Symposium

NAVAL HYDRODYNAMICS

F. S. SHERMAN
EDITOR

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Symposium on
NAVAL HYDRODYNAMICS

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PROCEEDINGS

SYMPOSIUM ON NAVAL HYDRODYNAMICS

September 24-28, 1956

FOREWORD

There are few disciplines having the far-reaching and intimate consequences in the whole spectrum of Navy operations, vehicles, and implements as does Hydrodynamics. Moreover, it is a contemporary characteristic of the field that the basic physical and mathematical problems now recognized as the central ones are also among the very ones faced by the designer of the modern Fleet and its myriad accouterments. In recent years, we have seen a rapid exploitation of scientific and technological breakthroughs in the development of Fleet components and Naval strategic and tactical concepts. It may be fairly stated that advances in Hydrodynamics have played a significant part in the evolution of the new Navy; on the other hand, the need for intensification of research in this discipline has been emphasized by the hydrodynamical problems which must be solved if the most effective Naval system compatible with the unprecedented developments in other fields is to be provided.

A result of the increasing importance of those aspects that have been characterized here as "Naval Hydrodynamics"—to highlight their significance in Naval applications—has been a continually growing need for meetings devoted exclusively to these aspects. To meet this need, and as one of the ways of fulfilling its mission of insuring maximum contribution of basic science to Naval effectiveness, the Office of Naval Research has developed a series of symposia of which these Proceedings are the initial outcome. This first "Symposium on Naval Hydrodynamics" differed in an important way from the usual scientific meetings in that the papers were specifically intended to contain critical surveys of various aspects of Hydrodynamics rather than the results of isolated pieces of research. In recognition of the widespread scientific interest and basic nature of the meeting, the National Academy of Sciences—National Research Council joined with the Office of Naval Research in co-sponsoring the Symposium.

In addition to the motivations mentioned above, there were several other objectives for the Symposium. In order to derive the greatest possible benefits for those both in research and in design, we sought the following goals in the several contributions: firstly, critical reviews and analyses of the status of each area including the results of important recent researches; secondly, an interpretation, where appropriate, of the significance of these results for future design applications; and, finally, guidance for future research based on the authors' conclusions and opinions regarding the most fruitful directions toward which future research might be oriented. The intimate role of the foundational scientific aspects of the field in Naval and marine applications is naturally revealed in the choice of topics finally selected for presentation. Thus, invitations were issued for papers on the basic physical and mathematical aspects (turbulence, cavitation, geophysical hydrodynamics) as well as more obviously applicatory topics (ship motions, water-based aircraft research, underwater ballistics problems, etc.).

A further, most important, motivation and purpose of this meeting was to develop an awareness—among those in related disciplines—of the many scientifically rewarding and technologically important problems still extant and the new ones that seem to arise faster than the classical ones are solved. In this way, it is hoped that additional activity in classical Hydrodynamics will be stimulated in this country and, as a consequence, increase the scientific potential ultimately needed in support of Naval problems.

During the early planning stages, as the scope and objectives began emerging clearly, there sprung up a spontaneous desire on the part of his colleagues, past and present, to choose the occasion of this first Symposium on Naval Hydrodynamics to honor Captain Harold E. Saunders, USN (Ret.) for his contributions in creating a stimulating atmosphere for research in Hydrodynamics in the U. S. Navy. Capt. Saunders, through his foresight and inexhaustible energy, can be credited—perhaps more than any other man—with the existence in the U. S. Navy at the start of World War II of the facilities and environment so urgently needed for dealing with the hydrodynamic problems suddenly faced by the Fleet and which had been neglected in prior years. In spite of, or perhaps because of, his uncompromising demand for the ultimate detailed solution of the most difficult problem, he holds the respect and affection of all those who know and worked with him, whether or not they agreed with him in all matters. While Capt. Saunders has received well-deserved honors from the U. S. Navy and various institutions, it was a privilege to make the Symposium banquet the occasion of a more personal appreciation by his friends and scientific colleagues. The testimonial, delivered by RAdm. A. G. Mumma, USN, Chief of the Bureau of Ships, is reproduced as an important part of this volume.

It is our privilege to express the appreciation of the Office of Naval Research and the National Academy of Sciences to the authors, each an authority in his field, for their graciousness in accepting our invitations. Special thanks are due Prof. L. M. Milne-Thomson for his willingness to accept the difficult assignment of presenting a critique of the theoretical foundations of the field.

Finally, it is a pleasure to acknowledge the contributions of the members of the team directly responsible for the Symposium. The technical program and physical arrangements for the formal sessions were the responsibility of the ONR Mechanics Branch. The many administrative details, the banquet, and arrangements for personal comfort were kindly undertaken by the National Research Council. Special recognition is deserved by Mr. Marshall P. Tulin for his contributions toward developing and formulating the technical program, Dr. Frederick S. Sherman for undertaking the task of editing the proceedings, Cdr. Waldo S. Simons, USN, and Mr. John M. Crowley and Dr. Howard M. Berger for the physical environment and aids and the Mrs. Lucille Rittue, Yetta Hassin and Audrey Steinman for their good humored secretarial assistance throughout the trying year leading up to the meeting. All are members of the ONR Mechanics Branch. Special thanks are also extended to Mrs. Alice McIntyre and Mrs. Catherine M. Parrish of the NAS-NRC for their assistance to the Executive Secretary, NRC Physical Sciences Division.

PHILLIP EISENBERG
Head, Mechanics Branch
Office of Naval Research
General Chairman

JOHN S. COLEMAN
Executive Secretary
Physical Sciences Division
National Academy of Sciences-
National Research Council
Co-Chairman

PREFACE

Thanks to the deft pens and rather widespread editorial experience of our contributors, these Proceedings have required very little editing on my part.

Discussions of the papers appear nearly as they were given orally, although most contributors availed themselves of the opportunity to polish the actual stenographic record. It is hoped that the resulting gain in clarity and brevity has offset any loss in color and informality.

My gratitude and admiration go out to Miss Ruth Benning, our steno-typist, for her herculean efforts in recording the oral proceedings, and to Mrs. Jane Cochrane and Mrs. Audrey Steinman for their aid in preparing the clean typescript.

Finally, an apology is due those who may have been led by past performance to expect an earlier publication date for these Proceedings. It is sincerely hoped that this volume will have a lasting value which will compensate for its tardy appearance.

F. S. SHERMAN

PROGRAM

Monday, September 24, 1956

Welcoming Address, RADM R. BENNETT, USN, Chief of Naval Research.

Session 1

Chairman: J. J. STOKER, Professor of Mathematics, Institute of Mathematical Sciences, New York University.

M. J. LIGHTHILL; W. H. MUNK, M. J. TUCKER and F. E. SNODGRASS.

Session 2

Chairman: K. S. M. DAVIDSON, Director Experimental Towing Tank, Stevens Institute of Technology.

G. P. WEINBLUM; J. V. WEHAUSEN.

Tuesday, September 25, 1956

Session 3

Chairman: R. W. L. GAWN, C.B.E., Superintendent, Admiralty Experiment Works, Haslar, England.

J. C. NIEDERMAIR; H. W. LERBS.

Session 4

Chairman: F. W. S. LOCKE, Hydrodynamics Consultant, Research Division, U. S. Navy Bureau of Aeronautics.

J. B. PARKINSON; R. N. COX and J. W. MACCOLL.

SYMPOSIUM BANQUET

Toastmaster: RADM A. G. MUMMA, USN, Chief, U. S. Navy Bureau of Ships.

Speaker: PROF. L. M. MILNE-THOMSON, C.B.E.

Wednesday, September 26, 1956

Session 5

Chairman: W. R. SEARS, Director, Graduate School of Aeronautical Engineering, Cornell University.

H. M. FITZPATRICK and M. STRASBERG; D. GILBARG.

Session 6

Chairman: E. H. KENNARD, Chief Scientist, Structural Mechanics Laboratory, David Taylor Model Basin.

M. S. PLESSET; H. G. SNAY.

Thursday, September 27, 1956

Session 7

Chairman: G. B. SCHUBAUER, Chief, Fluid Mechanics Section, National Bureau of Standards.

C. C. LIN; S. CORRSIN; G. K. BATCHELOR.

Closing remarks: F. J. WEYL, Office of Naval Research.

Friday, September 28, 1956

Tour of the DAVID TAYLOR MODEL BASIN, Carderock, Maryland.

Greetings: CAPT. E. A. WRIGHT, USN, Commanding Officer and Director.

Tour of the U. S. NAVAL ORDNANCE LABORATORY, White Oak, Maryland.

Greetings: CAPT. W. W. WILBOURNE, USN, Commander.

PLATE ONE

To Captain H. E. Saunders, USN, Ret.

by

Rear Admiral A. G. Mumma, Chief
Bureau of Ships

We honor tonight a very special member of the family of Hydrodynamicists, because of many, many reasons.

I wish this gentleman to stand while I make a few remarks concerning him, so that you can all get a good look at him.

Captain Harold E. Saunders, will you please stand?

It has been a personal pleasure of mine to have been associated with this gentleman for over fifteen years, and I would like to tell you a little of his history.

When he graduated from the Naval Academy in 1912, he had amassed an academic scholastic record there that had never been equaled, and has never been equaled since. The record he broke, scholastically, was that of David Watson Taylor. This resulted in prizes in seamanship, international law, ordnance, gunnery, and many, many other prizes, as a result of this scholastic achievement.

After graduation he specialized in Naval Architecture and Naval construction. He worked at the Mare Island Naval Shipyard in the Hull Division, 1916 to 1920, during the period in which we were engaged in World War I. During this time this Yard built fourteen destroyers and the USS CALIFORNIA.

Later he was assigned to the Bureau of Construction and Repair, in charge of submarine design and construction of submarines. He executed the preliminary design for the then revolutionary USS V-4, which was later renamed the USS ARGONAUT, and which was the largest submarine ever built at that time.

From 1933 to 1935 he was Force Constructor on the staff of the Battle Force, of the U. S. Fleet, and there effected revolutionary changes in our procedure for damage control of ships in warfare.

From 1938 to 1940 he was Liaison Officer for the Bureau of Construction and Repair, in connection with the design and construction of a new model basin at Carderock, Maryland, so that here we have the father of the David Taylor Model Basin. He served there as Technical Director for a period of six years, and then later, as Commanding Officer and Director and Technical Director, until 1947.

I would like to read you a little bit of what was said when he left that establishment, in 1947. This is a quote.

"Ten years ago Carderock was a barren, weedgrown field and 'the new model basin' existed only on paper.

"Today wide expanses of well-kept lawns surround many stately buildings that house the greatest research establishment of its kind in the world, the David W. Taylor Model Basin.

"In the recently ended World War II the fruits of our labors contributed largely to the victory and we have become well and widely known for our high standards in research.

"You, more than any other person, are responsible for these achievements; your knowledge and vision in planning, and your unremitting labor and infinite capacity for taking pains have made them possible. We, who under your firm but kindly guidance have shared in these achievements, are grateful for the privilege.

"Though you are about to relinquish your command, we who remain shall strive to uphold the high standards you leave us as our heritage. May that assurance and the consciousness of a job well done add to your happiness and contentment in the future."

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SOME PROBLEMS AND METHODS IN HYDRODYNAMICS

L. M. Milne-Thomson, C.B.E.

Brown University

When I received Admiral Bennett's invitation to address you tonight I felt almost as uncomfortable as General Burgoyne must have felt at Saratoga with the thought of the North behind me and hydrodynamicists massing in the South. Indeed I began to reflect on the distance from London to York which as you know, or ought to know, is a long way, and even non-stop trains take several hours to cover it. Once upon a time in one such non-stop train sat an American and an Englishman the only occupants of a compartment. For about an hour each looked out of the window. Then the American said "Do you mind if I talk to you" to which the Englishman replied "What about?" You see my difficulty. An after dinner speech which conveyed any information whatever is outside the range of my experience. An after dinner speech concerning Hydrodynamics must be unique in the history of dining and I therefore feel honored to be chosen as the first to make such history.

Hydrodynamics as an exact science started with Archimedes. It is true that he treated the particular case of zero velocity, but his work remains today a correct piece of applied mathematics and indeed a giant achievement for his time.

Scoffers have said that Archimedes' chief claim to fame is that he took a bath and then forgot to dress in spite of his principles.

It is not my purpose nor indeed am I competent to discuss hydrodynamics from the point of view of the historian, who, as you know, is an amiable gentleman advancing into eternity stern first.

Rather I should like to say a few words concerning some specific problems and methods available for tackling them. Omission of a name or a topic does not imply undervaluation by me. It merely proves that a choice has to be made. Many other choices would have been possible.

It is becoming increasingly evident that the most insight giving statement of the equations of motion of continuous media in general and of fluids in particular is by means of tensors.

There are at least two ways of regarding tensors, namely as quantities attached to a coordinate system, or intrinsically, the latter being to my way of thinking incomparably superior.

The intrinsic definition of a tensor $T^{(n)}$ of rank n is recursive.

A tensor of rank n is a linear operator which operating on an arbitrary vector $\mathbf{x}$, by scalar multiplication, gives rise to a tensor of rank $n-1$.

This definition, together with the statement that a tensor of rank zero is a scalar, completely characterizes tensors of all ranks.

Thus for example a vector $\mathbf{a}$ is a tensor of rank 1 since $\mathbf{a} \cdot \mathbf{x}$ is a scalar, or tensor of rank zero.

Similarly the dyadic product $\rho \mathbf{q}; \mathbf{q}$ combines with $\mathbf{x}$ to give $\rho \mathbf{q} (\mathbf{q} \cdot \mathbf{x})$ a tensor of rank 1 and so $\rho \mathbf{q}; \mathbf{q}$ is a tensor of rank 2, a 2-tensor, indeed the very important

momentum transfer tensor. As an application the equation of steady motion under no body forces can be written [1]

$$\nabla \cdot [\Phi - \rho(\mathbf{q}; \mathbf{q})] = 0, \quad (1)$$

where

$$\Phi = -pI - \frac{2}{3}\mu(\nabla \cdot \mathbf{q})I + \mu(\nabla; \mathbf{q} + \mathbf{q}; \nabla) \quad (2)$$

is the stress tensor, I being the idemfactor or unit 2-tensor.

From this form of the equation of motion, by integration over a sphere of large radius, an expression for the force on a moving solid is readily obtained. By applying Oseen's approximation at a distance we next analyse this force into a lift and a resistance or drag. The resistance is of particular interest since its expression is VF where V is the velocity of the body and F is an inflow of liquid into the sphere, predominantly an inflow into the wake behind the body. Moreover the result is of an asymptotic character improving in accuracy as the radius of the sphere is increased. The two-dimensional form of this result was obtained by Filon [2] 30 years ago.

Tensor expression also pinpoints the Lagrangian form of the equation of motion [3]

$$\frac{\partial; \mathbf{r}}{\partial \mathbf{r}_0} \cdot \left(\frac{\partial^2 \mathbf{r}}{\partial t^2} - \mathbf{F} \right) + \frac{1}{\rho} \frac{\partial p}{\partial \mathbf{r}_0} = 0, \quad (3)$$

where $\mathbf{r}$ is the position vector at time t of the particle originally at $\mathbf{r}_0$. The equation of continuity is

$$\rho \left(\frac{\partial; \mathbf{r}}{\partial \mathbf{r}_0} \right)_{III} = \rho_0, \quad (4)$$

where the notation indicates the third scalar invariant of the tensor derivative.

Integration from 0 to t leads directly to Weber's transformation

$$\frac{\partial; \mathbf{r}}{\partial \mathbf{r}_0} \cdot \mathbf{q} - \mathbf{q}_0 = - \frac{\partial \chi}{\partial \mathbf{r}_0}, \quad (5)$$

where

$$\chi = \int_0^t \left\{ \int \frac{dp}{\rho} + \Omega - \frac{1}{2} q^2 \right\} dt, \quad \mathbf{F} = - \nabla \Omega. \quad (6)$$

The Lagrangian form of the equation of motion has been applied to the one-dimensional motion of a gas and more recently to free surface problems which I shall mention later.

The point that I want to make here is that the Lagrangian form is not quite so repulsive as the three equations which result from its expression in coordinates would seem to indicate. In the hydrodynamic case we have $\rho = \rho_0$ and a consequent simplification. The equation, in my opinion, should repay further study.

The two great weapons of general fluid mechanics are the theorems of Gauss and Stokes [4] and their vector forms are suggestive:

$$\int_S d\mathbf{S} \circ X = \int_\tau \nabla \circ X d\tau, \quad \int_C d\mathbf{C} \circ X = \int_S (d\mathbf{S} \wedge \nabla) \circ X, \quad (7)$$

where, in the first the closed surface S encloses the region τ , and in the second the diaphragm S spans the closed curve C . Here the small circle indicates scalar, vector or dyadic multiplication and X is a general function of position, scalar, vector or tensor.

For the rate of change of circulation in a circuit which always consists of the same fluid particles we have

$$\frac{d}{dt} \text{circ } C = - \int_C d\mathbf{C} \cdot \frac{1}{\rho} \nabla p = \int_S d\mathbf{S} \cdot \mathbf{P} \quad (8)$$

where $\mathbf{P} = \nabla p \wedge \nabla \left(\frac{1}{\rho} \right)$, so that $\mathbf{P}$ is a vector along the intersection of surfaces of constant pressure and constant density. Also $\nabla \cdot \mathbf{P} = 0$ and so by Gauss's theorem $\mathbf{P}$ defines tubes of constant intensity. Thus we have the famous meteorological theorem of Bjerknes that the rate of change of circulation in C is measured by the number of $\mathbf{P}$ tubes which C embraces.

For plane flow the vector notation leads directly to the complex variable [5].

The use of the complex variable in two-dimensional problems has a long history, but it is only in recent years that full advantage has been taken of the methods of function theory as opposed to resolution into equations in x and y . What is beginning now to be more fully realized is that the variables most generally useful are not x, y but the conjugate pair $z, \bar{z}$. For example if $\varphi(x, y)$ is a plane harmonic function, it is the real part of a holomorphic function $f(z)$. The identity

$$\varphi(x, y) \equiv \frac{1}{2}[f(z) + \bar{f}(\bar{z})] \quad (9)$$

leads to

$$f(z) = 2\varphi(\frac{1}{2}z, -\frac{1}{2}iz) - \varphi(0, 0) + i\gamma, \quad (10)$$

where γ is an arbitrary real constant.

Again the circle theorem [6] states that the motion of an unbounded liquid whose complex potential is $f(z)$ when disturbed by the circle $|z| = a$ is governed by the complex potential

$$f(z) + \bar{f}\left(\frac{a^2}{z}\right), \quad (11)$$

for on the boundary $\bar{z} = a^2/z$, so that the boundary is a streamline. That no new singularities are introduced is clear from the fact that of the points z and a^2/z only one lies inside the circular boundary.

Combined with conformal mapping the circle theorem enables us to deal with the perturbation produced in disturbing the flow by a cylinder of any cross-section.

In the same line of thought we know that a stream function can be defined for any two-dimensional flow whether rotational or irrotational. Using the variables $z, \bar{z}$ we can denote this stream function by $\psi(z, \bar{z})$ and then the velocity is given by

$$u - iv = -2i \frac{\partial \psi}{\partial z}. \quad (12)$$

In the case of steady streaming past a fixed cylinder, ψ is constant on the boundary and so

$$\frac{\partial \psi}{\partial z} dz + \frac{\partial \psi}{\partial \bar{z}} d\bar{z} = 0. \quad (13)$$

The Blasius theorem for the force (X, Y) then gives [7]

$$X - iY = \frac{1}{2}i\rho \int_C (u - iv)(u + iv)d\bar{z} = -2i\rho \int_C \left(\frac{\partial \psi}{\partial z} \right)^2 dz, \quad (14)$$

where the integral is taken round the boundary C of the cross-section. But the equation of the boundary is of the form $f(z, \bar{z}) = 0$ and so $\bar{z}$ can be eliminated and Cauchy's

residue theorem used even in the case of rotational motion. For example a circular cylinder exposed to a stream V on which is superposed uniform shear flow of vorticity ω undergoes the lift $\pi\rho a^2\omega V$.

Just as the theorem of Stokes is fundamental in "solid flow," if I may use that expression, so is the form which the theorem assumes in two dimensions in terms of z and $\bar{z}$ a powerful tool. This I have named the complex Stokes' theorem [8]. The complex Stokes' theorem refers to a plane area S bounded by a closed curve C and states that

$$\int_C f(z, \bar{z}) dz = 2i \int_S \frac{\partial f}{\partial \bar{z}} dS. \quad (15)$$

Cauchy's theorem is a particular case, namely when $\partial f / \partial \bar{z} = 0$.

One immediate and important application is the calculation of the kinetic energy [9] of liquid in irrotational motion, with its application to virtual mass. Thus

$$\begin{aligned} \text{kinetic energy} &= \frac{1}{2} \int_S \rho q^2 dS = \frac{1}{2} \rho \int_S \frac{dw}{dz} \frac{d\bar{w}}{d\bar{z}} dS \\ &= \frac{1}{2} \rho \int_S \frac{\partial}{\partial z} \left(w \frac{d\bar{w}}{d\bar{z}} \right) dS = \frac{1}{2} i \rho \int_C w d\bar{w} \end{aligned} \quad (22)$$

where $w = \varphi + i\psi$ is the complex potential. If the region S is multiply-connected, suitable barriers must be introduced as part of the boundary C . Note that in virtue of the remarks made shortly ago the integral round the boundary can be evaluated by the residue theorem since $\bar{z}$ is a function of z .

The generalization of the complex variable to three-dimensions leads to Hamilton's quaternions. Alan Rose [10] defines a stream function $\psi(x, y, z; \xi, \eta, \zeta)$ as the flux across the triangle formed by the origin, the point $A(x, y, z)$ and the point $B(x + \xi, y + \eta, z + \zeta)$. If we define the vector

$$\Psi = (\psi_1, \psi_2, \psi_3) = \left(\frac{\partial \psi}{\partial \xi}, \frac{\partial \psi}{\partial \eta}, \frac{\partial \psi}{\partial \zeta} \right)_{\xi=\eta=\zeta=0}, \quad (23)$$

the velocity is

$$\mathbf{q} = -\nabla \wedge \Psi. \quad (24)$$

If the motion is axisymmetric and irrotational with velocity potential φ , the function

$$\varphi + i\psi_1 + j\psi_2 + k\psi_3 \quad (25)$$

satisfies the condition for it to be a right-regular quaternion function of the quaternion variable

$$w + ix + jy + kz \quad (26)$$

where w is an imagined coordinate whose axis is perpendicular to the axes of x, y, z and i, j, k are Hamilton's unit vectors.

Thus the theory of analytic quaternion functions is in principle available.

In this way, for example, it is possible to deduce the flow past a sphere in terms of the quaternion variable by a method entirely analogous to that for deducing the flow past a circle in terms of the complex variable.

