. Weight equal to that of table. 69. First support, $1\frac{2}{3}$ ft. from 150 lb. weight.	5. 3.1 lb./in.² 6. .002122; 12.73 in. ..
77. Maximum moment = 1920 ft.-lb.; maximum shear = 480 lb.
 78. Maximum moment = 4225 ft.-lb.; maximum shear = 950 lb.
 79. Maximum moment = $4666\frac{2}{3}$ ft.-lb.; maximum shear = $833\frac{1}{3}$ lb.
 80. Maximum moment = 2200 ft.-lb.; maximum shear = 375 lb.
 81. Maximum moment = 750 ft.-lb.; maximum shear = 200 lb.
 82. Maximum moment = $3333\frac{1}{3}$ ft.-lb.; maximum shear = 8000 lb.
 83. Maximum moment = 11,850 ft.-lb.; maximum shear = 4100 lb.
 93. Maximum moment = 7400 ft.-lb.; maximum shear = 1350 lb.
 94. Maximum moment = 24,000 ft.-lb.; maximum shear = 4000 lb.
 95. Maximum moment = 1250 ft.-lb.; maximum shear = 500 lb.

109. 35,280 ft.-lb.
 110. 869,500 ft.-lb.
 111. 7 to 4.
 112. $\frac{bh^2}{6}$; $\frac{bh^2}{24}$; $\frac{\pi d^3}{32}$.
 113. 8 in. I, 20 $\frac{1}{4}$ lb.
 128. 18.4 including weight of beam.
 129. Tension = 2480 + 500 lb./in.²
 Compression = 4100 - 500 lb./in.²
 130. 5 $\frac{1}{4}$ in.
 131. 16.3 ft.
 136. .192 in.
 166. 350 tons.
 167. 9 $\frac{1}{8}$ in. square.
 168. 5.82 in.
 169. 6 $\frac{1}{8}$ in.
 170. Rankine, 616 tons;
 Johnson, 627 tons.
 171. Rankine, 268 tons;
 Johnson, 267 tons.
 173. 132 tons.
 174. 15.
 175. 2 $\frac{3}{8}$ in. square.
 203. $d = 4.25$ in. for stiffness;
 $d = 6.6$ in. for strength.
 205. Minimum speed, 73 R. P. M.
 206. 2500 lb./in.²
 207. Shaft, 3 in. diameter.
 222. .13 in.
 223. 5911 lb./in.²
 224. 97 $\frac{1}{2}$.
 225. 2940 lb./in.²
 226. 3333 lb./in.²
 228. Side, $\frac{3}{8}$ in.;
 bottom, $\frac{9}{16}$ in.
 229. 6528 lb./in.²
 230. $\frac{7}{16}$ in.
 231. 5880 lb./in.²
 232. 4 ft.
 233. 1 in.
 234. 2 in.
 235. 4.
 236. 3677 lb./in.²;
 98.9 tons;
 13,550 lb./in.²;
 494.6 in.-tons.
 241. 1.25 in.
 114. 15-in. channels, 33 lb.
 115. 1 to 12.
 116. Section modulus = 24.
 117. 9 in. I, 21 lb.
 118. 9 in. I, 25 lb.
 119. 6 $\frac{1}{4}$ in.
 123. 330 in.⁴;
 2767 lb./ft.
 124. 3967 lb.
 125. 4 $\frac{1}{2}$ $\times$ 3 $\frac{3}{4}$ in.
 126. 28.1 lb.
 127. Stress = 5170 lb./in.²
 137. 101 $\frac{1}{4}$ lb./ft.²
 138. 12.68 ft. apart.
 139. .0373 in.
 156. Moment at wall = 1800 ft.-lb.;
 D maximum = .0776 in. at 5.06 ft.
 from free end.
 176. 4-in. channels, 5 $\frac{1}{4}$ lb.;
 plates, 6 $\times$ $\frac{1}{4}$ in.
 177. Top plate, 10 $\times$ $\frac{1}{2}$ in.;
 side plates, 8 $\frac{1}{2}$ $\times$ $\frac{1}{2}$ in.;
 angles, 3 $\times$ 3 $\times$ $\frac{1}{2}$ in.
 180. 3.4 in.; .9 in.
 181. $l = 17.7 d$.
 182. Diameters, 10 $\frac{1}{2}$ and
 7 $\frac{1}{2}$ in.
 183. About 3 in.
 184. 5 +.
 185. 20 in. I, 75 lb.
 190. 43.24 in.-lb.
 192. 3.68 in.
 193. 5.63 in.
 194. 7.114 in.
 195. 32° 28'.
 196. 2 $\frac{1}{4}$ H. P.
 197. 4.465 in.
 200. 4484 H. P.
 201. 9.2 in.
 202. 32.
 208. $p = 4235$ lb./in.²;
 $q = 4283$ lb./in.²
 209. $E = 30,346,935$ lb./in.²;
 $G = 12,158,485$ lb./in.²
 221. $\frac{1}{4}$ in.
 242. 2344 lb./in.²
 244. 139 lb./in.²
 245. .28 in.
 246. 1700 lb./ft.²
 247. $t = .108 r$.
 248. Flat, $t = 3.16$ in.;
 hemispherical, $t = .7$ in.
 250. 1.28 in.
 261. 70.8%; 52.4%.
 262. $\frac{1}{4}$ -in. plates,
 $\frac{3}{4}$ -in. rivets,
 3-in. pitch;
 $e = 75$ to 78%.
 263. $\frac{1}{2}$ -in plates;
 1-in. rivets,
 5.2-in. pitch.
 264. 82.5 lb./in.²
 267. 476 lb./in.²
 268. 450 lb./in.²
 269. 768 lb./ft.
 277. .3 ft.
 279. 5 times load.
 280. $F = \frac{W}{5}, \frac{W}{3}, \frac{W}{4}$.
 281. $F = \frac{W}{2} \left(\frac{a-b}{a} \right)$.
 282. 1 to 18.
 283. $F = \frac{W}{2} \left(\frac{R-r}{c} \right)$.
 284. $W = 90478 F$, neglect-
 ing friction.
 285. $\theta = \tan^{-1} \frac{\sin \beta}{\sin \alpha}$.
 286. 463.2 lb.
 288. Upper end, 13.6 lb.;
 lower end, 76.2 lb.
 289. 22 $\frac{2}{3}$ lb., 63 $\frac{8}{9}$ lb., 63 $\frac{8}{9}$ lb.;
 50 lb.
 290. 82 lb.