Here then is a method which may well merit further investigation.

Let us now turn to a problem which is important from the naval standpoint; the problem of virtual mass. To take the simplest case when a body of mass M moves

with uniform speed V in a straight line in inviscid liquid the total kinetic energy of the system is

$$\frac{1}{2}(M + H)V^2, \quad (27)$$

and the body moves as if the liquid were absent and the mass of the body were increased from its actual mass M to its virtual mass $M + H$. Here H is the added or hydrodynamic mass for this particular motion, and is the coefficient of $\frac{1}{2}V^2$ in the expression for the kinetic energy of the liquid.

It is only quite recently that a physical interpretation of hydrodynamic mass as an actual mass of liquid has been given by Darwin [11]. To understand Darwin's interpretation consider the particular case of a circular cylinder which moves along the x -axis from minus to plus infinity. Suppose that when the cylinder is at $x = -\infty$, a wall of blue dye is used to color the particles in a plane perpendicular to the direction of motion. Since the cylinder in its motion displaces a certain volume of the fluid the gap left behind must be filled up and it might appear reasonable to suppose that when the cylinder has attained the position $x = +\infty$, the wall of dye will have retreated a certain distance to the rear of its initial position.

Now the paths of the particles are elasticas, Fig. 1, so that a particle at A when the cylinder is at $x = -\infty$ will have drifted forward to B when the cylinder is at $x = +\infty$.

FIG 1

If then we consider those particles which at a given instant lie in an axial plane of the cylinder, perpendicular to the direction of motion, the positions of these particles when the cylinder is at $x = -\infty$ and at $x = +\infty$ define surfaces which, in the plane of the motion, are typified by curves A,A, and B,B, Fig. 2. Thus the particles on the curves A,A move forward, not backwards, to the positions B,B. The intuitive idea of reflux of a wall of dyed particles is false. Darwin's discovery is that the mass of the liquid enclosed between these initial and final positions is in fact the hydrodynamic mass of the cylinder for this particular motion.

The point can be established by integration since the coordinates of the points on the elasticas are expressible rationally in terms of Jacobian elliptic functions [12]. The argument is, however, capable of general formulation independent of the particular shape of the cross-section of the cylinder. The argument can also be extended to

FIG 2

three-dimensional motion. Thus we have here, I believe for the first time, a genuine physical interpretation of hydrodynamic mass as a mass of liquid entrained by the body.

The corresponding problem of interpretation with a free surface still awaits investigation, although some computations of the hydrodynamic mass itself have been made by Bloh [13] for spheres and ellipsoids half immersed and totally immersed.

The problem of rotation also offers opportunities for investigation. If we consider a rotating plate, in two-dimensional motion, it is found that two regions A and B exist in which liquid is trapped and moves round with the plate, not as a rigid body but consistently with irrotational flow, Fig. 3. This points to a method of interpreting hydrodynamic mass due to rotation.

Problems concerning free streamlines have been intensively studied in recent years in relation to the cavities formed behind bodies moving at high speed and the water entry of missiles.

The two-dimensional problem has been greatly simplified by Max Shiffman's [14] method of reflection across free streamlines whereby an image of the actual flow

FIG 3

is obtained in the free streamline, in effect a method of analytical continuation of the flow. Apart from the insight-giving character of this method, one of the integrations is already performed by geometrical considerations, thus greatly simplifying the technique. The method also affords a direct geometrical interpretation of the drag coefficient.

To give the simplest possible illustration, it is a provable theorem that the image of flow in an angle is flow outside an equal angle, Fig. 4. Thus for a jet running along a wall ABC we reduce the problem to flow in a channel formed by the original wall and a parallel wall $A' \infty B' C \infty$ got by translation along the bisector of the angle $A \infty B C \infty$.

FIG 4

FIG 5

FIG 6

FIG 7

The method also takes charge of reentrant jet problems such as the impact of a stream on a wedge, Fig. 5.

Applied to the impact of a stream on a lamina, Fig. 6, the image flow indicates a layer of sources on the downstream face, and thus offers at least a suggestion for tackling the corresponding problem of a stream impinging on a circular disc.

Riabouchinsky [15] 30 years ago discovered a method of finding the drag on a lamina exposed to a stream, Fig. 7. The method consists in placing an image lamina downstream, the two being joined by free streamlines.

This problem is capable of exact solution [16] in terms of Jacobian elliptic functions, where Prandtl's cavitation number, $\sigma = (p_\infty - p_c)/^{1/2}\rho u^2$, appears in the parameter of the elliptic functions. The case of an unlimited cavity can then be approximated by increasing the distance between image and plate.

Quite recently Garabedian [17] has undertaken the numerical study of the axisymmetrical problem, in particular that of the circular disc with an equal image disc behind it on the Riabouchinsky model. The two discs are joined by a free stream surface to enclose a region containing water vapor.

The equation satisfied by the stream function ψ is

$$\psi_{xx} + \psi_{yy} - \frac{1}{y}\psi_y = 0 \quad (28)$$

with $\psi = 0$ on the entire boundary and

$$-\frac{1}{y}\frac{\partial\psi}{\partial n} = 1 \quad (29)$$

on the free stream surface, where x and y are coordinates in a meridian plane.

From these we derive

$$\frac{\partial}{\partial n}\left(-\frac{1}{y}\frac{\partial\psi}{\partial n} + \frac{\kappa}{y}\psi\right) = 0 \quad (30)$$

on the free stream surface where κ is the curvature of the free streamline in the meridian plane.

The boundary conditions can be combined to give

$$-\frac{1}{y}\frac{\partial\psi}{\partial n} + \frac{\kappa}{y}\psi = 1 \quad (31)$$

on the free streamline and this boundary condition is therefore stationary in the sense that by (30) a normal displacement δn of the free streamline leads to an error of order $(\delta n)^2$. A convergent iterative process can be founded on this observation. Garabedian has carried this out, taking an initial position of the free streamline based on the curve afforded by the plane flow solution of Riabouchinsky.

Let us now turn to the problem of free surfaces when the liquid moves under gravity. By a free surface we shall mean a surface which always consists of the same fluid particles and on which the pressure is constant. The grand illustration in nature is the surface of the ocean.

But few simple complete solutions of this problem are known.

The only non-trivial cases which occur to one are Gerstner's trochoidal wave and Rankine's combined vortex. In the Gerstner wave the free surface is a trochoid and the motion is rotational. In Rankine's combined vortex the motion is rotational within a vertical cylindrical core and is irrotational outside the core, Fig. 8.

FIG 8

From the naval point of view there are three main problems whose exact solution still eludes us, namely the problems of periodic surface waves on deep water, surface waves on water of finite depth, and the solitary wave. The combination of aerial photographs with a knowledge of the relation between wave length and wave speed played an important part in determining depth and slope on the Normandy beaches.

Let us look at what is involved in the problem of the two-dimensional progressive wave of permanent type on water of finite depth. As long ago as 1925 Levi-Civita [18] stated the boundary value problem in the form

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{c^3} e^{-3\tau} \sin \theta, \quad \psi = 0, \quad q = ce^\tau. \quad (32)$$

Here θ is the inclination of the velocity vector to the horizontal.

By mapping the strip defined by a period on the unit circle and then obtaining a Taylor expansion, Levi-Civita established the existence of this type of wave.

One might inquire why such an existence theorem should be necessary when waves are to be seen any day.

But it is only fair to remember that the wave here considered is perfectly regular and is propagated in an inviscid fluid, conditions to which observed waves only approximate.

The intrinsic difficulty of the problem here envisaged is the non-linearity of the boundary condition, quite apart from the fact that the form of the boundary is part of the solution.

Putting $\theta + i\tau = \omega$, the linearized approximation is obtained by assuming

$$|\omega| = (\theta^2 + \tau^2)^{1/2} \quad (33)$$

to be small of the first order. From this assumption follows the usual theory of waves of small amplitude and slope.

There is, however, a serious limitation to the use of the linearized approximation. A wave will break at the crest when the fluid velocity there exceeds the velocity of the wave. The critical case is when the fluid velocity at the crest is equal to the velocity of the wave, that is to say $q = ce^\tau = 0$, so that $\tau = -\infty$.

It follows that no approximation based upon the assumption that τ is small can throw any light on the case of breaking.

A way to avoid this difficulty has been proposed by T. V. Davies [19] namely in Levi-Civita's boundary condition to replace $\sin \theta$ by $\sin 3\theta$.

This substitution replaces one non-linear boundary condition by another. It still preserves the essential feature but allows τ to be large.

The boundary condition then becomes

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{3c^3} e^{-3\tau} \sin 3\theta, \quad \psi = 0 \quad (34)$$

which leads to

$$e^{-3i\omega} = 1 - 3A \exp\left(\frac{2\pi i w}{c\lambda}\right) \quad (35)$$

where $w = \varphi + i\psi$ is the complex potential and $c^2 = g\lambda/2\pi$, where λ is the wavelength.

The condition for breaking at the crest is $u - iv = 0$ when $w = 0$ which since $u - iv = ce^{-i\omega}$, leads to $3A = 1$.

In the neighborhood of the crest the wave then forms a wedge of angle 120° .

Moreover if we write $A = 2\pi a/\lambda$ where a is small we recover the ordinary linearized theory.

Thus the method discovered by Davies yields an approximation which applies over the whole range from waves of small amplitude to those on the point of breaking at the crest.

FIG 9

The method has also been applied by my assistant B. A. Packham [20] at the Royal Naval College to obtain a unique solution in closed form of the problem of the solitary wave, Fig. 9. The solution is so simple that I must give it

$$e^{-3i\omega} = 1 - \sin^2 kch \operatorname{sech}^2 \frac{1}{2}k(w - ich) \quad (36)$$

$$\frac{c^2}{gh} = \frac{\tanh kch}{kch}, \quad 0 \leq kch \leq \frac{1}{3}\pi \quad (37)$$

In this result $k = 0$ corresponds to rest and $kch = \pi/3$ corresponds to breaking at the crest.

In connection with waves I should like to mention a striking phenomenon to which Dr. Gawn called my attention. He also demonstrated it to me at Haslar.

One method of making waves in a ship tank is by means of the periodic heaving motion of a cylinder. The result in general is to generate straight crested longitudinal waves.

If, however, the frequency of the oscillations of the cylinder exceeds a certain well defined critical value, transverse waves are also set up and a choppy sea results. You would hardly credit this without seeing it. I believe that no theoretical explanation of this phenomenon has so far been found. The suggestion that it is connected with the presence of the vertical walls of the tank is apparently without foundation. Here then is a matter to which research workers might usefully direct their attention.

What I have been describing is a method of approximation, very much superior to linearization but still an approximation.

It is therefore with gratification that we can turn to another method of approach to the free surface problem which reduces the attack on the two-dimensional case to the solution of a linear partial differential equation of the second order of parabolic type.

This method was originated by F. John [21] and reposes on the fact that on a surface of constant pressure the acceleration of the liquid relative to the acceleration due to gravity is a vector whose direction is normal to the surface of constant pressure. This statement is a simple consequence [22] of the equation of motion in the form

$$\mathbf{a} - \mathbf{g} = -\frac{1}{\rho} \nabla p. \quad (38)$$

If then the surface has the equation

$$z = f(\alpha, \theta), \quad z = x + iy, \quad (39)$$

where α is a real Lagrangian parameter, we have

$$\frac{\partial^2 f}{\partial t^2} + ig = ir(\alpha, t) \frac{\partial f}{\partial \alpha}, \quad (40)$$

where $r(\alpha, t)$ is an arbitrary function. This is the equation just mentioned. Every two-dimensional continuous free surface must satisfy this equation, whether the motion is rotational or irrotational, steady or unsteady.

When the motion is irrotational it is not difficult to deduce the equation satisfied by the velocity potential.

Further John has shown that the form $y = k(x, t)$ of the free surface can be prescribed. For a particle, x, y are functions of t and the constant pressure condition then implies that x as a function of t satisfied a non-linear ordinary differential equation of the second order.

In the case of steady motion the problem can be simplified still further to depend on the solution of an ordinary second order linear equation of the form

$$f''(\beta) + ig = iS(\beta)f'(\beta), \quad (41)$$

where $S(\beta)$ is an arbitrary function.

It follows from these considerations that two-dimensional free surface problems can be reduced to the study of a limited class of differential equations. Nevertheless progress will necessarily depend on divining the proper form of the arbitrary function involved.

On the other hand we have here a means of generating an unlimited number of free surfaces by assigning the arbitrary function. In particular putting $S(\beta) = \omega$, a constant, leads to a trochoidal free surface, from which we can proceed to a trochoidal progressive wave.

Unfortunately this wave has to be associated with a moving ocean bed since the singularities of the progressive wave are no longer fixed.

But let us take heart. A function devoid of singularities is also devoid of interest: it is a constant.

A different representation of steady irrotational flow is due to H. Lewy [23]. By proper choice of axes and units the surface condition can be expressed in the form

$$\frac{dz}{dw} \left(\frac{dz}{dw} - 2i \frac{dy}{dw} \right) + \frac{1}{2y} = 0, \quad \psi = 0, \quad w = \varphi + i\psi. \quad (42)$$

Let us regard this as an equation in the complex domain involving a complex function $y = -\eta(w)$, real when $\psi = 0$.

The equation has the solution

$$z = -i\eta(w) + \int \left[\frac{1}{2\eta} - \left(\frac{d\eta}{dw} \right)^2 \right]^{1/2} dw, \quad (43)$$

and provided the integral is real on some segment of $\psi = 0$, we have

$$y = -\eta(w) \quad \text{on} \quad \psi = 0, \quad (44)$$

and the point z describes a free streamline.

Lewy has also proved that the flow is analytic on the free surface in the steady case and has thus established that the formula gives the most general steady irrotational motion with a free surface.

M. J. Vitousek [24] has studied in detail flows obtained by attributing certain forms to $\eta(w)$ and in particular waves of trochoidal and cycloidal profile.

You will no doubt expect to hear something concerning the motion of a body through a fluid, typically the problem of the submarine well below the surface. The classical case governed by Kirchhoff's equations, when the vessel moves with velocity $u = u(t)$ and angular velocity ω in liquid otherwise at rest gives [25] for the force R_0 and the moment M_0

$$R_0 = -\frac{\partial K}{\partial t} - \omega \wedge K, \quad (45)$$

$$M_0 = -\frac{\partial L}{\partial t} - \omega \wedge L - u \wedge K, \quad (46)$$

where

$$K = \rho \int_S \varphi dS, \quad L = -\rho \int_S \varphi r_\wedge dS. \quad (47)$$

The integrals are taken over the surface S of the body. M. D. Haskind [26] has this year extended these formulae to include the case where the liquid has velocity $v = v(t)$ at infinity and acceleration $f = \frac{dv}{dt}$. If Φ is the velocity potential, he defines $\varphi = \Phi - v \cdot r$ and the corresponding force and moment are

$$R = R_0 + M'f \quad (48)$$

$$M = M_0 + v \wedge K + M' r_c \wedge f, \quad (49)$$

where R_0 and M_0 are calculated by the formulae above, M' is the mass of liquid displaced and r_c is the position vector of the centroid of the volume displaced.

The motion of a submarine when gravity is taken into account has been discussed this year by P. V. Harlamov [27]. The problem he studies is that of a body totally immersed with buoyancy equal to the weight of the body so that in general the weight and buoyancy form a couple. A thorough going discussion of directions of

permanent translation is made. The results are unsuited to reproduction here but I mention one as a specimen. If $\mathbf{r}$ is the position vector of the centroid of the displaced water relative to the centre of gravity of the body, permanent translation is impossible if the body is immersed so that the plane containing the vector $\mathbf{r}$ and one of the principal axes of the virtual mass ellipsoid is vertical but the vector $\mathbf{r}$ is not vertical.

Hydrodynamics, as appears sufficiently from the little I have been able to say presents many problems whose solution is to be desired. To keep in touch with the literature which is published in so many parts of the world is an increasingly difficult task. There is one obvious means by which knowledge could be diffused more quickly, namely by a greater readiness of authors to distribute their off-prints to those who are interested in the subject. Not every separatum will interest the recipient immediately, but nevertheless it makes him aware of what is happening. Were it not for Mathematical Reviews and Applied Mechanics Reviews much important work which is published in various journals and in many languages from Russian to Roumanian would be quite unknown. Moreover this work has often to endure a long delay before appearing in print and so to that extent is old before it is born.

I have quoted to you three pieces of work printed this year. Let me bring you bang up to date by quoting a result [28] which has not yet been published.

The equation of motion and equation of continuity of a viscous fluid, moving in a conservative field of force when incompressible, and under no external force when compressible, are satisfied identically by

$$\rho = \nabla^2 \chi, \quad \rho \mathbf{q} = - \nabla \frac{\partial \chi}{\partial t} \quad (50)$$

$$\Phi = \rho(\mathbf{q}; \mathbf{q}) + \left(V - \frac{\partial^2 \chi}{\partial t^2} \right) I + \nabla_{\wedge} \Psi_{\wedge} \nabla \quad (51)$$

Here Φ is the stress-tensor, $V = \rho \Omega$, where Ω is the potential of the field, χ is an arbitrary function and Ψ is an arbitrary symmetric two-tensor.

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II

RIVER WAVES

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1. *Introduction*

There are few pastimes more pleasant than water-watching, and fortunately most people live near enough to some river or stream to be able to enjoy it. Like other aesthetic pleasures, this is one that is greatly enhanced if one makes an effort to learn something of the underlying causes and effects. My aim in this lecture is to describe the results of reading and thinking undertaken in this spirit, and combined

with happy observation of many British rivers as well as respectful study through the printed page of some great American ones.

2. Frequency dispersion and amplitude dispersion

I shall have a lot to say about the effect of friction on river waves, but it is best to begin, I think, with a survey of water-wave motions in the absence of friction, as these are already fairly complicated [19, 15]. Waveforms on water tend to change shape as a result of two kinds of dispersion, which I propose to call frequency dispersion and amplitude dispersion. The word dispersion is normally used to mean frequency dispersion, according to which components of the wave of different frequencies are propagated with different velocities. Long waves, however, suffer also amplitude dispersion, according to which high values of the surface elevation are propagated with greater velocity than lower values. This amplitude dispersion, producing steepening of the front of the wave, is familiar also in sound waves. We shall see that there are important kinds of water waves for which the deforming effects of frequency dispersion and amplitude dispersion are in competition.

3. Gravity waves of small amplitude

To illustrate frequency dispersion one may draw a graph of wave velocity c against wavelength λ , beginning with Cauchy's relation*

$$c = \sqrt{\frac{g\lambda}{2\pi}} \quad (1)$$

which is valid for gravity waves of small amplitude if the depth exceeds half a wavelength. (In fact, in the Cauchy solution, the disturbance half a wavelength below the surface is reduced to only 4% of its surface amplitude.) This wave velocity c is the velocity of individual crests, but the energy in the wave is propagated at the group velocity**

$$U = c - \lambda \frac{dc}{d\lambda}, \quad (2)$$

which is $\frac{1}{2}c$ for these waves. This is of course why the group of waves generated say by throwing a stone into a pond has its energy travelling slower than the individual crests, which therefore have to disappear on reaching the front of the group and passing out of the energy-containing region. Similarly, new crests have to appear at the back of the group.

When the wavelength exceeds $2h$, where h is the depth, the bottom begins to influence the waves, but we still have a simple solution if the depth is uniform, when the wave velocity is

$$c = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}} \quad (3)$$

tending to the constant value $\sqrt{gh}$ as λ/h becomes large (see the righthand part of the lower full-line curve in Fig. 1). The group velocity

$$U = c \left(\frac{1}{2} + \frac{2\pi h/\lambda}{\sinh 4\pi h/\lambda} \right) \quad (4)$$

* Plotted as the broken line in Fig. 1 below.

** The most general proof of this statement is that given in the Appendix to Rayleigh's Theory of Sound, which applies to an arbitrary conservative system capable of transmitting sinusoidal waves of all frequencies.

Figure 1. Wave velocity c versus wavelength λ for waves of small amplitude on water. (Note that the relative position of the points 1.7 cm. and $2h$ on the wavelength scale is variable; so is that of the points 23 cm./sec. and $\sqrt{gh}$ on the wave-velocity scale.)

is then closer to c , and indeed one observes that the fading-out of wavecrests at the front of a group of waves on a pond becomes less pronounced when the group moves into the shallower water at the edge.

4. Capillary waves

For very small wavelengths surface tension becomes comparable with gravity as a force tending to restore the horizontal condition of the surface, and then g must be replaced (if T = surface tension, ρ = density) by

$$g + \frac{T}{\rho} \frac{4\pi^2}{\lambda^2} \quad (5)$$

throughout. Then (Fig. 1) c has a minimum of 23 cm/sec. at $\lambda = 1.7$ cm. and tends to infinity as $\lambda \rightarrow 0$. The group velocity U exceeds c for $\lambda < 1.7$ cm. I should remark, however, that for water of very small depth h (about half a centimetre) the rise in c due to surface tension and the fall due to finite λ/h occur for the same value of λ and largely cancel out, and there is then very little frequency dispersion.* This

* Expanding $c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{T}{\rho} \frac{2\pi}{\lambda} \right) \tanh \frac{2\pi h}{\lambda}}$ in powers of h/λ gives

$c = \sqrt{gh} + O(h^4/\lambda^4)$ provided $h = \sqrt{3T/\rho g} = 0.5$ cm. for water. To illustrate the conflicting tendencies better, the curve is shown for a slightly larger value of the depth (0.55 cm.) in Fig. 1.

makes water of the particular depth 0.5 cm. specially suitable for producing flows closely similar to gas flows.

5. Viscous attenuation

Viscous effects modify the flow near the free surface and near the bottom in boundary layers of thickness about

$$\sqrt{\frac{\nu\lambda}{c}} \quad (6)$$

(where ν = kinematic viscosity) and cause damping by a factor of e^{-1} in amplitude in a time

$$\frac{\lambda^2}{8\pi^2\nu} \approx (\text{wavelength in cm.})^2 \quad (7)$$

Hence the capillary waves with $\lambda < 1.7$ cm. travel only a short distance before being damped out by viscosity. The longer waves are not seriously affected, but of course in a river *turbulent* friction is a much more important source of damping for these, as we will be discussing later.

6. Wave-making resistance to two-dimensional obstacles

The most striking evidence of frequency dispersion in river waves is to be found by observing the wave pattern due to obstacles in the stream or steps in the bed. We may consider first two-dimensional obstacles or steps (see Fig. 2). In steady flow these can produce only waves whose wave-velocity c equals the velocity V of the stream, so that their crests remain stationary. The energy in the waves is propagated upstream at the group velocity U and swept downstream with the stream velocity V , which equals c . Thus the energy input due to the resistance of the obstacle travels upstream for capillary waves with $U > c$, which therefore are found in front. The energy travels downstream, however, for the gravity waves with $U < c$, which therefore are found behind, and for larger obstacles the bulk of the wave-making resistance of the obstacle leaks back through these at the speed $c - U$. On the other hand, for supercritical streams, with $V > \sqrt{gh}$, no gravity wave with $c = V$ exists, and so the wave-making resistance of a two-dimensional obstacle is lowered. Similarly, the drag of a barge on a narrow canal is reduced when its speed rises above $\sqrt{gh}$, and Scott Russell [28] describes how this was first discovered due to a horse taking fright, and galloping off pulling its barge behind it, when it was observed "to the proprietor's astonishment, that the foaming stern surge which used to devastate the banks had ceased, and the vessel was carried on through water comparatively smooth, with a resistance very greatly diminished." The proprietor went on to achieve "a large increase in revenue" as a result of this scientific observation.

7. Stationary waves on non-uniform streams

I might remark that Mr. J. C. Burns and I investigated [4] what relation corresponding to $c = V$ a stationary wave must satisfy in a stream whose velocity varies with depth according to a one-seventh power distribution typical of turbulent flow over a smooth bottom. For small λ/h the waves are on the surface only, so that for them to be stationary we must have $c = \text{surface velocity} = 1.143 V_m$ ($V_m = \text{mean velocity}$). As λ/h increases, c drops (Fig. 2) till it equals V_m for $\lambda/h = 10$ and is $0.966V_m$ when $\lambda/h \rightarrow \infty$; this lower value is because the oscillatory motion for very long waves tends to be slightly stronger in the low-speed region near the bottom.

VELOCITY

OF STREAM

OF CRESTS RELATIVE
TO MEDIUM

(a) Pattern of waves generated.

(b) Graph which plots against λ/h , for stationary waves of length λ on a stream of depth h , with one-seventh power distribution of velocity and mean speed V_m , the ratio to V_m of the wave velocity c which waves of that length would have on still water.

Figure 2. Two-dimensional wave making.

8. Three-dimensional wave pattern due to an obstacle

I will now discuss briefly the three-dimensional pattern of stationary waves due to an obstacle at a point in a running stream (Fig. 3). Oblique waves are then present, and for them to be stationary those of velocity c must set themselves at an angle

Figure 3. Three-dimensional gravity-wave making due to an obstacle O . Waves generated t seconds ago are shown. $OP = Vt$, $PQ = ct$, since $\angle QOP = \sin^{-1}(c/V)$, $\angle OQP$ must be a right angle; so locus of Q is a semicircle. Locus of R (mid-point of PQ) is therefore also a semicircle. Similarly waves generated t' seconds ago lie on semicircle through P' , and the different semicircles envelop a wedge of total angle 39° , within which all waves are found. Waves on boundary of wedge are set as shown.

$\sin^{-1}(c/V)$ to the stream. At a time t after they have been emitted they might be expected to be at a distance ct from the centre of the disturbance, which however has drifted a distance Vt downstream. In this case their crests would lie along a line through the disturbance, like a "Mach line" in supersonic aerodynamics. All the waves emitted at the same instant would be on their Mach lines, and the whole lot would lie on the semicircle through the obstacle O and the point P to which the water passing the obstacle at the instant of emission has drifted. However, this neglects the fact that each wave group proceeds at the group velocity, and has travelled a distance Ut , not ct , since it was emitted. As a consequence the waves have not got as far as the semicircle. For stream speeds which are well sub-critical all the waves produced are essentially "deep-water" waves with $U = \frac{1}{2}c$, and then the waves lie on an inner semicircle of half the diameter. Every wave, produced at any time, lies on one such semicircle, and all the semicircles lie in a wedge of total angle $2 \sin^{-1}(\frac{1}{2}) = 39^\circ$; a characteristic pattern is formed by the specially strong waves at the boundary of the wedge, which are set at an angle of 55° to the stream. However, as the stream speed V approaches $\sqrt{gh}$ this semi-circle gets elongated at the front, and the wedge expands. At supercritical speeds the wedge contracts again because only waves with $c < \sqrt{gh}$ are present, and it is then a genuine Mach wedge of semi-angle $\sin^{-1}(\sqrt{gh}/V)$.

I have no time to mention the details of the charming pattern of capillary waves which forms ahead of the obstacle, which is especially prominent if the obstacle is small, like Rayleigh's famous "fish-line" [19], but this can be understood by directly similar arguments.

9. Waves of greatest height

I pass now from the frequency spectrum to what I may call the "amplitude spectrum" of water waves. This is best displayed on a diagram (Fig. 4) in which the abscissa is λ/h and the ordinate a/λ , where a is the vertical height between crest and trough. The wave velocity c for small a/λ is exhibited under the diagram as a reminder.

Figure 4. Amplitude-wavelength diagram for water waves. Under it is shown the wave-velocity versus wavelength diagram for waves of small amplitude (Fig. 1) as a reminder.

Now, there are two kinds of *breaking* which can limit the possible existence of periodic waves (Fig. 5). At the deep-water end of the scale there is the formation of the "wave of greatest height" which when it occurs as a stationary wave on a stream brings the water locally to rest at its wedge-shaped crest.* Further increase in the energy of these waves goes into spray formation. Stokes [30] predicted this wave and also observed it, writing: "In watching many years ago a grand surf which came rolling in on a sandy beach near the Giant's Causeway, without any storm at the place itself, I recollect being struck with the blunt wedge-like form of the waves where they first lost their flowing outline, and began to show a little broken water at the very summit. It is only I imagine on an oceanic coast, and even there on somewhat rare occasions, that the form of waves of this kind, of nearly the maximum height, can be studied to

* Values of a/λ for waves of greatest height in Fig. 4 were obtained as follows: for deep-water waves from Michell [25], for the solitary wave from McCowan [24] and for intermediate cases from Kishi [16].

Figure 5. Illustrating the two kinds of breaking of water waves. Curve (a): wave of greatest height in deep water; further increase of wave energy produces spray at the crests. Curve (b): amplitude dispersion converts this sinusoidal wave on shallow water in a very short distance of travel into the breaking form (c) and then into a succession of bores.

full advantage. The observer must be stationed nearly in a line with the ridges of the waves where they begin to break." Stokes calculated also the form of waves of intermediate height, which are usually called Stokes waves (see Fig. 4).

10. Amplitude dispersion

By contrast, at the long-wave end of the scale, say for $\lambda > 8h$, the existence of periodic waves is limited by amplitude dispersion. By this I mean the tendency for each value η of surface elevation to be propagated with the speed $\sqrt{g\eta}$ appropriate to the local depth rather than with the speed $\sqrt{gh}$ corresponding to the mean depth. It is well known how this* leads to steepening of the front of the wave until it may become a "hydraulic jump" or "bore."

11. Cnoidal waves and the solitary wave

However, if the amplitude is not too great, this amplitude dispersion may be exactly balanced by frequency dispersion in such a way that periodic waves become possible. These are the "cnoidal" waves of Korteweg and de Vries [17], so called because their waveform is the square of the Jacobian elliptic function known as "cn." Their existence depends on the fact that the higher harmonics in the wave travel slightly slower than the fundamental, which for particular waveforms can just balance the steepening just mentioned. Changes due to this frequency dispersion are proportional to h^2/λ^2 , but changes due to amplitude dispersion are proportional to a/h ; as a result, any given cnoidal shape is possible for a fixed ratio $(a/h) : (h^2/\lambda^2)$. The limiting case here is the solitary wave with $a\lambda^2/h^3 \approx 25$. For amplitudes above that of the solitary wave we must have steepening and bore formation. For amplitudes just below it the cnoidal wave train is beginning to degenerate into a sequence of isolated solitary waves.

* Together with the fact that the particle velocity is forwards where η is greatest and backwards where η is least.

Figure 6. Integral curves of equation (9), showing spiral singularity and transition between curves by means of a hydraulic jump.

The two curves (Fig. 4) for the different kinds of breaking meet at a point representing the solitary wave of greatest height computed by McCowan [24]. The linearised theory leading to sinusoidal waveforms is valid only if the amplitude is small compared with that giving breaking. Conversely, it is only if the amplitude is large compared with that of the solitary wave of given λ/h that the so-called shallow-water theory which neglects frequency dispersion and predicts steepening and the formation of a hydraulic jump or bore is valid.

12. Occurrence of hydraulic jumps in steady flow

We must now discuss these bores in more detail. They occur also, of course, as stationary jumps in steady streams wherever they decelerate from supercritical to subcritical velocity. Then the equations of steady flow

$$vh = Q, \quad v \frac{dv}{dx} = g \left(\alpha - \frac{dh}{dx} \right) - \frac{f\rho v^2}{h}, \quad (8)$$

where Q is the volume flow per unit breadth, α is the slope of the bottom and f a coefficient of friction, can be written

$$\frac{dh}{dx} = \frac{\alpha h^3 - fh_c^3}{h^3 - h_c^3} \quad (9)$$

where h_c is the critical depth $\sqrt[3]{Q^2/g}$ for which $v = \sqrt{gh}$, and the flow is subcritical when $h > h_c$ (which for constant h means $\alpha < f$, a gradual slope) and supercritical when $h < h_c$ (which for constant h means $\alpha > f$, a steeper slope giving the so-called "torrent flow"). At a point where α decreases through the critical value f , the integral curves of (9) have a spiral singularity (Fig. 6) and no continuous solution exists.* The curves specified by the upstream and downstream conditions (the latter being the so-called backwater curve determined by the water-level in the down-

* On the other hand, at a point where α increases through the critical value f , the integral curves have a saddle-point singularity, and a solution representing continuous acceleration through the critical speed exists.

stream reservoir [27]) must be joined by a hydraulic jump at which h changes discontinuously but the momentum flow

$$S = v^2h + \frac{1}{2}gh^2 = \frac{Q^2}{h} + \frac{1}{2}gh^2 \quad (10)$$

is continuous. Such a jump is possible because S has a minimum value for $h = h_c$.

13. Mechanics of energy loss in bores

Thus, classical bore theory (due to Rayleigh [26]) assumes conservation of volume flow Q across the bore, and conservation of momentum flow S (more precisely, the decrease in momentum flow v^2h is balanced by the difference in horizontal pressure force $\frac{1}{2}gh^2$ upstream and downstream). Energy, however, is lost at the bore; the total head $\frac{1}{2}v^2 + gh$ decreases by

$$\frac{1}{4}g \frac{(h_2 - h_1)^3}{h_1 h_2}, \quad (11)$$

and Rayleigh suggested this was due to friction.

Now it is certainly true that vigorous turbulent dissipation and churning-up of the flow is the main mechanism of energy loss in strong bores, but weaker bores have a different structure. They carry a train of waves behind them, whose position is stationary relative to the bore, and Favre [10] found that these waves exhibit no breaking, and that the flow appears perfectly smooth (Fig. 7) for depth ratios $h_2/h_1 < 1.28$.

Figure 7. Weak bore with train of waves behind.

As an example, the tidal bore formed on the British river Severn often carries such a train of waves along behind it. It is clear that the energy dissipation then takes place principally by radiation through this stationary wave train at a speed equal to the difference between the wave velocity and the group velocity—exactly as in the waves generated by a two-dimensional obstacle in a sub-critical stream. A calculation of amplitude by Lemoine [20], assuming sinusoidal waves, was based on the amount of energy lost per unit time (which is ρQ times (11) per unit span), and gave

$$\frac{a}{h_2} = 1.15 \left(\frac{h_2 - h_1}{h_1} \right), \quad \frac{\lambda}{h_2} = 3.0 \left(\frac{h_2 - h_1}{h_1} \right)^{-1/2} \quad (12)$$

The reason why fairly long waves are produced is that their velocity must equal the flow speed behind the bore, which is only just subcritical. As a result, a substantial amplitude is needed to carry away a fairly small amount of energy, simply because the difference between group velocity and wave velocity is small for such long waves. In support of this theory of Lemoine there is the observation that the waves form one by one behind the bore after it is first created—as the energy transmitted by the group of waves travels backwards.

14. Cnoidal wave trains behind bores

However, Mr. Brooke Benjamin and I thought it desirable to reexamine this theory on a more accurate basis [2], partly because* the ratio $a\lambda^2/h_2^3$ is too great for sinusoidal-wave theory to be applicable (it is about 10) and so the long waves behind the bore should really be treated as cnoidal waves. (Another reason was that although the 1.15 in (12) is close to the mean of the observations of many workers, it has been common to find on occasion values either much larger or much smaller.) To understand the subject better we re-worked the theory of cnoidal waves as follows. The momentum flow per unit span (divided by the density) is in general

$$S = \int_0^\eta \left(\frac{p}{\rho} + u^2 \right) dy \quad (13)$$

and for irrotational flow this can be calculated, for given volume flow Q per unit span and total head $R = \frac{1}{2}(u^2 + v^2) + gy$, as

$$S = R\eta - \frac{1}{2}g\eta^2 + \frac{Q^2}{\eta} \left(1 - \frac{1}{3}\eta'^2 + O\left(\frac{h^4}{\lambda^4}\right) \right) \quad (14)$$

Thus, taking in frequency dispersion to the first approximation only, we have a differential equation

$$\frac{1}{3}Q^2 \left(\frac{d\eta}{dx} \right)^2 + g\eta^3 - 2R\eta^2 + 2S\eta - Q^2 = 0 \quad (15)$$

as the equation for all stationary long waves.

The character of the solutions is easily seen by thinking of x as the time, and the equation as that of a particle of mass $\frac{2}{3}Q^2$, velocity $d\eta/dx$, potential energy equal to the cubic, and zero total energy. The particle would oscillate if the cubic has two zeros between which it is negative as in Fig. 8, curve B; this behavior oscillatory in x

Figure 8. Different possible positions of the graph of the cubic occurring in equation (15).

* Note that our a is twice the a of reference 2.

corresponds of course to waves (cnoidal waves, in fact). In the extreme case of curve A the amplitude becomes zero and we have a steady subcritical stream. In the other extreme case of curve C we have two solutions, either a steady supercritical stream or an oscillation taking an infinite time—this corresponds to the solitary wave. For positions of the curve beyond A or C there is no solution; at least, the depth would become zero in a finite distance.

Now in the bore problem we start upstream with a uniform supercritical flow, so we see that if R and S remain unchanged the only possible downstream flow is the solitary wave. To this extent Lemoine is proved wrong. However, you observe that it is necessary to reduce the total head R by viscous dissipation by only the slightest amount to raise the curve to one of type B. A reduction by the full value given by Rayleigh would of course raise it to one of type A, representing a uniform subcritical flow; no greater loss of head is possible without simultaneous loss of momentum. But if energy intermediate between zero and the classical Rayleigh value is dissipated, then the curve becomes one of type B and cnoidal waves will be present behind the bore. We find that these will have the amplitude calculated by Lemoine if 20% of the classical energy dissipation is accomplished by friction, but that amplitudes from zero to 1.3 times Lemoine's value are possible for other values of this percentage. These considerations give a reason for the wide scatter of the experimental results.

15. Energy-momentum diagram for steady flows

Next, following up these general ideas, Brooke Benjamin and I and Mr. S. C. De [2, 6] constructed a diagram (Fig. 9) giving the values of R and S for different kinds of steady flow with given volume flow Q . The abscissa and ordinate on this diagram are R/R_c and S/S_c , where R_c and S_c are the total head and momentum flow for a critical stream of volume flow Q ; actually, both R and S have minimum values for such a uniform stream with $v = \sqrt{gh}$. On this diagram the cusped curve represents uniform streams with the Froude numbers $v/\sqrt{gh}$ as marked, subcritical on the left, supercritical on the right. The broken-line barrier represents "waves of greatest height." Waves are possible in the thin region between this and the other two barriers. Weak, sinusoidal waves occur only near the left-hand barrier, cnoidal waves occur in the region below Z, Stokes waves in the region above Z. The solitary wave lies on the right hand branch of the cusped curve below Z, since this can appear out of a uniform supercritical stream without any loss of momentum or energy. The point Z represents the solitary wave of greatest height. For every point in the diagram the amplitude and wavelength have been computed (those for cnoidal waves by Benjamin and myself and those for Stokes waves by De). Hence the diagram can be used to find the characteristics of the waves generated by adding or subtracting any given amounts of energy or momentum from a given uniform flow.

In the problem of the weak bore, for example, one starts from a point below Z in Fig. 9, representing uniform supercritical flow, and moves to the left in proportion to the energy removed. If some momentum is lost also by friction at the bottom one moves also slightly down. Again, in the problem of wave resistance to a two-dimensional obstacle, one can use the diagram to obtain a theory for large-amplitude waves by starting at a point on the left-hand curve representing uniform subcritical flow and moving down (if the obstacle is streamlined so that only momentum is lost), or down and to the left (if dissipation occurs so that energy also is lost). In any problem, if one lands beyond the broken-line barrier the waves have exceeded their greatest possible height and must lose energy by breaking, which carries the point away to the left. Strong bores represent a transition from the isolated solutions above Z, representing uniform supercritical streams, to the range of physically possible flows to the left of the

Figure 9. Energy-momentum diagram for steady flows, including stationary-wave motions.

broken-line barrier. In these, obviously, much energy must be lost in spray and turbulence. Brooke Benjamin has recently shown that many transitions to wavy flow regimes resulting from changes in depth or breadth of the stream can also be understood by use of this diagram.

16. Attenuation of long waves by turbulent friction.

I must pass now to flows which extend over much greater lengths of river than those we have considered so far—over lengths so great, in fact, that wave attenuation by turbulent friction becomes important. If one begins by writing down the equation for long waves of small amplitude on a stream of subcritical velocity, V , resisted by a frictional force $f \cdot \rho V^2$ per unit area, one finds that for either direction of propagation they are damped approximately like

$$e^{-fVt/h} \quad (16)$$

so that for $f = 0.005$, say, their amplitude is reduced by e^{-1} in the time taken for the stream to travel 200 times its depth. This result appears for upstream propagation in Abbott's treatment of the formation of tidal bores [1] and for downstream propagation

in Whitham's treatment of flood waves [22]. I shall describe the further ramifications of this result separately in the two cases.

17. Conditions for the formation of tidal bores

One of the problems of river waves which most need an improved mathematical treatment, with the object of throwing light on what are the essential physical processes involved, is that of the propagation of the influence of tidal motions in an estuary into the upstream part of the river. Despite all the beautiful experimentation which has been done with tidal models, the physical mechanisms governing what part of the tidal energy passes upstream are still not fully clear. An interesting beginning, however, has been made by Mr. M. R. Abbott [1], and I should like to describe his work.

He uses a simplified method due to Dr. G. B. Whitham [32] for calculating wave propagation with amplitude dispersion. In this method, one first calculates the flow using the linearised equations for small disturbances. One then allows for amplitude dispersion by taking the paths of individual waves as calculated, and modifying each path by giving it at every point a wave velocity appropriate to the local depth and flow speed in the presence of the wave.

In this method, therefore, one must begin by considering the variation of wave amplitude on linearised theory. This is a combination of attenuation due to friction (equation (16)), and amplification due to reductions in the breadth and depth of the river as the wave passes upstream. If only the attenuation were present the amplitude would decay too rapidly for steepening to the extent of bore formation to occur. Therefore the reductions in breadth and depth are essential. Their effect can be calculated only very roughly. If no energy is reflected back down the river as the channel becomes narrower and shallower we have Green's law [12, 19]

$$\frac{a_2}{a_1} = \frac{b_1^{1/2} h_1^{1/4}}{b_2^{1/2} h_2^{1/4}}. \quad (17)$$

This can give very substantial amplification. At the other extreme, if the narrowing occurs suddenly, so that the maximum energy is reflected, then we have the much smaller amplitude increase

$$\frac{a_2}{a_1} = \frac{2}{1 + \frac{b_2 \sqrt{h_2}}{b_1 \sqrt{h_1}}} < 2. \quad (18)$$

Intermediate values occur in intermediate conditions. The condition for Green's law is that the length scale of the tidal wave is small compared with the length scale in which the breadth or depth changes by a large fraction. Now, the basic time scale (period/(2 π)) for tidal waves is 2 hours, so that for a typical wave in a river of moderate depth the length scale would be around 20 miles. This is unlikely to be small compared with the length scale for changes in breadth and depth. For greater rivers both scales would be larger but the conclusion would be much the same. However, for want of a better approximation, Abbott used Green's law of amplification combined with the attenuation due to turbulent friction and deduced a condition for a bore to form. This requires that the amplitude shall remain sufficiently high for sufficiently long, assisted by narrowing of the river and in spite of turbulent energy loss—sufficiently. I say, for neighboring waves to run into one another due to amplitude dispersion. The condition may be expressed in terms of the total variation of tide level in the estuary, and for the Severn Abbott predicted bore formation if the tidal variation exceeds 39 feet. This is close to what is observed; the tidal variation is 41 feet at the spring tides and 22 feet at the neap tides and bores appear only during a few days before and after the highest tide. Again, Abbott's theory gives the position where the

bore is formed correctly as 12 miles upstream of the estuary; for comparison, one obtains only 4 miles in a theory neglecting friction. The agreement on both points is really far better than it should be in view of the fact that the tidal length scale is too great for Green's law to be applicable.

However, there is another observed fact which may explain this mystery. The theory indicates that where the bore forms the local depth η has a vertical tangent but has already been rising for some time. What is observed is that the bore when it arrives is the initial increase of depth from the low-water mark. This indicates that perhaps the wave as a whole has not obeyed Green's law, but that only the steepened part, where the effective length scale has been greatly reduced by amplitude dispersion, has been capable of extracting the full Green's-law amplification out of the reductions in breadth and depth. The rest of the wave has been left behind, its energy being reflected downstream (as well as dissipated by turbulence), and only the concentrated part of the disturbance has reached the upper river. This hypothesis would explain why the use of Green's law for this concentrated part gives a good result for the criterion, and predicted position, for bore formation. However, it is clear that a more exact analysis of this problem is greatly needed.

18. *Effect of friction on the downstream propagation of flood waves*

I come now to the effect of the turbulent frictional dissipation of long gravity waves on the downstream propagation of flood waves; with this problem of how the effect of variations over a period of days in the runoff into the river is felt far downstream we reach perhaps the extreme long-wave end of the spectrum of river waves.

Now, it is clear that any local increase in flow will cause a signal to be propagated downstream at the velocity $\sqrt{gh}$ of long gravity waves. However, we have seen that this signal is damped by a factor of e^{-1} in the time that it takes the river to travel a distance of 200 times the depth, say in 10 minutes or so. Here there will in general be no compensating amplification, by reductions in breadth or depth, and therefore in about an hour the signal moving downstream at the speed of gravity waves will be completely damped out. The question then arises: what has happened to the additional fluid put in at the beginning? It must be found somewhere in the river! (Note in passing that such an appeal to continuity could not be made in the upstream propagation problem, where a rise in the estuary water-level can perfectly well be accommodated through a steady flow in the river, the so-called "backwater curve.") No steady flow, however, can pass down the river a flow quantity whose value at one upstream point is varying).