291. 288.7 lb. between pipes ;
144.35 lb. against sides.
292. Left, 174,110 lb.;
right, 235,992 lb.
295. 6.83 tons.
297. Connecting rod, 47,580 lb.;
cross head, 6730 lb.
Maximum on crank pin 48,030 lb.
298. $AK = 86$ tons, $AB = 76.3$ tons,
 $EF = 121.6$ tons, $ED = 82.9$ tons.
299. $AD = 99.2$ tons, $BC = 95.6$ tons,
 $AB = 87.7$ tons, $AC = 85.5$ tons,
 $DE = 49.5$ tons, $FC = 117.0$ tons,
 $DF = 51.7$ tons.
303. $CD = 296,500$ lb., $DE = 12,842$ lb.,
 $EF = 288,500$ lb.
304. $BH = 49,400$ lb., $BE = 33,450$ lb., $HE = 31,090$ lb., $HC = 33,500$ lb.,
 $CE = 24,657$ lb., $CK = 41,094$ lb., $KE = 10,000$ lb., $KA = 51,094$ lb.,
 $AE = 16,574$ lb.
305. 1390 lb.
306. $AB = 203,100$ lb., $AD = 208,300$ lb., $AG = 234,375$ lb.,
 $BM = CN = 130,200$ lb., $EO = 208,300$ lb., $BC = 62,500$ lb.,
 $DE = 31,250$ lb., $CD = 121,875$ lb., $EF = 40,640$ lb.

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