The answer to the question can be obtained in any particular case by getting a complete solution of the equations for small disturbances. This has been done for a channel of uniform breadth and slope and a velocity-squared resistance law by Whitham, who included it as section 3 of our joint paper on kinematic waves [22]. This proves that the maximum of the disturbance travels downstream at the velocity $\frac{3}{2} V$ (where V is the stream velocity); this of course is in general far less than the velocity of gravity waves $V + \sqrt{gh}$. Such a theory shows that, although the characteristics of the partial differential equations represent propagation at the speeds $V + \sqrt{gh}$ and $V - \sqrt{gh}$, nevertheless signals propagated along those characteristics are so damped that for changes extending over really long periods they are unimportant, and only the propagation speed $\frac{3}{2} V$, with which the bulk of the energy travels, is vital to the problem.

This propagation speed is that of the Kleitz-Seddon flood waves, which can however be treated separately under far less restrictive conditions both as to the character of the river and the size of disturbance which is treated, and which therefore,

after the investigation showing how in a typical case they come in time to dominate the scene, must now be described in detail.

19. Kinematic waves

Although such flood waves were discovered 98 years ago by Kleitz (unpublished), also treated very ably by Boussinesq [3] in 1877 and greatly illuminated by James Seddon [29] in 1900 with his studies of the Mississippi and the Missouri, Dr. Whitham and I felt that the subject was still sufficiently little-known, and contained a sufficient number of doubts and ambiguities, to warrant a new treatment [22]. In this treatment we considered such flood waves as a special case of a general class of wave motions which we ventured to call "kinematic waves."

Kinematic waves exist in any one-dimensional flow system if, to a sufficient approximation, there is a functional relationship between

- (i) the flow q (quantity passing a given point in unit time),
- (ii) the concentration k (quantity per unit distance), and
- (iii) the position x .

Thus we assume that

$$q = q(k, x) \text{ or } k = k(q, x). \quad (19)$$

Note that q/k is the mean velocity of flow, V say, so that the assumption also fixes V as a function of flow or of concentration at every place in the system.

In the application to river waves, "quantity" is taken as volume of water; q is volume flow rate, and k is the cross-sectional area of the water. For any river there is a series of possible steady regimes, each with uniform flow rate q all along the river, and k (or V) is a definite observed function of x for each regime; if this is known for every value of q we have our function $k(q, x)$. I shall have more to say about the mechanisms that govern the relationship between k and q at a given position x , but a common one is of course the balance between gravity and friction. In a flow in which q is varying there is obviously a time lag before the changes in k at a point catch up with the changes in q , but for sufficient slowly-varying flows it is reasonable to neglect this time lag.

On this basic assumption, the properties of kinematic waves follow from the equation of continuity alone,

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (20)$$

It is for this reason that the name "kinematic" is suggested, in contrast to the classical wave motions which we would call "dynamic," depending as they do on Newton's second law of motion. Another contrast is that kinematic waves possess only one wave velocity at each point, while dynamic waves possess at least two (forwards and backwards relative to the medium). This is because equation (20) is a first-order equation. On multiplication by

$$C = \left(\frac{\partial q}{\partial k} \right)_{x = \text{constant}} \quad (21)$$

it becomes

$$\frac{\partial q}{\partial t} + C \frac{\partial q}{\partial x} = 0 \quad (22)$$

This means that the flow q is constant on waves travelling past the point with velocity C . Mathematically, the equation has one system of "characteristics" (given by $dx = C dt$), and along each of these q is constant.

The kinematic wave velocity C is the slope of the flow-concentration curve for fixed x . This fact is known in the literature of flood waves as the Kleitz-Seddon law.

20. Other fields of application of kinematic wave theory

Other fields of application in which extensive use of kinematic wave theory has proved possible are the theory of traffic flow on long crowded roads [23], where one starts from the assumption (see Fig. 10) that at each point of the road the mean velocity is a function of vehicle concentration (falling from a maximum velocity at zero concentration to a zero velocity for a certain critical concentration k_J — J for Jam); and, again, the theory of sedimentation of small particles in a liquid, where also the velocity falls as the concentration increases, this time as a consequence of Einstein's law for the viscosity of suspensions. This theory was worked out in 1952 by Professor G. J. Kynch [18], and it was only after Dr. Whitham and I had published our paper on the general theory that we realized that Professor Kynch had anticipated many of our results, at least as far as this application is concerned.

I might remark also that at an early stage of our work we felt that the fact that, in flood waves, the energy travels not at the gravity-wave velocity but at the slower

Figure 10. (a) Mean velocity V and flow q plotted as a function of concentration k for rural roads in U.S.A.: data due to B.D. Greenshields. *Proc. Highw. Res. Bd.* 14, 448 (Washington, 1935). (b) Flow-concentration curve for flood waves, showing construction to determine speed of kinematic shock wave.

speed of the kinematic waves has so much in common with the theory of group velocity (in systems with what I have called frequency dispersion) that it might in some obscure way be a special case of that theory. We could not sustain this view for long, because no frequency analysis comes into the floodwave problem, but later we saw that the two problems were differently related, both being in fact special cases of the general kinematic wave theory.

To see this for the case of wave propagation with frequency dispersion, one has to apply a law of conservation of number of wave crests. This gives that the flow q is the number of wave crests passing per unit time, which is the frequency (c/λ), and that k is the number per unit distance, namely $1/\lambda$, so that the frequency remains constant along kinematic waves travelling at the speed

$$\frac{dq}{dk} = \frac{d(c/\lambda)}{d(1/\lambda)} = c - \lambda \frac{dc}{d\lambda}, \quad (23)$$

which is in fact the group velocity (2). Conservation of number of wave crests is not,

of course, accurately true; but, once sufficient dispersion has occurred to render small the frequency change in a single wavelength, then the appearance or disappearance of crests through the occurrence of horizontal points of inflexion must become very rare.

21. Kinematic shock waves

Now kinematic waves themselves do not suffer frequency dispersion, but they do suffer amplitude dispersion, and it is the consequences of this fact that Dr. Whitham and I have felt were inadequately covered in the literature, making a new treatment necessary. Kinematic waves are, as we have seen, totally different from long gravity waves, and are important for problems with time scales over which long gravity waves would be completely damped out, but they have this property of amplitude dispersion in common. Thus, the wave velocity C is a function of the flow q carried by the wave, as well as of the position x . Since the mean velocity V increases with concentration, the graph of $q = kV$ against k is concave upwards, and $C = \partial q / \partial k$ increases with k , and also with q . Hence waves carrying high values of the flow travel faster than waves carrying low values, and it is possible for the former to overtake the latter, giving discontinuities in the flow q . We suggested the name "kinematic shock waves" for these discontinuities, since their process of formation is exactly like that of shock waves in a gas. It is similar also to the process of bore formation in long gravity waves which we discussed earlier.

Now, if a discontinuous wave moves downstream with velocity W , it is easily seen that the flow across the moving wave is $q - Wk$, and it follows by continuity that the law of motion of such a kinematic shock wave requires that this expression be continuous across it, giving

$$W = \frac{q_2 - q_1}{k_2 - k_1}. \quad (24)$$

This speed W of the kinematic shock wave is the slope of the chord (Fig. 10) joining the points (k_1, q_1) and (k_2, q_2) on the flow-concentration curve representing the upstream and downstream conditions. In the limit of a weak shock wave it coincides with the slope of the tangent, that is, $\partial q / \partial k$, the velocity of continuous waves.

Now one must not suppose from the use of the word "discontinuity" that these kinematic shock waves are very concentrated things, similar to bores. Actually, owing to the time lag needed for adjustments of flow and concentration to catch up with one another, such a wave, though discontinuous on the simple theory, may extend over many miles. It is in fact the so-called "monoclinical flood wave" which in the special case of uniform flow upstream and downstream has been computed for special river models by Thomas [31] and Dressler [7] as well as by Whitham [22]. Our suggestion for treating more general cases is that a first approximation to the flow be obtained by kinematic wave theory, and that the results be then improved by replacing such discontinuities as arise by calculated profiles of this kind, choosing those appropriate to the computed strengths of the kinematic shock waves (see lower diagram in Fig. 11). The justification for this is that it is only in the places where the flood-wave profile has been considerably steepened by amplitude dispersion that the time lags for adjustments of flow and concentration to catch up with one another will be important.

22. A new method for calculating flows with kinematic shock waves

It is unnecessary to give here any details of special calculations involving kinematic waves, but I will describe a new method of determining the position of the kinematic shock wave as a function of time, which is simpler than the methods given in our original paper.

To fix the ideas, suppose that the flow q is measured as a function of time at a particular station $x = x_1$ in a river, and that we want to predict what the flow variation

with time will be at some other station $x=x_2$ far downstream. Then each value q of the flow at x_1 will remain constant along a wave which will take a time

$$T(q) = \int_{x_1}^{x_2} \frac{dx}{C(q,x)} = \int_{x_1}^{x_2} \frac{\partial k}{\partial q} dx = \frac{\partial}{\partial q} \int_{x_1}^{x_2} k(q,x) dx \quad (25)$$

to reach x_2 . This $T(q)$ can be calculated in various ways, but one way is to notice that $T(q)$ is the derivative with respect to q of the total volume of water between x_1 and x_2 when the river is flowing steadily at the rate q .

The deformation of the curve is now easily sketched (Fig. 11). Each point (q,t) on the flow-time curve at x_1 becomes $(q,t + T(q))$ at x_2 . We then have to answer the question: what happens if waves carrying high values of q overtake those carrying low values, leading to a curve of the form illustrated? The simple answer is

Figure 11. (a) Construction to determine position of kinematic shock wave. Lobes shown must be of equal area so that total area under (q,t) curve remains unaltered.

(b) Replacements of kinematic shock wave by calculated "monoclinal flood wave" profile.

that the kinematic shock wave (or discontinuous jump in q) must be fitted in at that value of t for which the total area under the (q,t) curve, that is the total flow across $x=x_2$, remains unaltered by the substitution.

To see this, one need observe only that the (q,t) curve has been deduced from a theory based on the equation of continuity. Therefore the area under it, or total flow across $x=x_2$ in a certain time interval must equal the total flow across $x=x_1$ in the same time interval, plus the volume of water between the stations at the beginning, minus the volume of water left at the end.* However, in the real motion the total flow

* Mathematically this says only that

$$\int_C (qdt - kdx) = \iint_S \left(\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} \right) dxdt = 0 \quad (26)$$

where C is a curve in the (x,t) plane and S its interior.

across station 2 must also equal the total flow across station 1 plus this difference in the volume between the stations at the beginning and end of the time interval. Therefore the true discontinuous flow must have the same area under its (q,t) curve as the calculated continuous flow which it replaces.

By this rule the development of the flood as it progresses downstream is easily calculated, and the kinematic shock wave is found first to grow in strength, and later on to decay (as the total length of the wave increases).

23. *Mechanisms governing the flow-concentration relation*

I should like to discuss now the problem of what the flow-concentration curve should be taken to be in the kinematic-wave theory of flood movements; this question is the same as asking: what is the steady regime in the river for different uniform values of the flow? Here the point I want to make is one which was already shown clearly by James Seddon in his article [29] of 1900, namely, that this question can only be answered observationally. The reasons need not be discussed at length here as the principal arguments are clearly set out in Seddon's very readable article as well as summarised in the paper by Dr. Whitham and myself.

The main point is that the equations of hydraulics [11, 27], which were developed largely for flow in man-made conduits of regular cross-section, are misleading for natural rivers. For a conduit one could certainly apply kinematic wave theory, using for example the Chézy law of frictional resistance for relating velocity to mean depth, which gives for a rectangular section

$$q \propto k^{3/2}, \quad C = \frac{dq}{dk} = \frac{3}{2} \frac{q}{k} = \frac{3}{2} V \quad (27)$$

Or one could use the Manning frictional law, which gives $C/V = \frac{5}{3}$ or with either law and a different shape of cross-section one can get different values of C/V . But the resulting theory could not be applied to the irregular bed shape of any real river. As a matter of fact, if one knew precisely the dynamics of the flow there would be little point in using a kinematic-wave theory, except as a rough guide, because one could put the complete equations of motion on to an electronic computer.

However, the variations in bed shape along the river contribute many mechanisms controlling the flow rate which are ignored by the hydraulic equations; for example, local flows similar to that through a vertical slit, or over a submerged weir. To illustrate the point further I shall give details of just one of the many difficult problems arising in steady river hydrodynamics, that of the occurrence of reversed-flow regions. My object in discussing it is mainly to emphasize the complication of the subject and to suggest that at present a dynamic theory is too difficult, but that a theory like kinematic wave theory, based on one assumed relation which is especially susceptible to direct observational determination, is the best point from which to start in the problem of flood waves.

24. *Reversed-flow regions in steady river flow*

Now, the standard works on hydraulics often consider flows in which the depth h varies considerably across the width of the river, but they do not usually discuss clearly what is the associated distribution of the flow velocity v across the width. Sometimes it is tacitly assumed that v is approximately uniform, but for small ratios of depth to width the bottom effectively prevents the existence of eddies which are anything like big enough to diffuse a nearly uniform velocity over the width of river (in the way that happens, say, in a pipe). An alternative hypothesis is that at each point across the

width gravity and friction are balanced, leading (say, for the Chezy resistance law) to a relation

$$v = K \sqrt{h\alpha} \quad (28)$$

where α , the slope of the free surface, is constant across the river and can be determined by integration from the given total flow rate q . This approach has a solid basis for channels of uniform section.

However, in rivers one often observes a quite different kind of flow, which is actually reversed in the regions of shallow water (Fig. 12). This is connected with the general tendency of flow to separate, not only from solid obstacles, but also from local areas of high resistance, for example a row of trees used as a wind break.

The explanation is that if separation did not occur there would still be only a slow flow in the area of high resistance. On the other hand, the velocity just outside it (on the other side of the river) needs to be greater than its values upstream and downstream if the total flow rate q is to be passed. As this high velocity slows down (farther downstream) a rise in water level must occur, by Bernoulli's equation, and this can bring to rest the slower fluid which is in the area of high resistance. Hence separation and reversed flow set in.

It would be possible to treat these flows by equations of the usual boundary-

Figure 12. Separated flow resulting from presence of region (shown dotted) of shallow water.

layer type, because the condition of constant water elevation across the river (corresponding to the condition of constant pressure across a boundary layer) is very nearly fulfilled. The only difference from the usual boundary-layer equations would be the occurrence of a turbulent friction term. The condition for separation should emerge from such a treatment, but to my knowledge this has not so far been carried out.

Similar reversed flows occur in estuaries, where at high tide the presence of a submerged sandbank is often revealed by a line of foam which has collected along the locus where the forward and reversed flows meet.

25. Improvements of kinematic wave theory

At this stage it can be reasonably objected that I am advocating the use of a kinematic-wave approach to the prediction of flood movement, based on an experimental flow-concentration relationship, but am giving no suggestions on what should be done if changes are taking place somewhat too rapidly for the time lag between adjustments of flow and concentration to be accurately negligible. This serious criticism I must now try to meet.

The situation is similar to one that has arisen in the aerodynamics of rarefied gases, when one tries to apply the theory of sound waves of finite amplitude and shock waves to problems with characteristic length scales of only a few mean free paths, so that such shock waves as occur are really thick. Important advances in this subject were made recently by Prof. Eberhard Hopf [13] and Dr. Julian Cole [5], and I gave a survey of it as my contribution to the G. I. Taylor anniversary volume [21]. There the effect of lag in establishing equilibrium can in many cases be treated by taking a second approximation to the flow equations in the form of the Burgers equation, which has a

convenient exact solution for any initial conditions. It may be that a development along these lines, involving a suitable approximation to the non-linear terms in the equation of continuity, and the incorporation of a second-derivative term to represent the effect of lag, will prove practicable.

Alternatively, there is the solution given in our original paper, based on making a direct observation of how the flow varies with the rate of increase of concentration at a point for given concentration. We showed that if the ratio of the change of concentration to the change of time-derivative of concentration required to produce a given flow change is taken constant (say as an average of the observed values), then such an approach would necessarily give the kinematic shock waves their correct position and velocity.

As a final alternative, it may be that one who wants real accuracy rather than just a bird's-eye view of the picture may always need to go, as at present, to models, with the detailed characteristics of each stretch of the river reproduced in detail in the model.

26. Roll waves

As my last topic, I come to what Dr. Whitham found [22] when he applied the theory of the attenuation of gravity waves by turbulent friction to torrents. If $F = V/\sqrt{gh}$ is the Froude number of the stream, then on the model described earlier the attenuation factor

$$e^{-fVt/h} \quad (29)$$

has to be replaced by

$$e^{-\frac{fVt}{h}(1-\frac{1}{2}F)} \quad (30)$$

so that when F reaches 2 attenuation ceases. This is due to the fact that on the particular model used the kinematic wave velocity $C = \frac{3}{2}V$ with which the main part of the flow is carried becomes equal to the gravity-wave velocity $V + \sqrt{gh}$, so that gravity waves thereafter carry the flow without attenuation. It seems that for any system this will happen when C reaches $V + \sqrt{gh}$.

What is not so clear at a first glance is what happens when the kinematic-wave velocity actually exceeds the gravity-wave velocity. However, Whitham's analysis com-

Figure 13. Roll waves (after Dressler [71]).

bined with earlier work by Jeffreys [14] and Dressler [7, 8, 9] shows that these flows with $C > V + c$ are unstable and tend to break up into the so-called "roll-waves," which can be regarded as alternations of kinematic shock waves* and bores (Fig. 13). Good two-dimensional roll-waves are observed only with a fairly flat bottom of regular slope, but a similar instability involving, if necessary, three-dimensional disturbances occurs in any sufficiently rapid torrent.

On a small scale, one can easily observe roll waves on a thin film of water flowing down a vertical glass surface; it would be interesting to try to develop the theory of these capillary roll waves, taking surface tension instead of gravity as the restoring force and laminar instead of turbulent friction.

27. Conclusion

In conclusion, I hope that the steeplechase on which I have led you over, and round, the difficulties of this extensive subject has at least suggested to you that interesting questions abound in the study of river waves. There may be some in the audience who like myself have devoted much of their working life to the quite different question of trying to unravel the mysteries revealed by the beautiful schlieren and interferometric photographs which are taken in supersonic wind tunnels and shock tubes. To them I would suggest that, fascinating though these subjects are which relate to the behaviour of gases under very extreme conditions, there is sometimes an even greater delight to be derived from fluid motions in which one can become immersed not only metaphorically but also literally.

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DISCUSSION

J. J. Stoker

All of you will agree, I am sure, that Professor Lighthill has given us a very stimulating discussion. While it was confined to a very small section of water wave phenomena, it still shows that there is a huge variety in even small portions of that subject, and a variety that stems not only from variety in the physical phenomena, but also from variety with respect to possible ways of mathematical treatment.

Professor Lighthill has already alluded to the fact that he and I might not agree in all respects, about some of the things that he said concerning wave motions in rivers.

I have a few slides giving results of calculations which a group of people at our Institute have been making during the last three years. We have been involved in computing flows in large rivers and reservoirs, with the object of making flood predictions for lengthy periods of time, and over long stretches of rivers. For example, we have dealt with floods in the Ohio River between Wheeling and Cincinnati; that is a stretch close to 400 miles in length. We have also computed flows through the junction of the Ohio and Mississippi at Cairo, Illinois, taking forty mile lengths of the rivers on both sides of Cairo upstream in the Ohio and Mississippi, and the same distance downstream in the lower Mississippi. Calculations have been made for flows in Kentucky Reservoir at the mouth of the Tennessee River. This reservoir lies between two dams, the Kentucky Dam on the downstream side, just about where the Tennessee runs into the Ohio River, and the Pickwick Dam, some 180 miles upstream from there.

Now it is true that kinematic waves, which Professor Lighthill described so nicely here, certainly give a correct general picture of what happens in a large river. There is no doubt that the main disturbance is one which goes much slower than the propagation speed of wavelets, and the main disturbance does indeed travel downstream in general, and not both up and down stream, in spite of the fact that the basic theory formulated by the exact differential equations leads always to wave propagation in both directions.

However, we would be in very great difficulties in the Ohio River if we tried to base our calculations on "kinematic waves," because even the main disturbance sometimes goes up-stream in the Ohio, because the main flow is very much influenced by the flow from tributaries and the local drainage areas. That part of the flow is so large sometimes that the main flood travels in the wrong direction, so to say. We find it necessary to use the full differential equations, both the momentum equation and the continuity equation, and consequently we must put up with the fact that the differential equations have two real characteristics which require us to use small time steps for calculation with the UNIVAC.

In the Kentucky Reservoir if one wants to make predictions for three weeks, there is ample time for many waves to propagate back and forth from the dams at both

ends, so that propagation upstream is quite vital in this case, also, and thus one needs to consider the waves that are going in both directions.

I will cut this short by showing a few slides giving the results of calculations at the junction of the Ohio and Mississippi, and in the Kentucky Reservoir.

The first slide (Fig. 1) shows stage at Cairo, at the junction of the Ohio and Mississippi, and this is a prediction for sixteen days. (Stage means height of the river.) One sees that the flood stages are predicted quite accurately. This was done on the UNIVAC, and the calculation requires a few hours of UNIVAC calculating time. That

is a lot of calculating, since the UNIVAC can do a great deal in two hours time. In addition to the initial state of the system (i.e. the initial stage and velocity at all points), the flows at the upper end of the Ohio and Mississippi are assumed known over the whole time of the prediction. The second slide (Fig. 2) shows the results at Hickman, below Cairo, in the lower Mississippi, and again this shows stages for a sixteen-day forecast. At Hickman, the relation between stage and discharge was known. This is an impedance condition which simulates the effect of the rest of the Mississippi below that point. As one sees, the observed flood and the calculated one agree very well in this case. The difference is a matter of inches, even in a period of two weeks.

The third slide (Fig. 3) gives our results for the Kentucky Reservoir. The lowest curve is for Kentucky Dam, at the mouth of the Tennessee. The next one above is Perryville, somewhere near the center of the reservoir, some eighty miles upstream from Kentucky Dam. Pickwick Dam is the dam at the upper end of the reservoir. The length of this reservoir is about 186 miles.

Again we were given initially the state of the reservoir. That is, flow velocity and depth or stage were known throughout the reservoir at the initial instant. The discharges out of Kentucky Dam and into the reservoir from Pickwick Dam were given,

Stages in the Junction Problem for 16 Day Forecast

and also the flows from tributaries, which contributed as much as 30 per cent of the flow sometimes. By integrating the differential equations we find the results that you see on the slide. Again computed and observed values are very close—the difference between the observed and the calculated stages is a matter of inches.

G. K. Morikawa

I wish to make a brief comment on Prof. Lighthill's interesting paper, in particular, about his "kinematic waves" (Lighthill and Whitham, 1955). An unexpected feature of his proposed theory of flood movement is that shock (or bore) formation is included as an essential part of the theory. This approximate theory might appear rather strange at first glance since, in river flows, resistance plays such an important role. (Of course, in flow with no, or with small, resistance we know that shocks are always imminent.) For the kinematic wave approximation, all derivative terms (inertia plus gravity-surface slope) in the momentum eqn. (20) (see Ref.) are ignored, leaving only the gravity-bottom slope to balance the resistance force in the momentum equation. The neglect of the restoring force provided by the gravity-surface slope term allows bore formation.

It is well known that bores are formed only under very severe conditions such as those encountered in very steep rivers or at large tidal estuaries. In slow, long rivers, bores are not formed even under the extreme conditions of flood stage during which time heavy tributary flows into the main stream cause relatively (compared to the bottom slope) large surface gradients. These steep gradients generally are damped as they travel downstream. However, I hope that the "kinematic wave" approach, which seems quite attractive from the viewpoint of its possible simplicity, will be pursued further in the study of flood movement and other physical problems.

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M. J. Lighthill

In reply to Dr. Morikawa, there are different kinds of "shock waves." One is the dynamic shock wave, which we call a bore, and the other is the kinematic shock wave, which is very extensive, covering many miles. You could see, from one of the curves Professor Stoker showed, this rapid rise at the beginning of the flood.

Now, of course, my answer to Professor Stoker is that I agree with him. On the other hand, he has the advantage of having come to the end of his programme, and being able to show the final results. The kind of thing that Whitham and I have suggested hasn't yet been applied, and of course I wasn't able in the lecture to discuss all the modifications we propose. When run-off, for example, is present, as Professor Stoker was describing, we incorporate it in the theory in a perfectly straightforward manner (described in our paper). We also attach importance to the inclusion of second-derivative terms which in our paper we called diffusion (rather like diffusion in a gaseous shock wave, which increases its thickness) but which in my lecture I treated only in a brief section.

This theory, as it were, hasn't been fed into the data on which Professor Stoker worked.

Now, I am not suggesting that it would give such good results as the kind of thing Professor Stoker does, because obviously if you spend a lot of money, if you use very big machines, and very long and complicated models, you are bound to get better results. To a large extent it is a matter of compromise. By applying kinematic waves, and then improving the theory, you may be able to get rough results which have some

value fairly quickly. On the other hand, you can go to the construction of a model of the river, or, what is the equivalent of a model, use of a digital computer. You are obviously best-off to use the model, if you want an accurate answer.

J. J. Stoker

I would say that if you once decide to solve these problems by integrating differential equations by one scheme or another, then the biggest job is the processing of the data, obtaining the resistance coefficient, getting the cross-section areas, coding all of this, and setting up a procedure. It could be that by using simplified differential equations, the whole thing might be done on a smaller computing machine, which would be less costly in its operation. On the other hand, as I have pointed out, it doesn't take very long anyway to make these calculations.

M. J. Lighthill

Three years!

III

REMARKS ON THE OCEAN WAVE SPECTRUM *

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ABSTRACT

A record of water level against time at a fixed point off La Jolla can be interpreted as the result of superposition from waves of all possible frequencies. A typical analysis into component frequencies shows five distinct bands of activity with frequencies higher than those of the tides: ripples around 20 c/s, ordinary gravity waves around 0.1 c/s, a narrow band of swell at 0.065 c/s, "surf beat" around 0.015 c/s (due to non-linear transformation of ordinary waves in shallow water), and surges with frequencies around 0.001 c/s. Quantitative estimates of energy density are given. The measurement of the directional properties of waves is discussed in some detail.

INTRODUCTION

"Remarkable progress would be attained if only naval architects and oceanographers would get together and study ship motion at sea." You may have come across statements like this; they frequently appear in the literature of the last hundred years. Offhand this sounds quite reasonable. But we doubt whether the proposed marriage of the two professions would have resulted in a healthy offspring if it had taken place prior to the last decade. The oceanographer simply could not offer an *adequate description* of the sea surface.

The latest and most successful of matchmakers is J. King Couper of the Bureau of Ships. We think his success is due, at least in part, to the circumstance that we are now more than halfway towards an adequate description of the surface in terms of the energy spectrum. The first step was to resolve the sea surface at a point in terms of elementary wave trains of various frequencies, but regardless of wave direction. This has been done. The next step is to obtain the directional pattern as well. Work on this problem is underway. When it has succeeded, an important goal will have been attained: the two-dimensional spectrum of the sea surface.

Two reviews have recently been published: Pierson (1955) emphasizes how much is known about what waves are like; and Ursell (1956) insists that nothing is known about how they are generated. Our remarks will be confined to a few selected topics: first a few basic (and well known) features of ocean wave spectra; then low frequency waves, because we find them interesting; and finally, the directional spectrum, because it is an urgent topic.

* Contributions from the Scripps Institution of Oceanography, New Series, No. 950.

DISCRETE AND CONTINUOUS SPECTRA

Suppose we were to convert the fluctuating water level above a point into a fluctuating voltage, pass this through a filter which responds fully to frequencies between $f - \frac{1}{2}\delta f$ and $f + \frac{1}{2}\delta f$ and not at all to others; then square the filter output and average. Designate the final output by $E(f) \delta f$. If the ocean waves consisted of one or several *discrete* frequencies, $f_1, f_2, \dots$, then we should expect a large output $E(f_1) E(f_2) \dots$ when the filter is tuned to these frequencies, and a very low output otherwise. What we find in fact is that $E(f)$ varies more or less smoothly with f , as if energy were distributed continuously among all possible frequencies. No matter how narrow we design the width δf of the filter, the result is the same. This leads us to suspect that the ocean wave spectrum is a continuous and not a discrete spectrum.

In practice we must always deal with records of finite length, T , and the best we can do is to analyze for harmonics of this record length. This means that we can sample only for frequencies separated by T^{-1} . But this discreteness in sampling has nothing whatever to do with whether the true spectrum is discrete or continuous. In the former case we would find that the energy content is insignificant except for just two or three lines in the immediate vicinity of the discrete frequencies present; in the latter case we would find individual energy values to be badly scattered, but if T is sufficiently long so that we can crowd (say) 10 or more values into the frequency range to be analyzed, $T = 10/\delta f$, then the average of these ten values is a meaningful measure of the energy density in this range.

The important thing is that all pertinent measurements of ocean waves with periods shorter than the tides have been in agreement with the expected results from continuous spectra, and none have given evidence for discrete spectral lines.

One observation all of us have made suffices to dispose of the idea that the ocean surface can be represented by a single-frequency, single-directional wave train. Consider sun glitter on the sea surface (Cox and Munk, 1956). The glitter pattern is composed of millions of tiny images of the sun, an image occurring wherever the sea surface has the slope (tilt and azimuth) required to reflect the sun's rays into the observer's eye. The radial grid lines in Fig. 1 give the required azimuth of tilt (to the right and left of the sun), the quasi-elliptical lines the required tilt, drawn for a solar elevation of 30° . Suppose the water pattern were represented by a train from a direction 60° to the right of the sun, with a maximum slope of 20° . The resulting glitter pattern is shown in the diagram on the left of Fig. 1. It lies entirely along the 60°

Figure 1. The glitter pattern due to a single sinusoidal wavetrain approaching from a direction 60° to the right of the sun, and a photograph of a real glitter pattern.

azimuth line. The glitter is brightest where the slope reaches the maximum values, these being the most probable slopes of a sinusoidal wave. Actual conditions are just the opposite. A real glitter pattern as shown in the photograph does not follow any particular azimuth line, and it is brightest in the center, where the required tilt is zero. The observed result is consistent with a continuous two-dimensional spectrum.

THE SPECTRUM FROM .2 to 200 C/KS

Most of our work deals with frequencies that are dismissed as D. C. by the radio engineer. For unit frequency we use cycles per 1000 sec., or c/Ks. Fig. 2 shows a typical La Jolla frequency spectrum (non-directional). The ordinate E in cm^2 per c/Ks is referred to as the "Energy density." The range between 50 and 200 c/Ks comprises the ordinary sea and swell spectrum, and has been extensively studied. The remaining part of the spectrum is based on recent work which has not been presented

Figure 2. A typical frequency spectrum of the variations in the level of the sea surface above a fixed point at La Jolla, California.

previously. The low frequency records were obtained by means of a sensitive pressure transducer, the Vibrotron. Some analyses have been performed with an analogue computer in England, others with a high speed digital computer in the United States.¹ It is encouraging that the results agree.

Apparently there are five separate bands of high activity. This is a remarkably complex pattern. For comparison, the spectrum of sound background in the sea decreases monotonically between 10 c/s and 100,000 c/s by roughly 6 db per octave.

Of course, Fig. 2 is a description, not an explanation. But we have reasons to believe that each of the five peaks involves different physical factors, and that the resolution by frequency is a fruitful one. Also, that each of these bands is a factor in Naval Hydrodynamics when interpreted in its broadest sense.

The usefulness of the spectral representation has been limited largely to problems where each frequency could be considered as being independent of, and superimposed on, other frequencies; in other words, linear problems. The topic of ocean waves pre-

¹ We are greatly indebted to B. Ferber, D. Shumway and M. Covher of CONVAIR, San Diego, for their part in analyzing many records.

sents important non-linear problems, such as the breaking of ocean waves. The previous speaker, Dr. Lighthill, has been particularly successful in coming to grips with non-linear problems. But as far as we know, all such work has been confined to monochromatic, one-directional waves. What is the situation for a continuous spectrum? Not even the limits for the linear approximation have been stated explicitly. For the monochromatic case one simply assumes that the height-length and height-depth ratios must be small. Is the equivalent requirement that the rms slope must be small, or would infrequent occurrence of steep slopes change the situation materially. These remarks are in part directed at Professor Stoker, the chairman of this session. Is it safe to apply any of the work on the formation of bores to the actual ocean unless we understand more about the effect of non-linearity for continuous spectra?

RIPPLES

The tiny peak to the right of Fig. 2 represents enhanced activity for waves a mm or so in length. The result is preliminary, and numerical values of energy density are not yet available. It is based on work by Charles Cox. In this range of the spectrum, capillary forces are more important than gravity. Some of this activity must be related to tiny wavelets on the lee side (forward) of wave crests. An oil slick will wipe out this part of the spectrum.

It has been shown by Eckart (1953) and others that sound waves or electromagnetic waves impinging on the sea surface will be backscattered by ocean waves of about half the wavelength of the incident radiation. The problem of RADAR sea return and acoustic surface reverberation involves this high-frequency spectrum.

The visibility of objects beneath the surface from an observer above the surface also appears to be critically related to this part of the spectrum (Duntley, 1950).

SEA AND SWELL

This part of the spectrum is closely related to meteorologic activity. Sea is generated by local winds, swell by distant winds. In La Jolla we usually have a peak at about 65 c/Ks (period 15^s), presumably due to swell arriving from generating areas 5000 miles to the south. More often than not there is a second peak due to meteorologic activity in the Northern Hemisphere. Numerical values in the figure represent typical conditions. On occasion the energy might be 10 times higher or 10 times lower. The cutoff to the right is drawn as $E \propto f^{-6}$ in accordance with Neumann's proposed formula (Neumann, 1954) for the spectrum of a fully developed sea.

It is a strange, if you wish an unhappy circumstance, that this peak is associated with wave lengths of dimensions commensurate with those of ships. We shall return to this subject later.

SURF BEAT

This band was discovered several years ago by means of wave recorders tuned to low frequency (Munk, 1949; Tucker, 1950). At a depth of 20 to 30 feet the energy of the surf beat band is about 1 per cent of the sea-swell band; at greater depths and large distances offshore this band appears to be considerably reduced. The mechanism of generation is not clear, but some kind of non-linear interaction of the sloping bottom with the incoming groups of waves appears to be involved. Obviously our pressure mines, which are tuned to frequencies lower than the swell, must not be triggered by surf beat.

SURGES

These differ greatly from locality to locality, depending critically on offshore topography. Some of the activity is in the form of trapped modes, with the continental

shelf acting as a wave guide (Munk, Snodgrass and Carrier, 1956). Crests and troughs are normal to shore, and the direction of propagation parallel to shore. Amplitudes decrease rapidly seaward, and are insensible at a distance of one wave length from shore. Such wave motion was theoretically described by Stokes more than one hundred years ago, and referred to as "edge waves." (see also Eckart, 1951; Ursell, 1952; Isaacs, Williams and Eckart, 1951.)

Edge wave activity is related to meteorologic activity, but not in the same sense as sea and swell. There is some indication that these waves are generated by traveling disturbances of atmospheric pressure, such as result from internal waves in the atmosphere. If so, they represent a curious coupling of the height and sharpness of the atmospheric inversion layer, on one hand, to the slope and width of the continental shelf on the other hand.

All our measurements so far have been confined to the continental shelf and borderland. One week from today we plan to occupy a station off the coast of Mexico and beyond the borderland to see whether such activity can be found also in the open sea. If so, this would open up the possibility of an old dream; could deep sea surges, which travel at speeds of about 400 knots, serve as a storm warning?

SHIPS AND WAVES

The aspect of waves which is probably of greatest interest to the present gathering is their effect on ships. To a first order of accuracy, each degree of freedom of the ship may be regarded as an independent filter which responds in a prescribed way to the various wave components. For example, in its rolling motion a ship behaves rather like a pendulum and responds only to those components of the complex wave pattern which pass the ship with more-or-less the resonant frequency. It is obvious, however, that wave frequency is not the only factor since we all know that the rolling of a ship depends greatly on the direction of approach of the waves, usually being greatest when the waves approach from 2 or 3 points abaft the beam. The frequency at which the waves pass the ship will also depend on the ship's speed and its direction relative to the direction of wave travel.

If we regard the sea as being composed of a large number of simple sinusoidal, unidirectional wavetrains (see below), it is in principle possible to calculate the response of the ship to each of these, and the mean square response will be the sum of the mean square amplitudes of each individual wavetrain multiplied by the appropriate response factor. When generalized to a continuous spectrum, this is equivalent to regarding the ship as a filter with a two-dimensional filter function depending on the ship's geometry and velocity. The spectrum of the ship's motion is the integral of the appropriate product of this function with the two-dimensional spectrum of the waves, and can be calculated, at least in principle. From this, many useful statistical parameters can be calculated by methods developed by Rice (1944) for one-dimensional problems, and by Longuet-Higgins (1957) for two-dimensional problems. For example, r.m.s. amplitude, mean period, greatest expected amplitude during an hour's period of observation, number of times a given amplitude will be exceeded and whether the cook will be seasick can all be predicted. A convenient summary of these relationships is given by Cartwright and Rydill (1956).

A great deal of theoretical work has been done on the response of ships to waves, the effect of a simple sinusoidal wave train traveling in a single direction having received most attention, though more complicated wave patterns have recently been studied by St. Denis and Pierson (1953), by Cartwright and Rydill (1956) and others. (See, for example, several papers in *Ships and Waves*, 1954.) Considerable progress has been made in checking these theoretical results using models in tanks, but the acid test will be measurements using real ships in a real seaway. For these tests it will be necessary to measure the characteristics of the wave pattern in both frequency and direction of travel, and a knowledge of typical sea conditions based on such measure-

ments is also necessary to enable the naval architect to design ships with the best performance in waves.

What is the present position with regard to the measurement of waves at sea? A shipborne wave recorder is available (Tucker, 1955) which enables the frequency composition and the mean direction of wave travel to be measured (Cartwright, 1956), but as far as the authors are aware no one has yet succeeded in measuring the complete wave characteristics in frequency and direction with anything approaching the necessary detail, and this remains one of the most important problems in wave studies. We shall now discuss the problem in more detail, and then some possible methods of measurement.

THE 2-DIMENSIONAL WAVE SPECTRUM

We have seen that the wave spectrum is continuous in direction. However, for most problems, both the conception and the mathematics become considerably simpler if we start by considering the sea as being made up of a large number of simple wavetrains, so that the elevation $\zeta(x, y, t)$ above a point in the mean sea surface with coordinates (x, y) is

$$\zeta(x, y, t) = \sum_n C_n \cos(\omega_n t - k_n x \cos \theta_n - k_n y \sin \theta_n + \epsilon_n) \quad (1)$$

where $k_n = 2\pi/\text{wavelength}$, $\omega_n = 2\pi f_n$ and ϵ_n is a phase angle which is assumed to be random and usually disappears when appropriate averages are taken. At some stage in the problem the expressions can be written in terms of $\frac{1}{2} C_n^2$, which is proportional to the energy per unit area of the sea surface associated with the n^{th} wavetrain. If the energy of all the wavetrains contained in an interval $\delta\theta$, δf is summed, then

$$\sum_{\delta\theta\delta f} \frac{1}{2} C_n^2 \simeq E(f, \theta) \delta\theta \delta f, \quad (2)$$

where $E(f, \theta)$ is the energy per unit frequency range and per unit angle. The number of elementary wavetrains can now be considered to be infinite, so $\delta\theta$ and δf to tend to zero, so that

$$\sum_n \frac{1}{2} C_n^2 \rightarrow \int E(f, \theta) d\theta df. \quad (3)$$

This process was used, for example, by Rice (1944) to solve one-dimensional problems. $E(f, \theta)$ is called the spectral density function, and is the characteristic of the waves which we desire to measure.

For a given f , $E(f, \theta)$ is periodic in θ with a period of 2π , and may therefore be represented as a Fourier series of harmonic components:

$$E(f, \theta) = a_0(f) + \sum_n [a_n(f) \cos n\theta + b_n(f) \sin n\theta] \quad (4)$$

where n has integral values from 1 to ∞ , $a_n(f)$ and $b_n(f)$ define the n^{th} harmonic and are given by

$$a_n(f) = \frac{1}{\pi} \int_0^{2\pi} E(f, \theta) \cos n\theta d\theta \quad (5)$$

$$b_n(f) = \frac{1}{\pi} \int_0^{2\pi} E(f, \theta) \sin n\theta d\theta \quad (6)$$

Most methods for measuring $E(f, \theta)$ do not give a direct estimate of its value (smudged by the instrumental resolving power) but rather values of the individual

angular harmonics a_n, b_n . For example, pressure fluctuations at one point give a_0 , the total energy; the shipborne wave recorder (see below) gives a_0, a_1 and b_1 ; a current meter and pressure recorder give a_0, a_1, b_1, a_2 and b_2 . From a_0, a_1 and b_1 one could determine, at least in principle, three parameters of a wave pattern, such as energy, mean direction and angular spread (Longuet-Higgins, 1954). In practice these values are subject to instrumental and statistical uncertainties, and it will be difficult to distinguish between a narrow beam and a unidirectional beam.

The problem can be put in another way. A unidirectional wavetrain of unit energy has all its harmonics equal, so that if it comes from direction $\theta = 0$, an instru-

Figure 3. The directional sensitivity pattern of a device measuring the zero order and first two angular harmonics of the wave spectrum. The equation of the curve is $r = 1 + \cos \theta + \cos 2\theta$.

ment measuring the zero order and first two harmonics will show $a_0 = a_1 = a_2 = 1$, $b_1 = b_2 = 0$. Recombining these,

$$R(\theta) = 1 + \cos \theta + \cos 2\theta. \quad (7)$$

This is plotted in Fig. 3, and is the exact equivalent of the directional sensitivity pattern of a radio aerial or sound receiver. It shows a lobe covering approximately $\pm 60^\circ$, and the accuracy of detecting mean direction and beam width of the waves is determined by this lobe. A beam width which is narrow compared to this lobe will be difficult to distinguish from a unidirectional beam.

Figure 4. The directional sensitivity pattern of a device measuring the zero order and first six angular harmonics of the wave spectrum. The equation of the curve is $r = 1 + \cos \theta + \cos 2\theta + \dots + \cos 6\theta$.

Instruments measuring higher harmonics have a narrower lobe. The directional sensitivity pattern for an instrument measuring the first 6 harmonics is shown in Fig. 4.

A locally generated sea is unlikely to contain important harmonics above the 5th or 6th, and an instrument giving the first six harmonics would therefore be adequate for measuring storm waves. Information approximately equivalent to this can be obtained for a limited range of frequencies using an array of six conventional wave recorders suitably spaced in an L-shaped pattern. The calculation of the spectrum from the recorder outputs requires a rather formidable amount of computation unless automatic equipment is used (see below).

Swell from a small distant storm may be a very narrow beam, and harmonics up to the 20th or 30th might be required to define it fully. For most purposes, however, a knowledge of the first 5 or 6 angular harmonics will be sufficient.

The first estimate of directional distribution was made by Arthur (1949). He observed that the high surf activity experienced during the landings in Sicily during World War II could not be accounted for if the wind-generated waves travelled only in the direction of the mean wind, but required that substantial energy be radiated up to 30° to each side of the mean wind direction.

PRACTICAL METHODS

At the present time the most practicable method of gaining some directional information is probably the shipborne wave recorder (Tucker, 1955; Cartwright, 1956), which is the basis of some very useful comparisons of ship and wave motion by Cartwright and Rydill (1956). This instrument may be regarded as measuring the height which the water surface would have near the center of the ship were the ship not there. The directional information is obtained by steaming the ship on a series of different courses (e.g., 12 courses at 30° spacings) and measuring the Doppler shifts of the various frequencies recorded. It is possible to measure only the first harmonic of the angular distribution integrated over all frequencies present, though in the special case where there are two or more separate bands of frequencies (e.g., long period swell and short period locally-generated sea), these can be studied separately. In principle more detail can be obtained, but impossibly long records are required to give sufficient statistical reliability.

Methods depending on the correlation of the outputs of various measuring devices are practicable with present techniques. The combination of a pressure measuring device with a current meter measuring the two horizontal components of orbital velocity gives a_0 , a_1 , b_1 , a_2 , b_2 . The same information could be obtained by other similar methods, such as a free floating buoy measuring the three components of its acceleration (Barber, 1946). Barber (1954) has described a method for measuring the two-dimensional spectrum using a number of correctly spaced wave recorders and simple correlating equipment which works directly on the recorder outputs. Using three wave recorders in a limited fetch, Barber observed a "beam width" of $\pm 45^\circ$. Barber and Doyle (1956) have described a simplified version of this which measures the mean direction of a narrow beam using just two measuring heads and which is suitable for measuring the direction of approach of swell.

Another series of methods depends on photography of the sea surface. Since these are instantaneous measurements, they all have a 180° ambiguity, and energy traveling in directions opposed by 180° is added together. Barber (1949) used a photograph of the sea surface as an optical diffraction grating. If the optical density at each point of the photograph were proportional to the elevation of the sea surface, the diffraction pattern would be a 2-dimensional analysis of the wave pattern, with the radius from the center of the pattern being proportional to $1/\text{wavelength}$ and the intensity being related to $E(\theta, f)$ in a simple manner. The difficulty is, of course, that the optical density of the photograph is a function of the slope of the surface, the angle of observation and of the distribution of the brightness over the sky. Cox (1955) has examined this problem in more detail, and finds that if a photograph of the sun's

glitter pattern on the sea surface is taken, it is possible to deduce the angular distribution of the long wave components from diffraction analyses of two small areas of the photograph. (In practice, two independent photographs of the required areas would be taken.) This technique has some theoretical and practical limitations, perhaps the most important being that it is only applicable to the long-wave part of the spectrum. It has not so far been applied to a practical example.

It is possible that Barber's simple diffraction analysis could be applied to an artificial "photograph" of the sea surface prepared from stereoscopic photographs in such a way that the optical density is proportional to the corresponding surface elevation.

Pierson (brief remark in his 1955 reference) is analysing a number of stereoscopic photographs of the sea surface. Perhaps he will tell us later how he is analysing the measurements and something about the results. The first step is to convert the photograph into "maps" of the elevation of the sea surface. Marks (1954) has described one way in which these maps may be analysed by averaging surface elevations along a series of parallel lines to produce a one-dimensional record which contains contributions from components traveling in a known narrow range of directions. By altering the direction of the lines of averaging, a 2-dimensional analysis may be made. Methods using stereoscopic photographs are unlikely to become routine, since the labor of analysis is great.

A related method is based on recording surface profiles along different courses using an airborne radio altimeter. Perhaps Dr. Longuet-Higgins can inform us about the present status of this technique?

Cox and Munk (1954) have shown that it is possible to obtain some directional information from a simple photometric analysis of an out-of-focus photograph of the sun's glitter pattern (see Fig. 1). Their method gives only the second harmonic of the spectrum of the surface slopes integrated over all frequencies, and as such is of limited utility. (Their principal aim was to determine the statistical distribution of slopes.) They found a beam width of $\pm 45^\circ$.

Another class of methods depends on the use of directional primary measurement. Eckart (1953) and Crombie (1955) have shown that if a beam of radio waves is directed at the sea surface at almost glancing incidence, the back-scattering is due to the component of the sea waves with half of wave-length of the incident radiation and which is traveling in the direction of the receiver. Since the sea-waves are moving, the back-scattered radiation has a Doppler shift in frequency which enables it to be separated from direct coupling between the transmitter and receiver and from back-scattering due to static objects. The range of directions included in the back-scattering depends on the beam-width of the transmitting aerial, and this can only be made small if the aerial is several wavelengths long. The method is therefore probably practical only for short waves.

Groves (private communication) is examining the possibility of using towed electrodes as a directional wave measuring device. The use of towed electrodes for measuring ocean currents is now a routine matter (Longuet-Higgins, Stern and Stommel, 1954). The average value of the recorded e.m.f. is due to the average motion of the cable in the Earth's magnetic field. Wave frequency oscillations, now smoothed, might give useful information. Groves finds that with normal electrode separations the results are rather difficult to interpret, particularly when the device is near the magnetic equator. If a cable with a length of several wavelengths is used the results become easier to interpret, but the device may then be impracticable. The cable has to be kept near the water surface and sufficiently slack to enable it to move with the water particles, but at the same time it must be kept in a straight line.

To sum up, through several ways of obtaining information about the direction of wave travel has been suggested, the more practicable methods give very limited information, and no satisfactory way of obtaining a detailed 2-dimensional spectrum is yet available.

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DISCUSSION

M. S. Longuet-Higgins

The paper presented by Dr. Munk has rightly stressed the importance of measuring the whole two-dimensional spectrum of the sea surface. I would like to describe to you more fully the method he has mentioned that we are developing at the National Institute of Oceanography, and to point out some of its advantages.

The principle of the method is as follows. Suppose the surface elevation is represented in the form

$$\zeta(x, y, t) = \sum_n C_n \cos(u_n x + v_n y + \sigma_n t + \epsilon_n), \quad (8)$$

where (x, y) are horizontal co-ordinates, and t is the time. The wave-numbers (u_n, v_n) are assumed to be distributed densely throughout the (u, v) -plane, and the frequency σ_n is a function only of the wavelength;

$$\sigma_n = \sigma(u_n^2 + v_n^2)^{1/2} \quad (9)$$

as in free gravity waves. The phases ϵ_n are randomly distributed between 0 and 2π and the amplitudes C_n are such that over any small interval of wave-number $du dv$

$$\sum_{du dv} \frac{1}{2} C_n^2 = E(u, v) du dv, \quad (10)$$

where $E(u, v)$ represents the spectrum of the surface. If, by some means, we take the intersection of the surface with a vertical plane $x \sin \theta = y \cos \theta$ then the resulting curve will have an autocorrelation function $\psi(R)$ given by

$$\psi(R) = \overline{\zeta(x, y, t) \zeta(x + X, y + Y, t)} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(u, v) \cos(uX + vY) du dv, \quad (11)$$

where

$$X = R \cos \theta, \quad Y = R \sin \theta \quad (12)$$

Assuming this function to be obtained for all directions θ and hence all values of X and Y , we may, by taking the cosine transform, obtain the function

$$\frac{1}{2}[E(u, v) + E(-u, -v)], \quad (13)$$

which is the even part of the energy spectrum.

Suppose also that we are able to measure $\partial \zeta / \partial t$ at all points along the plane section. Then from the relation

$$\overline{\frac{\partial \zeta}{\partial t}(x, y, t) \zeta(x + X, y + Y, t)} = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma(u, v) E(u, v) \sin(uX + vY) du dv \quad (14)$$

we may find, by taking the sine-transform, the function

$$\frac{1}{2}[\sigma(u,v)E(u,v) - \sigma(-u, -v)E(-u, -v)] \quad (15)$$

But $\sigma(u, v)$ is known, even function. Thus on division we find

$$\frac{1}{2}[E(u,v) - E(-u, -v)] \quad (16)$$

and hence, by addition of (13) and (16), the spectrum $E(u, v)$ itself.

It is planned to measure $\zeta(x, y, t)$ along an arbitrary straight line $x \sin \theta = y \cos \theta$ by means of a radar altimeter mounted in an aircraft and directed vertically downwards to the sea surface. A second altimeter, mounted directly aft of the first, and therefore passing over the position occupied by the first a short time before, would provide a measure of $\partial\zeta/\partial t$ along the same path.

The method has certain advantages over stereophography. In the first place a large number of wavelengths may be included in each straight cross-section of the surface, so enabling a more accurate determination of the autocorrelation function to be made. In the second place the complete directional spectrum $E(u,v)$ is obtained, not merely its even part $\frac{1}{2}[E(u,v) + E(-u,-v)]$.

Even without performing the correlation analysis, the data may be made to yield some useful information. For example, it can be shown (see reference [1]) that the average number N_θ of zero-crossings of the surface $\zeta(x,y,t)$ along a plane section in direction θ is given by

$$N_\theta^2 = \frac{1}{\pi^2 m_{00}} (m_{20} \cos^2 \theta + 2m_{11} \cos \theta \sin \theta + m_{02} \sin^2 \theta), \quad (17)$$

where

$$m_{pq} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(u,v) u^p v^q du dv. \quad (18)$$

Thus N_θ is a maximum and a minimum in two directions at right-angles. The ratio

$$\frac{N_{\theta \text{ min}}}{N_{\theta \text{ max}}} \quad (19)$$

is equal to the r.m.s. angular spread in direction of the energy about the mean direction, provided this is small (see reference [1]). So by measuring N_θ as a function of θ the r.m.s. spread in direction may be found.

Similarly, the number N_1 of crests and troughs per unit distance in the direction θ can be shown to be given by

$$N_1^2 = \frac{1}{\pi^4 N_{00}^2 m_{00}} (m_{40} \cos^4 \theta + 4m_{31} \cos^3 \theta \sin \theta + \dots + m_{04} \sin^4 \theta). \quad (20)$$

From this the moments $m_{40}, m_{31} \dots m_{04}$ can be determined. If we write

$$\Delta_4 = \begin{vmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04} \end{vmatrix} \quad (21)$$

and

$$3H = m_{40}m_{04} - 4m_{31}m_{13} + 3m_{22}^2 \quad (22)$$

then it can be shown that the ratio

$$\frac{\Delta_4^2}{H^3} \quad (23)$$

vanishes only when the wave system consists of two intersecting systems of long-crested waves, and equals unity when the surface is isotropic. So this ratio is a measure of the "bimodality" of the spectrum with regard to direction. Upon the same parameter depend also the distribution of the "total" curvature of the surface and the number of maxima, minima and saddle-points per unit horizontal area.

Reference 1.

Longuet-Higgins, M. S. "The statistical analysis of a random, moving surface." *Phil. Trans. Roy. Soc. A.* 249, p. 321 (1957).

R. W. L. Gawn

The determination of the frequency and amplitude of the complex undulations of the ocean surface is of fundamental importance to those who design ships, those who operate them and to those who plan operations.

Dr. Munk mentioned the importance of the information as regards ship motion but the designer's interest is much wider. The ship must stand up to all the hazards of rough weather and I would like to make a few remarks on the designer's problems in this connection. Gravity waves impose heavy stresses and slamming, objectionable motion and speed loss and further limit efficiency if operational positions, e.g. gun positions in a naval ship are exposed to water or spray.

In the past and that not far distant the information available on ocean waves was scanty and it is remarkable that the designer was able to achieve a good measure of success. He assumed for strength calculations that the height of an ocean wave was one twentieth the length, although later thanks to the observations of Vaughan Cornish and others he could accept a smaller ratio for longer waves.

Until a few years ago that was about all that was known of use to the designer and it is refreshing now to hear of the enterprising work and the ingenious and painstaking analysis which Dr. Munk and others are carrying out.

There is scope for a closer identification of the length or period of the wave and the related height with the probability of occurrence in different oceans. While height is the dominant term in the energy estimates mentioned in the paper, period or length is dominant for ship motion since it influences both the excitation and the magnification. The Authors mention that the gravity waves are centered around 10 seconds period or a length of about 500 feet. It is emphasized that the range of interest to the ship designer is really from about 100 to 1500 feet length or about 4 to 17 seconds period and it is hoped that the Authors' further investigations will include a detailed study of this particular field accordingly. It looks as though from the wonderful instruments described by Dr. Munk and the work of the last speaker this should be realized.

As regards instrumentation, one requirement is a handy apparatus that can be taken to a ship and dropped in the water as necessary. As a matter of fact thinking on these lines we have made a simple buoy at Haslar which transmits radio signals of its vertical movement. Whether it is effective or not I am not yet in a position to say but a preliminary trial was promising. I am hoping that it is at sea just about now so that we should soon know if it is at all promising.

J. D. Pierson

Before proceeding to a few comments on this excellent paper, I would like to bring a third party to the "marriage" of the oceanographers and ship builders. The

designers of aircraft also have a great interest in the surface of the sea. Water-based aircraft are moving out of the bays and rivers further into the open sea.

Generally, aircraft are too fragile to operate in all sea conditions. The sea becomes too rough (both in a local and overall sense) once in a while, but on the other hand the hybrid nature of the water-based aircraft enables us to avoid the worst conditions. On the water, at rest or slowly taxiing, the seaplane experiences a long sequence of waves representing the full spectrum of the sea. However, in the transition conditions of landing and take-off, which are critical for the aircraft design, only a very small portion of the whole sea wave spectrum is encountered.

Thus, we may be looking for somewhat different information than ship designers. We are both concerned with the occurrence of high water loads and seaworthiness as they may be related to the energy spectrum for the seaways. Yet because of the short transient nature of the seaplane takeoff and landing, a further analysis of the sea spectrum may be profitably employed.

For example, as the various components of the sea add to give the larger waves—do high crests generally follow immediately the deep troughs? The extent to which this may occur on a probability basis would be helpful in determining the conditions for which the hull loads should be obtained.

Another problem suggested by the type of operation of water-based aircraft is the description of the sea so that small areas at a time could be defined in terms of the overall spectrum existing at the time. Along this line we may be able to contribute through the measurement of the sea surface from the air. Although airborne direct wave measuring systems have not yet been satisfactorily demonstrated, there are several now under development which have considerable promise. Measurements of the sea roughness by aircraft will not only provide a tremendous increase in data over what can be obtained at fixed locations but also provide quick data for portions of the total sea spectrum for small transient areas.

W. J. Pierson, Jr.

I should like to add to Dr. Munk's remarks a very brief summary of Project SWOP (Stereo Wave Observation Project), a joint effort of the Office of Naval Research, U. S. Navy Hydrographic Office, Woods Hole Oceanographic Institution, and many operating arms of the Navy. A final report on the project is soon to appear under the title "The directional spectrum of a wind-generated sea as determined from data obtained by the Stereo Wave Observation Project."

The plan of the project was to take simultaneous stereo photographs and wave pole records of the sea surface. The directional spectrum could be computed from the stereo photographic data, and the frequency spectrum could be computed from the wave pole data. Theoretical considerations led to the conclusion that the total volume under the directional spectrum should equal the total area under the frequency spectrum and thus a check of the two methods would be obtained. After the computation of the directional spectrum and the wave pole spectrum, it was discovered that the two quantities given above did not agree. Several sources of discrepancy have been discovered and eliminated from the data. First of all, the wave pole spectrum did not have enough area under it to represent the true wave record. The spectral components for many important frequencies were attenuated because the wave pole had a tendency to move up under the crest of passing waves and move down in the troughs. A correction was derived for this effect, and brought the two results into closer agreement.

The stereo data unfortunately were obtained from topographic-base film instead of stereo-base film and there was some differential shrinkage in the record. The errors in the stereo data are due to this differential shrinkage, to a background curvature of the plane of the sea surface, and to spot height reading error. They all combine to make the volume under the directional spectrum quite a bit greater than the area under the

frequency spectrum. The errors due to the spot height reading and to background curvature have been eliminated, and at the moment work is under way to eliminate the error due to differential shrinkage. The directional spectra have been computed and indications are that such a correction can be made and that the results of the wave pole observations and the stereo observations can be made consistent.

Some tentative conclusions can be made even with the data in their present form; and even if no further attempt were made in analyzing the data, many valuable conclusions could be drawn. The error due to differential shrinkage shows a definite pattern which can be removed qualitatively from the spectrum. The conclusions would then be that the theoretical Neumann spectrum is well verified by the corrected wave pole spectrum and that the directional spread of the spectrum is symmetric about the wind direction and falls to half power at $\pm 30^\circ$ to the wind direction.

Unfortunately none of the slides and drawings to accompany the final report are at present in a form ready for inclusion in this report. They need correction (in some cases) and further checking. Therefore the reader is requested to write for the forthcoming report which will describe this work in complete detail.

R. C. Vetter

It may be appropriate to add, to the various ingenious methods mentioned by Dr. Munk, one more method for determining some two-dimensional characteristics of ocean waves. If two identical transparent copies of a vertical photograph of the sea surface are super-imposed and illuminated by a beam of light, one may determine the correlation function of the density variations in the transparencies by recording the change in light intensity as one transparency is translated with respect to the other. The two dimensional power spectrum of the density patches in the transparencies can then be derived from this correlation function. This method is being used with some success by Dr. Uberoi of the University of Michigan in studies of turbulence in gases. Some of the same transparencies that were used in Project SWOP (see Dr. Willard Pierson's comments) have been sent to Dr. Uberoi for analysis. It seems highly unlikely that the two two-dimensional power spectra obtained from Dr. Pierson's SWOP Project and Dr. Uberoi's method of analysis will be alike since there is no simple relationship between the brightness of the sea surface and the height of waves. However, it will be interesting to see in what respects the two spectra are different or alike.

A tremendous effort has gone into the determination of a two-dimensional power spectrum by Project SWOP. I am sure that the results will justify this effort many times over. One wonders if an easier way can be found to give similar information. At present, none of the optical methods appear to be satisfactory. The value of more information about the two-dimensional characteristics of real ocean waves is considerable and I am sure that everyone, including Dr. Pierson, will thank the ingenious person who finds an easier way of getting it.

Walter H. Munk

Vetter's suggestion is closely related to Barber's diffraction method (PRACTICAL METHODS, p. 3 of our paper). In particular, the reference to Cox (1955) is pertinent.

IV

CONTRIBUTION OF SHIP THEORY TO THE SEAWORTHINESS PROBLEM

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Introduction

In an earlier survey [1] on the behavior of ships in a seaway we quoted A. Krylov's opinion on his famous memoir presented to the INA [2] and later published in the Enzyklopaedie der Mathematischen Wissenschaften, . . . "it is evident that these

methods are quite sufficient to solve all the questions which arise regarding the influence on the seagoing qualities of a ship". . . .

We stated then that the great scientist has heavily overestimated the bearing of his investigation. However, his claim provides us with a program which we shall use as the backbone of the present lecture. We shall first give a short review of the present status of knowledge on the seagoing properties of ships as pictured by recent research work. This task is much simplified by the fact that recently in this country excellent papers have been published of original as well as of synoptical character, for example, by Korvin-Kroukowsky [4] and Lewis [5]. These publications are assumed as known, and only short reference will be made to them. Emphasis will be laid on investigations which may have escaped attention in this country although, unfortunately, the present writer's attempts to procure literature were not too successful.

We shall try to indicate some trends in the methodology of research and some important results without going into details of the analysis. Secondly, the significance of scientific results for *future* design will be discussed after having indicated sketchily some shortcomings of the present state. Here again an excellent foundation is given in reference 5. Beside promoting problems of design, a consistent theory of ship behavior will have to furnish basic information for ship operation. The task of linking explicitly design and operational problems in research, appears to present a strong challenge, but is beyond the scope of this lecture.

Finally, some ideas are developed as to future research in our field.

Seaworthiness we define as the property of a ship to perform her duties safely and well in a seaway. For a more detailed definition of seakindliness reference is made to an article by Kent [7].

Ship theory embraces in general the application of geometry and rational mechanics, especially of general mechanics and hydrodynamics, to problems presented by design and operation of ships. A narrower concept of ship theory, excluding problems of structural strength, will be especially useful for our present review, although obviously a clear-cut separation between ship dynamics and strength cannot be made, problems of elastic vibrations by definition belonging to ship theory. In fact, the unity of shipbuilding science is stressed at this point.

Introduction of the concept—"ship mechanics"—as analogue to flight mechanics when dealing with the behavior of a ship in a seaway, is suggested, to exclude any overemphasizing of geometrical topics, a characteristic of earlier ship theory.

When speaking about ships we have actually to deal with two different classes of vessels—Archimedean and hydrodynamic, which in turn each divide into two subclasses—surface and submarine ships, and planing and hydrofoil craft. From the point of view of analysis this means that we may have to create different *mechanical* models, each of which has to embody the most important properties of the vessel in question. So for the displacement vessel the appropriate hydrodynamic singularity is the source (sink, doublet), while the hydrofoil is treated by the vortex concept. We must add that beside ships, other floating or waterborn bodies have to be considered and should profit from the activities in our field.

If governmental stipulations [6] were completely indicative of the general trend in design practice, the present speaker should confine himself to a discussion of elementary stability principles. Fortunately, design practice is a mental activity which only to a certain extent is regulated by legal prescriptions, and the actual practical contributions of ship theory in our field are more impressive than can be deduced from such regulations. Especially the navies have in general displayed a more progressive attitude.

Denying any possibility of the *sacrificium intellecti* in our profession the author admits that in the light of *present* requirements of shipbuilding and shipping practice the importance of some scientific investigations and results may somewhat shrink. The situation becomes completely different, however, when we accept the author's contention that a considerable development of the ship is to be expected in the future.

It is already a truism that a strong incentive for further research is to increase the sustained ship speed in heavy seas. Numerous problems which at low speeds of advance could be treated hopefully by hydrostatic methods require more detailed hydrodynamic consideration at higher Froude numbers.

Several years ago we used to paraphrase Hilbert's well known remark on physics by stating that ship theory is becoming too difficult for naval architects. On the other side, because of the complex character of problems, it may be too complicated for mathematicians and physicists. The best approach may attempt to establish a fruitful cooperation between different directions of research.

The last five or six years have seen a tremendous increase of interest in our field, and considering the current rapid rate of progress, the author wishes to point out that the picture sketched by him refers to a transitory condition only, and even so is essentially incomplete.

I. THE INVESTIGATION OF SHIP BEHAVIOR IN A SEAWAY

Giving full credit to experimental research in the model and full scale range, we find it nonetheless advantageous to describe the present state of knowledge primarily in terms of analytic methods.

1. The Seaway Concept

Times are bygone, fortunately, when an amateur-oceanographer like the author could gain an easy reputation by measuring big waves. In what follows we confine ourselves to some remarks on the subject from a naval architect's point of view, since problems of oceanography have been dealt with by Professor W. Munk.

One of the most important recent achievements was the detection of long wave components by measuring ship motions [8]. An impressive development of the theory of the irregular seaway has taken place especially in this country and in Great Britain; apparently, parallel important work has been done by the Russian School although from the meager information available [9] it seems that the results reached are less definite.

Successes in oceanography have an important bearing on naval architecture. Obviously, no rational theory of seaworthiness of ships can be developed without a thorough knowledge of actual seaway phenomena; in fact, the lack of the latter was one of the reasons why the study of ship behavior remained somewhat academic and failed to find a wider response in practice.

Design procedures require the knowledge of *safety limits and of average performance* of a ship in a seaway.

Let us remember by what crude assumptions experienced model investigators attempted to cope with the latter problem: an "equivalent" regular wave was arbitrarily substituted for the actual seaway.

Under these circumstances, the work by St. Denis, Pierson and Neumann proved to be extremely successful after the first shock over the unfamiliar mathematical formulation had faded away [10]. The link with reality appears to be established, although from a hydrodynamic viewpoint Neumann's seaway formula may be

no more than a successful heuristic approach. Thus only by the introduction of the irregular seaway concept can problems of average performance, especially of sustained average sea speed, be handled in an appropriate manner. This fact will allow us to appraise correctly the importance of seakindliness of vessels with respect to other basic qualities of a ship when the necessary statistical data will be collected. Again, there is a brighter outlook in this respect after pertinent research facilities have been developed.

There appear still to exist some differences in opinion on the fundamentals of the irregular seaway concept and the resulting ship motions. However, for the present purpose I shall assume that the principle of superposition is valid for elementary waves as well as ship motions and restrict myself to the study of ship behavior in a regular seaway till we later come to a discussion of design problems.

Attempts to establish *safety limits* for ships by statistical observations had been made long before a consistent theory of the actual seaway existed. It is evident that the statistical approach is a necessary condition for dealing with the safety problem but it is questionable if it is a sufficient one. Thus it is the present writer's contention that the old-fashioned approach based on the investigations of transient conditions should not be completely disregarded, although it is admitted that no proposal so far exists as to how to formulate these conditions.

The present theory of ship behavior in an irregular seaway as a linear theory is not fit to deal with the safety problem as far as large displacements, such as those in roll, are concerned. The same objection must be raised against the linear theory of ship motions in a regular seaway.

2. *Preliminary Survey on the Development of the Theory of Ship Behavior*

A. Krylov's study of ship behavior, based on the undisturbed wave hypothesis, remains classic because of its general character and its consistent formal approach. Recently his method has been reviewed by at least six authors. It should be kept in mind that Krylov's work was originally stimulated by a study of extraneous forces causing bending moments and shearing forces, thus indicating the close interconnection of shipbuilding science branches.

One could surmise that Krylov, by neglecting the influence of speed of advance on forces, and by rather aggressively disregarding the importance of damping in heave and pitch, might have detrimentally influenced further development. However, his scientific achievements have been grasped by the profession so slowly, that their weak points did little harm. Indeed, by failing to use Krylov's theory as a guide for model research, some experimenters were derailed into a rather sterile empiricism. Thus the need for sometimes tedious computations has occasionally slowed progress in understanding basic phenomena in our field.

The determination of exciting forces as functions of the waterline form, (to some extent of the sectional area curve), of the ratio $\lambda^* = \lambda/L$, and then of the heading angle, was the main problem attacked by Krylov. While Krylov himself still cherished the idea that he had solved the problem of ship behavior completely, his moderate adepts were critical enough to consider their investigations as attempts to obtain some general ideas of the phenomena. At the prevailing state of knowledge that was already a useful task.

By 1930 a standard form of linearized equations of motion for a ship in a seaway was established [11].

Simple equations for heave and pitch are

$$(m + m_{33}) \frac{d^2 z}{dt^2} + N_{33} \frac{dz}{dt} + \rho g A_w z = \rho g r_m A_w E_{33} e^{i(\omega t - \epsilon_{33})}, \quad (1)$$

$$(J_y + m_{55}) \frac{d^2 \psi}{dt^2} + N_{55} \frac{d\psi}{dt} + \rho g D \overline{M_L G_\psi} = \rho g \vartheta_m D \overline{M_L G} E_{55} e^{i(\omega t - \epsilon_{55})}. \quad (2)$$

"Relative motion" in heave is considered by the following expression:

$$m \frac{d^2 z}{dt^2} + m_{33} \frac{d^2}{dt^2} (z - r) + N_{33} \frac{d}{dt} (z - r) + \rho g A_w z = \rho g r_m A_w E_{33} e^{i(\omega t - \epsilon_{33})} \quad (3)$$

where

$$\frac{d^2 r}{dt^2} = -\omega_0^2 r_m e^{i\omega t}, \quad \frac{dr}{dt} = i\omega_0 r_m e^{i\omega t}.$$

Known hydrostatic coupling terms were neglected for simplicity, although more recent research in hydrodynamics has led to the conclusion, that this involves an appreciable loss of accuracy [4]. It was realized comparatively early that nonlinearities of the restoring moment in roll must be considered, although the application of this finding to forced motions was slow. Special phenomena later required more attention to coupling terms and finally rheo-linear effects were treated.

The computation of *damping forces* by hydrodynamic means was initiated by Schuler [12] and Holstein [13], and put on safe ground by Havelock [14].

Although quite a bit had been achieved in this way there was an uneasy feeling about the applicability of the formal apparatus to concrete design problems, stemming from the distrust in the Froude-Krylov hypothesis underlying the computation of the *exciting forces*.

It turned out that the wholly submerged body was a mechanical model easier to handle than the surface ships. The classical theory of solids moving in a liquid developed mainly by Kirchhoff and Kelvin was rediscovered for our purposes after it had been considered (by our profession) as a kind of mathematical exercise only. Progress was essentially due to the application of the concept of hydrodynamic singularities. Lagally's theorem and its generalization by Cummins [15] were important steps in this direction. Reference is made, for example, to the work by Haskind [16] who asserted, not quite justly, that he had found the solution of the hydrodynamic problem neglected by Krylov.

In the present writer's opinion the procedure of building up equations of motion by intuitive synthesis will continue, notwithstanding its obvious shortcoming, i.e., the need to add new terms in the pertinent equations when dealing with special applications, such as ship stabilization [17]. Nonetheless, there exists an encouraging analogy with flight mechanics, the theory of which has reached a comparatively very satisfactory state by much the same path. It should not be overlooked that the directional stability of ships has been successfully attacked only after methods borrowed from aerodynamics have been applied.

The use of hydrodynamic singularities is one method of attacking the boundary value problem involved. Since, however, one ordinarily uses a first approximation only when substituting effects due to singularities for effects caused by bodies, it makes sense

to speak about a "singularity theory" as distinct from a "rigorous theory" based on a more exact solution of the boundary value problem.

Beside the work of Havelock [18] we mention the well-known investigation by Ursell [19] on the heaving of a cylinder, as an example of a rigorous procedure which becomes indispensable when added masses, the magnitudes of which depend upon the flow field in the neighborhood of the body, are investigated.

The determination of the ship behavior in a seaway as a boundary-value problem was formulated in its full generality first by F. John, whose papers on the motion of floating bodies [20] are already classical. It is impressive that this investigation led immediately to important practical results in a special case, thus defying classification as only a mathematical exercise.

Stoker and Peters [21] attacked the explicit solution of the boundary problem in a more general way, restricting it, however, to symmetric motions of the "Michell" (thin) ship. From a methodological point of view this work produced important progress, which is enhanced by a study [22] dealing with the "Stoker-Peters" problem on arbitrary heading angles. It has been pointed out by the present writer that the results achieved by these authors in their first paper indicate that the Michell ship cannot be considered as an adequate mechanical model of an actual ship as far as heaving and pitching motions are concerned. However, quite recently Stoker and Peters have obtained farther reaching results in the case of the general problem [23]. Although explicit solutions apparently can be worked out only with great difficulties, their method represents the ideal goal aimed at in our field of research, since it derives the motion of the ship from the pressure forces on its hull.

In the case of plane motions, methods of solving the boundary problem have been established by Ursell and Grim [24]. The results obtained represent already an indispensable part of useful knowledge in our field.

A significant present trend is to treat problems as two dimensional and to solve them as rigorously as possible, although solutions for the three-dimensional potential of pulsating and advancing singularities are known. A strip model is then applied. Because of the inherent limitations of the plane-motion concept it is expected that this trend may be changed rather soon, although Haskind, in an interesting recent paper [25], takes advantage of the simpler procedures used in the two-dimensional case, and at the same time gets rid, as far as possible, of their shortcomings.

Having given what is thought due credit to strategists of a rigorous approach, we must still pay attention to problems whose treatment by rigorous methods may still prove impossibly cumbersome. An important example is the problem of hydrodynamic impacts associated with the emergence and reentry of parts of the hull, especially of the bottom from and into the water [26].

Unfortunately, we have not been able to deal adequately with the resistance problem in this paper. This is partly due to the unsatisfactory status of knowledge in this field, and partly to our inability to obtain literature on valuable work by Japanese researchers, in time for this writing. Research conducted at the present writer's institute at a simplified level has so far led to promising, but not yet conclusive results.**

3. More Detailed Survey of Problems Connected with Ship Behavior

The brief sketch of the scientific status in our field leads to the conclusion that beside the promising first efforts to establish a general rigorous theory we have still essentially to rely, probably for a long time, on partial solutions of our problem, resting to a large extent upon inductive and rather intuitive methods. We shall therefore discuss in more detail results obtained by the latter approach.

** Reference is made to recent publications by Hanaoka and Maruo.

The ship will be considered as rigid and the fluid as ideal. In fact, according to our present ideas viscosity plays an important part almost only in the damping of roll. The insight that viscosity effects can be disregarded in most cases represents in itself an important achievement.

A large part of current investigations deals with forced motions of ships in calm water; effects due to wave action are added using the principle of superposition.

3.1. Dimensionless Parameters

In the absence of a consistent theory the determination of similitude parameters is a necessary prerequisite for research and application; in a later stage appropriate expressions are indispensable working tools. At the same time, this absence of theory hinders discovery of the most advantageous and significant similitude parameters for a particular problem. Up to the present time, varied experiences and personal preferences have produced a rather wide variety of dimensionless parameters for the study of ship mechanics, and some kind of standardization appears desirable. Once these parameters will have been standardized, it should become customary to characterize a vessel's oscillatory properties by them in the same way that this is being done for the resistance by the Froude number. (The author's preferences in choice of parameters is indicated in the list of symbols and throughout the remaining text.)

For purposes of any general discussion, it is convenient to have in mind some standardized ship geometry or series of shapes. Submerged elongated vessels can be approximated by bodies of revolution (especially their most distinguished of such—the spheroid), even a moderate departure from axial symmetry being permissible; and it appears that in the case of surface ships a two-dimensional treatment can be founded with success upon the well-known class of F. Lewis sections [27] which have been used independently by Grim and Haskind.

A section of the Lewis class is determined by the ratio $\frac{B}{2H}$ and the area coefficient β .

The velocity potential which governs fluid perturbations caused by motions of such bodies has been established also in the presence of a free surface.

3.2. Added Masses

The determination of added masses has become a central problem in the study of ship behavior, since the computation not only of inertia but also of exciting and even damping forces depends upon it.

To my knowledge, Haskind [28] was the first to establish theoretically the influence of a frequency parameter, say $\omega_B^* = \omega\sqrt{B/g}$, upon the magnitude of added mass values $m_{ij}(\omega_B^*)$ already found experimentally by Holstein [13]. Unfortunately, our information rested on a part of Haskind's publications only, so that the method by which he obtained his important results in the three-dimensional case remains unknown, although further details of his work have become available [25].* In the meanwhile contributions have been made by Ursell, Grim, and Havelock. Japanese authors [29] investigated the submerged spheroid under the free boundary condition $\Phi = 0$, while

the other extreme case $\frac{\partial\Phi}{\partial z} = 0$ has been treated by Eisenberg [30].

There are several methods in use to formulate dimensionless added mass coefficients; the most popular are those found in Lamb and those proposed by F. Lewis. However, especially in the case of rotational motions there is still a wide variety of definitions.

* In the two-dimensional case of heaving the result $m_{33}(0) \rightarrow \infty$ is obtained while Haskind's investigation yielded a finite value supported by experiments.

Reference is made to some results obtained by Grim [31] for sections of the Lewis class and circular segments. These findings have been recently supplemented by values $m_{44}(0)$, i.e. of the hydrodynamic moment of inertia of cylindrical bodies in roll for a frequency parameter $\omega_B^* \rightarrow 0$, which change somewhat the shape of curves communicated earlier.

For the elliptic section we have the well-known result $\frac{\bar{m}_{22}(\infty)}{\bar{m}_{22}(0)} = \frac{4}{\pi^2}$ which pictures the conditions $\Phi = 0$ and $\frac{\partial \Phi}{\partial z} = 0$ respectively; and for a flat plate Haskind [25] obtained the relation $\frac{\bar{m}_{44}(0)}{\bar{m}_{44}(\infty)} \approx 1.6$. In Grim's presentation the inertia coefficient K_{44} is referred to the moment of inertia of the homogeneous elliptic halfcylinder.

The variability of the added mass coefficients with the frequency parameter is impressive. Nonetheless, Haskind proposed recently to use in heave and pitch the approximations $m_{33} = \bar{m}_{33}(\infty)$ and $\bar{m}_{55} = \bar{m}_{55}(\infty)$; in sway and roll $m_{22} = m_{22}(0)$ and $m_{44} = m_{44}(0)$, since free heave and pitch oscillations are in general short perioded as compared with those of sway and roll. This means in effect that the earlier practice of substituting the deeply immersed double body for the ship should be applied as first approximation. We do not agree with this proposal which, in our opinion, is permissible not as a first *approximation* but as a first *orientation* only. In the later part of our study it will be pointed out that it can become quite important to determine added mass values more rigorously.

In [25] expressions are communicated for added masses $\bar{m}_{22}$, $\bar{m}_{44}$, $\bar{m}_{24}$ of bodies belonging to the class of Lewis' forms. Previously, only $\bar{m}_{33}$ was known. The total added masses of the ship are determined by a strip method. Approximate expressions are given in the form $m_{33} = \bar{m}_{033} L \frac{2\alpha^2}{1+\alpha} m_{22} = \bar{m}_{022} L \frac{2\beta_o^2}{1+\beta_o}$ with $\bar{m}_0$ the added mass of the midship section, α the waterline area coefficient, and β_o the area coefficient of the *longitudinal* section generally close to 1. These formulas are based on a rather rough assumption on the average distribution of $\bar{m}_{33}$ etc. and on Chapman's parabolas as waterline equation; a correction coefficient has to be used for the finite length of the ship.

Haskind suggests use of the condition $\frac{\partial \Phi}{\partial z} = 0$ to determine added masses in the aperiodic motions of sway and yaw, which again appears to be rather arbitrary.

The vertical added mass of half a sphere oscillating at the free surface has been calculated by Havelock [32]. Its dependency upon the frequency parameter is similar to that found by Haskind for a ship; the ratio $\frac{m_{33}(0)}{m_{33}(\infty)} \approx 1.6$, is somewhat lower than for the latter.

The magnitude of added mass values in transient conditions depends upon the time history. There exists a solution [33] for the wholly immersed circular cylinder moving sidewise from rest with a constant acceleration, which yields an added mass value $\bar{m}_{22}(\infty)$, i.e. corresponding to the assumption $\Phi = 0$. Suggestions have been made to approximate transient motions by parts of a sine curve and thus to determine a frequency parameter. This procedure, obviously, is questionable.

So far, no consistent theory is known to us which deals with the influence of speed of advance on added masses of bodies moving at or near the free surface. Haskind indicates however, that following his theoretical and experimental work, this influence is not strong in case of the heave motion, and recent experiments by Golovato [34] support this statement reasonably well.

3.3. Damping

Beginning with Froude [3], considerable efforts have been made to determine the damping of roll by experimental methods. It is known that Froude possessed sound ideas on the wave damping in this case, but his plotting of extinction curves without deriving therefrom orthodox dimensionless damping coefficients has hampered progress.

To the present writer's knowledge, pertinent experiments on heave and pitch were first carried out by him at the Berlin Tank on a very full and medium full cargo ship model. They yielded added mass coefficients of about unity for heave and somewhat less for pitch. Dimensionless damping coefficients were $\kappa \approx 0.4 \rightarrow 0.5$ for both motions; there being no measurable difference in these values for $F = 0$ and $F = 0.20$. These results were later checked by Igonet [35].

Experiments conducted in Göttingen by Schuler and his school to check theoretical findings yielded a frequency dependence of damping as well as of added masses. Havelock [18] interpreted an important result due to Holstein by indicating the underlying source distribution, and treated the three-dimensional case of heave and pitch by alternating sources. Almost all work mentioned below refers to the damping experienced by bodies moving in calm water and performing forced oscillations. Such is in fact, the mechanical scheme used at present in this kind of research, (except for *model* investigations in a seaway). Similar studies were independently performed by Kochin and followers primarily on the damping of submerged bodies including shallow water effects. Haskind gave an extended general treatment of the subject and developed formulas for the heave and pitch damping of thin ships in a regular very long seaway. We mention Brard's important work on pulsating singularities advancing in calm water. In the two-dimensional case Ursell and Grim satisfied more accurately the boundary conditions on the body which otherwise remained somewhat undetermined.

Further work of Haskind [25] discusses approximate methods to determine hydrodynamic forces. Finally, Havelock [36] has made an investigation on the limits of applicability of the strip method for determining the damping of submerged bodies as function of the frequency parameter. So far no systematic investigations are known on the magnitude of damping coefficients in a regular *seaway*.

a. Damping in the two-dimensional case

A strip method is now widely in use to determine pitch and heave damping of ships; it can be applied to roll also. Results have been given in the form of rate of energy dissipated, wave amplitude at infinity, and damping coefficient N , from which the dimensionless damping coefficient κ can be derived.

In Holstein's case of a uniform alternating source distribution, the wave amplitude referred to the heaving amplitude amounts to

$$\frac{r_m}{z_m} \approx \omega^2 \frac{B}{g} \exp \left[- \frac{\omega^2 H}{g} \right] \quad (4)$$

Therefrom

$$\bar{N} = \frac{\rho g^2 r_m^2}{\omega^3 z_m^2} \approx \rho \omega B^2 \exp \left[- \frac{2\omega^2}{g} H \right] \quad (5)$$

$$\kappa \approx \frac{\omega B}{\sqrt{(1 + K_s) H g}} \exp \left[- \frac{2\omega^2}{g} H \right] = \omega_B^* \sqrt{\frac{B}{H} \frac{1}{1 + K_s}} \exp \left[- \omega_B^{*2} \frac{2H}{B} \right] \quad (6)$$

Holstein's result has been generalized by St. Denis [37] for arbitrary section forms.

We note that for small ω , the damping and its dimensionless coefficient is proportional to ω . The same applies to results found by Ursell for the circular cylinder

and by Grim for the class of Lewis sections. There is a good agreement between Ursell's and Grim's findings in the case of the circular cylinder; however, a wide discrepancy has been stated between Grim's results and those found by the "source theory" for larger frequency parameters ω_B^* . In the latter case, one should have more confidence in Grim's and Ursell's results since they rest upon a more advanced approach; recent measurements by Golovato support the correctness of Grim's findings. Further investigations will be conducted to settle this problem finally.

From the point of view of ship design, the range of frequency parameters close to synchronism only is of primary importance, say corresponding to $0.6 \leq \Lambda \leq 1.2$, or expressed in ship parameters roughly $0.5 \leq \nu_B^* \leq 2$.

b. The Three-Dimensional Problem

The velocity potential is known for the harmonically pulsating source and doublet advancing with a constant speed, from which in principle a solution can be built up for a system picturing the actual ship [38]. Neglecting the speed of advance Havelock [18] has given his well-known analysis of heave and pitch. The damping coefficients for an elliptic waterline are:

$$N_z = -\frac{\pi \rho}{4g} B^2 L^2 \omega^3 \exp[-2\omega^2 H/g] N_z^* \quad (7)$$

$$N_\psi = -\frac{\pi \rho}{16g} B^2 L^4 \omega^3 \exp[-2\omega^2 H/g] N_\psi^* \quad (8)$$

for small ω $N_z \sim \omega^3$, as against $\bar{N} \sim \omega$ in the two-dimensional case. N_z^* and N_ψ^* are dimensionless coefficients.

For a wholly submerged body, oscillations are pictured by dipole distributions. In the range of small ω

$$N_z \approx \omega^5, \quad N_z^{(s)} \approx \omega^5, \quad N_\psi \approx \omega^7, \quad (9)$$

where $N_z^{(s)}$ is the damping coefficient of the body calculated by a strip method. While it had been earlier established [16] that for heave the strip method is appropriate for $\omega^{*2} = \frac{\omega^2 L}{g} \geq 6$, it follows that in pitching the corresponding lower limit of applicability is higher, say $\omega^{*2} \geq 8$. It should be remembered that for application damping values become decisive in the neighborhood of synchronism only.

Haskind's results for low frequency parameters are given in a different form [16]. However, an agreement with Havelock's expressions can be obtained.

From [25] we reproduce two diagrams. The full lines in Fig. 1 represent dimensionless damping coefficients of heave, $C_{33} = C_z = \frac{N_{33}}{\pi \rho \sqrt{gL} B^2}$, for $F = 0$, as

a function of the frequency parameter ω^* , apparently based on Reference [16], for a full ($\alpha = 0.8$) and a fine waterline ($\alpha = 0.64$). The marked points in this diagram refer to damping at the Froude numbers indicated. This diagram and another showing experimentally-obtained heaving amplitude ratios plotted versus ω^* , support Haskind's assertion, that within a wide range the damping coefficient is roughly independent of the Froude number. Phase lag curves corroborate the statement.

In principle there is, however, no reason to expect that Haskind's finding is more than an approximate rule which may be valid within certain ranges of the frequency parameter and Froude number. Eggers, of my institute, has investigated this question using Brard's results for simple combinations of singularities and found that outside of the range considered by Haskind important deviations from the rule can occur. Of special interest in this connection appears to be Japanese work [29].

Figure 1. Dimensionless damping coefficient in heave, $C_{33} = \frac{N_{33}}{\pi \rho B^2 \sqrt{gL}}$, as a function of the frequency parameter ω^* . (From Haskind, Reference 25.)

There are some experimental findings for submerged bodies where a definite speed dependence of damping at a constant frequency was established.

Thus the problem of dependency upon the Froude number is not yet settled, although recent experimental results by Golovato indicate that one is entitled to accept Haskind's result as a useful working hypothesis within the most important speed range. Conditions may be different in the case of roll. Japanese writers [29] have indicated an increase of wave damping with increasing Froude number. Further, effects due to viscosity become important.

3.4. Simplified Theory of Heave and Pitch

We shall classify all investigations based on the Froude-Krylov hypothesis as simplified and, when in addition coupling terms are neglected, as simple. It is quite significant that investigations of such a character still appear today.

A consequent work of this character has been published recently by Gerritsma [39] on heave and pitch in longitudinal waves. Coupling terms following Korvin-Kroukovsky have been considered separately; but their contribution is not large in the present case.

Regarding the importance of Smith effect and relative-motion corrections to the simplified theory, I take note of recent research on diffraction and do not quite agree with a remark made by Cartwright and Rydill [40], that it may not pay to consider anything else beyond the simplest theory for interpreting full-scale experiments in an actual seaway.

3.5. Investigations on Exciting Forces

The classical theory of motion of bodies in an unbounded fluid, with approximations introduced to deal with a non-uniform and non-steady flow, showed that the

heaving force depends upon such factors as K_{23} - K_{11} and $\frac{U}{c}$.

The shortcomings of the Froude-Krylov hypothesis have first been overcome in the case of submerged bodies moving uniformly and horizontally close to the corrugated free water surface. Later Havelock [41] gave a solution for the spheroid, and Cummins [42] applied his generalized singularity theorem to determine the hydrodynamic part of the exciting force experienced by bodies of revolution. The distortion of the wave shape is not considered by this approach. The application of Cummins' method to general problems of ship mechanics is promising, although some difficulties still exist in the case of rotational motions [43].

The question arises if use can be made of the results found for the rectilinear horizontal motion of submerged bodies, when dealing with surface ships. The problem of heave and pitch has been attacked by Korvin-Kroukovsky. By using an analogy with submerged bodies, he obtained expressions for the exciting force per unit ship length which agree in structure with the corresponding ones for wholly-submerged

bodies. A dependence of the total force and moment upon the speed ratio $\frac{U}{c}$ arises.

Although this dependence is not supported by Korvin's own experiments, and its existence is denied by Haskind [25], one would feel inclined to assert by analogy that some speed effect should exist. At least one curve published by Haskind appears to support

Figure 2. Coefficients of exciting force and moment in heave and pitch, E_z and E_ψ , as functions of $U^* = \frac{U}{c}$ (From Haskind.)

this conjecture. (See Fig. 2.) Difficulties presented by free surface effects have been discussed by Professor Korvin himself; some of them dealing with the strip method used for computation of damping have been clarified in the meantime by Havelock. It appears that in the most important range of natural periods, the strip method has its definite merits.

We discuss now the method used by Haskind to determine exciting forces [44] [45] [25]. Haskind's calculation is based on Krylov's work and on the consideration of diffraction effects. In quite an unusual way the ship is pictured by an infinitely long cylinder, whose axis makes the angle χ with the wave direction.

The form of the potential of the free wave suggests a complete potential constructed as the sum of a part which solves the problem of forced motions in calm water, and a part which solves the diffraction problem.

Forces due to wave diffraction can be interpreted as due to damping and added mass effects which, however, differ from the common concepts. For side waves the concepts coincide, but denoting the diffraction terms by a superscript (d) one obtains

$$\begin{aligned} \bar{N}^{(d)}(0) &= 0, & \bar{m}^{(d)}(0) &= 0, & \frac{2\pi}{\lambda} \cos \chi &\neq 0. \\ \bar{N}^{(d)}(\infty) &= 0, & \bar{m}^{(d)}(\infty) &= 0, & \lambda & \end{aligned} \quad (10)$$

Thus the general character of the curves $\overline{m}^{(d)}$ is different from that of $\overline{m}$, while the damping curves $\overline{N}^{(d)}$ and $\overline{N}$ display a certain similarity.

Approximate relations for $\overline{N}_{jm}^{(d)}$ are given when $\frac{2\pi}{\lambda}y|\sin \chi|$ is small. Formulas are developed from which the forces $\overline{Y}^{(d)}$, $\overline{Z}^{(d)}$, $\overline{M}_x^{(d)}$ per unit length can be calculated when the transverse dimensions are small compared with the length of the ship, and therefrom by a strip method *all resultant forces and moments experienced by the vessel* (except, obviously $\overline{X}^{(d)}$).

In the case $\chi = 0$ or π in which we are strongly interested, the present simplified method leads to a purely vertical force per unit length, which coincides in type with that for the well-known relative-motion effect. However, instead of $\overline{m}_z$ the diffraction mass $\overline{m}_z^{(d)}$ appears. If we assume for the time being that especially for larger $\frac{\omega_0^2}{g}$, $\overline{m}_z^{(d)} < \overline{m}_z$, one sees that the relative-motion correction over-emphasizes the actual effect.

In [25], the inconsistency of the relative-motion correction is emphasized. Very probably, however, this criticism applies to an erroneous computation of the velocity and acceleration field in a wave.

Exciting forces and moments calculated by Haskind depend *linearly* upon the orbital velocities of the wave particles. The important hydrodynamic reactions due to yaw and sway of an advancing ship cannot be determined by the method proposed.

Finally, we consider an unpublished investigation by Grim already referred to [24], "Die Schwingungen von schwimmenden Zweidimensionalen Koerpern." For simplicity we treat here all hydrodynamic forces involved in heave, sway and roll. While emphasis is laid on the determination of hydrodynamic characteristics when $\omega_0 \rightarrow 0$, the method is extended to finite small frequency parameters $\frac{\omega_0^2 B}{g}$.

Using Lewis' sections the velocity potential for forced roll in calm water is established when $\omega_0 \rightarrow 0$, and the hydrodynamic forces are determined. The potential for the lateral motion (sway) is obtained for $\omega_0 \rightarrow 0$ and added masses calculated; results agree with those given by Haskind.

The equation for roll without damping,

$$\left[J_{ox} + J_{zx} - \frac{(m_o \overline{OG} + m_y h_q)^2}{m_o + m_y} \right] \ddot{\phi} + D \overline{GM} \phi = 0, \quad (11)$$

agrees with the findings by Woznessensky based on Haskind's work [25]. J_{ox} is the moment of inertia of the ship referred to the waterline point O (axis of symmetry of the load water line); $\overline{OG}$ is the distance of the center of gravity; and h_q is the lever of the lateral added mass force. The position of the axis of rotation is derived herefrom.

A strip method is suggested to compute pertinent values for the actual ship. For small ω the *damping* due to the side motion and the roll can be approximately determined by adding an appropriate potential. The results are compared with those obtained for elliptical and circular sections by a more rigorous method [24], and a reasonable agreement is stated. In connection with an earlier investigation by Ursell, it is interesting to state, that for the profile considered the sway contribution to damping is much larger than that due to roll. Thus the loss of wave damping effect which has been found by Ursell for certain sections may not be so tragic as anticipated.

The influence of the effective added moment of inertia upon the roll frequency is communicated as function of $\overline{GM}$. Beside, the influence of large bilge keels on the period is shown as the result of a rather intricate investigation. In addition, expres-

sions are given for the lateral and rotational damping caused by such bilge (or center) keels, showing that this damping increases with the fourth power of the keel height.

The results for the ship without bilge keels are checked by another method based on a conformal transformation.

The same method is then applied to determine the heave motion in calm water. Expressions are obtained for the added mass which in principle agree with Haskind's findings; the known result for the damping is restated.

A bold attempt is further made to compute the damping (for example for an elliptic section) when the rolling angle is no more considered as small.

Boundary conditions are now fulfilled at the actual condition and not at the state of rest. The magnitude of the damping coefficient obtained in this way may be appreciably higher than by the standard approach.

The exciting hydrodynamic forces caused by *regular waves* are determined by assuming first that the body is fixed. When the body is released these generalized exciting forces determine orbital motions of the body.

By considering quadratic terms in $\frac{\omega^2}{g}$ in the development of the wave potential,

further results are obtained. It can be shown within the range of validity of the development made that the heaving motion does not exceed appreciably the orbital motion even at synchronism. This statement, which represents an extension of some results due to Ursell, settles the question raised by Chadwick [4].

Expressions for the exciting moment of roll are extremely interesting. They depend appreciably upon the shape of the cross section. It is imaginable, at least in principle, to design sections which experience low exciting moment at synchronism.

Unfortunately, it was impossible so far to establish agreement between the fundamental equations of *forced* roll used by Woznessensky [9] and by Grim. A lot remains to be done to reach a satisfactory state of knowledge in this seemingly outworn field of research, and results of high interest from the viewpoint of theory as well as practice can be expected from further work.

3.6. Coupling of Motions and Equations of Motion

This problem has been discussed already at some length at the VII International Tank conference. In the meanwhile important contributions have been made, to some extent inspired by the analogy with flight mechanics, but otherwise following lines established primarily by Haskind. Havelock [46] has recently reconsidered the problem of coupled free heave and pitch treated by Haskind, neglecting damping of the motions. He shows that the influence of coupling on natural periods is negligible.

The interesting technical problem which involves the discussions of forced oscillations so far remains open. It was raised by Grim's experiments showing large heaving effects due to forced pitching.

Korvin-Kroukovsky has tried to give a plausible physical interpretation of coupling terms in the combined heave and pitch equations

$$m'_{33}\ddot{z} + N_{33}\dot{z} + cz + m_{35}\ddot{\psi} + N_{35}\dot{\psi} + g\psi = Fe^{i\omega t} \quad (12)$$

$$m'_{55}\ddot{\psi} + N_{55}\dot{\psi} + C\psi + m_{35}\ddot{z} + N_{35}\dot{z} + Gz = Me^{i\omega t} \quad (13)$$

From his general analysis of forces he obtains

$$m_{35} = m_{53}, N_{35} = N_{53}.$$

The coefficients are calculated from added mass and damping values in heave per unit length of the ship by integration. It is further shown that C and g should be split up into $C = C_1 + C_2$ and $g = g_1 + g_2$ where C_1, g_1 are due to hydrostatic and C_2, g_2 to hydrodynamic effects. These latter effects have not been considered by Haskind. Prof. Korvin emphasizes some shortcomings in his analysis which can be eliminated

only by a more profound analysis. Such an analysis apparently cannot rest on Haskind's earlier approach [16] where diffraction effects have not been considered.

Under these circumstances the experimental method is appropriate. Haskind and Riemann [28] have proposed a method by use of which one is enabled to evaluate coupling coefficients from test data under plausible assumptions.

Experimental information on the damping coefficient N_{35} has been obtained by Golovato [34], using a TMB vertical oscillator. In a rather important range of

frequency parameter $\omega_B^* = \omega \sqrt{\frac{B}{g}}$ which, however, in general lies below synchronism of pitch and heave, there is a striking coincidence with Havelock's findings, and outside of it a strong departure. Special effects in the low frequency range are determined at the critical speed ratio $\frac{U}{c} = F\omega_o^* = 1/4$. At its peak value, the moment $N_{35}\dot{z}_m$ is

not negligible as compared with the exciting pitching moment $M_y = \theta_m \rho g I_y E \psi$. The same applies in still higher degree to the term $m_{35}\ddot{z}_m$; but such low frequencies can be reached under exceptional conditions only.

We expect still more difficulties for unsymmetric motions. For example, Chadwick [17] showed that the so-called yaw-heel effect is essentially due to the rate of sway (drift angle) and not to the centripetal acceleration of the ship moving on a curved path as earlier thought [47].

It is reasonable to assume that the couplings of roll back into sway or yaw are negligible.

The papers reviewed represent important steps forward in some directions. However, complete solutions are still lacking even in the *calm water* case, at least in the official literature, and the general problem of arbitrary motions of a ship in a regular seaway is quite open.

What difficulties must be expected when proceeding to the seaway problem can be estimated from the author's tentative treatment on directional stability [48] which unfortunately, is mutilated by misprints. Especially the computation of resistance presents a rather formidable task.

The establishment of equations of motion requires a thorough investigation on the geometry as well as on the mechanics of the problem. Already Krylov has indicated the difficulties arising from the fact that angular displacements of roll cannot be considered small. Further, Ursell has shown that the concept of a permanent axis of roll breaks down in certain cases. Szebehely has discussed the problem of the apparent pitching axis.

One could further follow lines suggested by Haskind [16] which already led to success in the case of heave and pitch. Useful if not general results may be expected, however, in a rather cumbersome form.

A scheme for linearized equations can be established using procedures familiar in flight mechanics. Leaving aside the motion in the direction of the axis and assuming that the symmetric oscillations of heave and pitch are described by Equations (12) and (13), the following expressions can be given for the asymmetric group:

$$m'_{22}\ddot{y} + N_{22}\dot{y} + m_{26}\ddot{\theta} + N_{26}\dot{\theta} + m_{24}\ddot{\phi} + N_{24}\dot{\phi} = Y, \quad (14)$$

where, probably, the terms with $\ddot{\phi}$ or $\dot{\phi}$ can be neglected; and

$$m'_{44}\ddot{\phi} + N_{44}\dot{\phi} + C_{44}\phi + m_{42}\ddot{y} + N_{42}\dot{y} + m_{46}\ddot{\theta} + N_{46}\dot{\theta} = M_x, \quad (15)$$

and

$$m'_{66}\ddot{\theta} + N_{66}\dot{\theta} + m_{64}\ddot{\phi} + N_{64}\dot{\phi} + m_{62}\ddot{y} + N_{62}\dot{y} = M_z. \quad (16)$$

In [16] the terms with $\ddot{\phi}$ or $\dot{\phi}$ may again be small. These equations do not represent much more than a skeleton for research.

Gyrostatic terms discussed in Ref. 1 are omitted in the process of linearization. This agrees with a recommendation made by Chadwick and can be considered as a necessary first step from a formal point of view. But the appraisal of the order of magnitude of inertia and hydrodynamic forces is a serious task in itself. Very probably in the case of roll the term $(m'_{66} - m'_{55}) \dot{\varphi} \dot{\theta}$ is important because of the high value of the difference in added masses $m_{66} - m_{55}$ as compared with m'_{44} . In general by neglecting such coupling terms peculiar effects may be omitted.

Some further remarks on nonlinearities appear to be appropriate.

It is natural to treat the roll simultaneously with sway and heave as done so far, but separate investigations are needed because of the basic importance of *transverse* stability and the nonlinear character of the motion at large angles. In fact, studies on roll serve as a prototype when dealing with nonlinear ship motions.

Efforts are being continued to determine the actual stability in a seaway. It has been shown that stable models capsized when advancing on oblique courses in regular waves although some experiments do not support results obtained by elementary calculations with respect to heavy losses of metacentric height on wave crests in a following sea. Theoretical work has not progressed appreciably either in this field or with respect to roll at large angles. Baumann [49] has discussed the latter problem under simplifying assumptions which admit the solution of the equation of forced roll in a closed form.

Equations with higher order terms in the restoring expressions for heave and pitch have been solved by a step by step method [50], but results so obtained do not present especially interesting features.

Nonlinear effects in roll damping are well known [51]. Recent experiments on forced heave have yielded similar pronounced effects [34]. So far they have not been detected by model investigations in regular seaways of various but moderate height, which generally yield a reasonably linear relation for motion amplitudes.

4. Hydrodynamic Impacts

Within the last five years the application of hydrodynamic impact theory to problems of naval architecture has successfully developed. At present, shock phenomena are considered to be as important as the approximately periodic effects experienced by a ship in a seaway.

A first step has been to borrow from sea plane mechanics the fundamental theory due to Wagner; however, quite a bit remains to be done to arrive at quantitative results in the case of a ship.

Special studies had to be devoted to the investigation of conditions under which "slamming" occurs. With regard to notation, by "slamming" we denote essentially impacts which act in a direction not too far from the vertical and mostly on bottom parts of the ship. However, important impact effects can arise in other cases, e.g. when a rather blunt forebody heads into steep waves, when green water on deck hits the bridge superstructure or when wave impacts are exerted on the side of the ship. All that is needed for the occurrence of such phenomena is a high relative normal velocity between a flat part of the ship and a free water surface bounding a considerable amount of liquid.

It is, therefore, preferable to restrict the use of the expression "slamming" to the "vertical effects" only mentioned earlier and to use the expression hydrodynamic impact as the more general concept. Reference is made to the work of V. Szebehely and collaborators at TMB [52], which rests on analysis, model experiments and full-scale research. Interesting experimental studies have been performed by Akita and Ochi [53].

Although approximate calculations have been made to determine the influence of elasticity on the impact process in seaplanes, no comparable work has been performed in ship theory nor has Cauchy's Law been observed when experimenting. Pos-

sibly this elastic influence may not be too large if ship bottoms have a reasonable rise of floor.

Apparently without knowing Wagner's work, Bagnold has developed a physically consistent qualitative theory of wave impacts on hydraulic structures, especially moles. He assumes that air cushions can be sealed off at vertical walls by breaking waves. These cushions may collapse. Such effects suffice to explain qualitatively the occurrence of large impacts as has been demonstrated by model experiments. There may be some parallelism with Japanese work hinted at by Akita.

By accepting such a mechanism it is possible to cope with boundary conditions at a wall in a manner which is at least physically consistent, although the present writer does not see unsurmountable difficulties in assuming the rise of large impacts due to shallow water wave momentum by direct water action only.

5. Full-Scale Experiments and Trials

We distinguish between 1) collection of data on ship behavior on board a ship without ample instrumentation, and 2) full-scale experimenting in the proper sense, which is performed by competent staffs using adequate instruments, and which, at least occasionally, can rely upon special trials.

We list some purposes of full-scale experimenting:

a. Collection of statistical data on the seaway, on ship motions, accelerations, impacts, speed including propeller and engine performance, pressures, stresses etc.

b. Determination of *limiting* values of some items mentioned which can be admitted from the point of view of safety and seakindliness. For example, the establishment of upper *limits* of permissible acceleration has to rely almost exclusively on full-scale experience at least at present where facilities similar to those used in flight research are not available.

c. Exploration of effects so far unknown caused by the seaway.

d. Check of results obtained by analysis and model research under actual conditions; investigation of scale effects.

e. Acquaintance of scientific staffs with actual conditions and ship operation at sea.

As an example of the first approach we mention a rather wide enterprise organized by the Hamburg Tank before the last war. Results obtained contributed some general information which to a certain extent was deduced earlier from the simple linear differential equations. An attempt was made to classify ship behavior as a function of a vertical prismatic coefficient which agrees with our present ideas on damping properties of hulls.

a. Earlier, full-scale experiments on the M. S. "San Francisco" became famous. The ship was adequately staffed and instrumented. However, because of lack of experience, no thorough theoretical and experimental preliminary work was accomplished in the field of hydrodynamics before starting the expedition. Nonetheless, the trip was successful, mainly for two reasons:

1. the extreme luck that, close to the end of the voyage, exceptionally heavy seaways were encountered.

2. the outstanding ability of one of the leaders of the expedition, in interpreting experimental data obtained in the field of strength of the ship.

b. Such luck was absent in the case of the "Ocean Vulcan". Although investigations extended over a long time, the ship was lavishly instrumented, the scope of data obtained especially with respect to pressures is ample, and its evaluation is exemplary—no revolutionary information resulted from these thorough tests. Probably the comprehensive Report 8 [50] is being more frequently consulted because of its beautiful diagrams based on rather exhaustive routine computations than because of the experimental material collected. Further, the author's attempts to consider the influence of non-linear effects deserve attention.

c. Similar remarks apply to the Japanese report on the "Nissei Maru" [54]. Although interesting data on power, oscillations and stresses have been collected, the piece de resistance are model experiments designed to elucidate the full-scale work.

Since statistical methods have not been used in all three cases so far discussed, the latter must be considered obsolete from a methodological standpoint.

d. Modern investigations in this sense are being conducted by the David Taylor Model Basin and other agencies in this country.*

e. A prototype for future full-scale research has been supplied by Cartwright and Rydill by their recent paper "The Rolling and Pitching of a Ship at Sea" [40]: For the first time a satisfactory correlation between seaway and ship motions, which earlier investigators have aimed at, has been reached. The statistical aspect used is somewhat different from that proposed by St. Denis and Pierson and by Woznessensky insofar as it is based from the beginning on Fourier series—an approach, the validity of which has been questioned by the authors quoted.

Cartwright and Rydill show how some modes of ship motion can be related to the seaway. Waves have been measured by a ship-borne wave recorded [56]; the analysis is primarily based on an automatic Fourier analyser. To check theory, linear equations of motion are used in a "simple" and "advanced" form. Both are based on the Froude-Krylov hypothesis. Only roll and the coupled pitch and heave motions are treated, for arbitrary heading angles.

Numerous graphs show the comparison of results obtained by experiment and calculation. The agreement is from reasonable to good, especially considering the fact that basic parameters had to be estimated.

The statistical results are thought by the authors to be just as important as the spectral analysis. Almost any aspect of the motion of the ship can be dealt with in statistical terms once the connection between the energy spectrum and the statistics has been established.

The determination of damping by auto-correlation following Tukey deserves attention, and a proof is given by the authors that the auto-correlogram under special conditions coincides with the extinction curve.

6. *Seaworthiness of the Hydrodynamic Class of Vessels*

So far we have considered displacement ships only, because of the overwhelming importance of this class of vessels. However, from the viewpoint of future development, this attitude is not correct. A consistent theory should be able to furnish information on the inherent seaworthiness qualities of all existing classes of ships. We are still far from this goal.

While in the field of aircraft building there is a strong correlation between research and development, the hydrodynamic class of vessels—planing and hydrofoil boats—has for a long time been developed widely by a trial and error procedure. This applies especially to gliding craft; but even in the case of the hydrofoil boat, the success of which depends to a large extent on its seaworthiness qualities, the rate of investment of scientific work did not correspond to the importance of the task.

We review briefly the status of our knowledge of the behavior of hydrofoil systems in a seaway.

Even accepting the most drastic simplifications as to hydrodynamic effects involved (quasi-steady conditions) the problem is tedious because of the large number of significant variations: wholly submerged and piercing hydrofoils, number of foils and distribution of lift amongst them, the air gap between the bottom of the hull and the free surface in planing conditions, elastic cushioning etc. Model experiments yielded a decisive superiority of hydrofoil craft in a head sea over high-speed displacement and

* After completion of this report an important paper has been presented by Jasper before the SNAME [55].

semi-gliding and gliding boats. Because of difficulties in experimenting they were seldom conducted in a following sea.

We mention first an elementary investigation on the behavior of hydrofoil systems in heave and pitch motion [57]. Assuming that we have at least two foil systems (at the bow and at the stern) we easily see that the coupling of heave and pitch is an essential feature of the problem involved. Expressions for the natural periods and the damping are readily obtained; the latter can be made much stronger than with displacement ships. This has to be considered an advantage of the hydrofoil principle. Conditions are in general less favorable in a following sea, the main reason being the direction of the orbital velocities.

Transverse stability of piercing foil systems so far has not caused much trouble. Experience has shown that it is easy to cope with the problem of roll; so little has been done in this direction.

At present a thesis is being prepared which deals with all six degrees of freedom, the hydrodynamic part of the investigation being kept as elementary as possible.

One can expect that such theories may serve at least as guide for planning experiments. Investigations of this character may be linked with those on displacement vessels with fins, and may yield a valuable extension of our knowledge on future lines of development of high-speed seaworthy vessels. Another important sideline is the connection with seaplane research.

Quite recently an important step forward in the field of hydrofoil theory has been made by P. Kaplan [58]. The theory developed by him refers to unsteady motions of a completely submerged foil of infinite span near a free surface. Explicit results are given for the lift force and mean wave drag for two important special cases, assuming a constant speed of advance:

1. foil performing harmonic heaving oscillations in calm water.
2. foil advancing rectilinearly, head-on under sinusoidal waves.

The amplitude of lift force is rather frequency dependent. Values obtained by quasi-steady considerations grossly overestimate the actual conditions even at moderate Strouhal numbers.

It is to be expected that this theory, notwithstanding its limitations, will prove extremely useful in promoting hydrofoil boat development.

By far the largest part of our scientific knowledge concerning planing vessels (hydrogliders) is due to investigations on seaplane performance.

So far no theory exists of the behavior of hydrogliders in a seaway and literature on experiments is extremely meager also. It is imaginable that useful results can be developed from the impact theory due to Wagner, and such a theory supported by experiments should enable us to reach conclusions as to advisable dead rise angles, L/B ratios etc.

The present writer does not consider this task especially important since it is obvious that planing vessels must remain much inferior to hydrofoils with respect to seaworthiness. It is recommended, however, that the hydroglider problem be treated as a byproduct of research on hydrofoil and Archimedean vessels.

II. THE SIGNIFICANCE OF RESEARCH RESULTS FOR DESIGN

1. General Remarks

Recently in a technical journal engineers were described as people toiling over meager problems left over by physicists and mathematicians and frequently suffering from an inferiority complex.

Because physics is developing with increased speed in its "central" part, many classical problems like hydrodynamics and theory of elasticity, are left aside and taken over by engineering sciences. In addition, however, engineers have to fulfill other tasks, for example, to consider and coordinate the multitude of problems arising in a tech-

nical field and to evaluate and interpret results in such a way that they can be used in design. The extensive algebraic, numerical and experimental work needed to establish quantitative relations and to appraise the importance of various parameters is a necessary prerequisite for making suggestions which may lead to an increasing safety and seakindliness of ships. There should be no inferiority complex about such kind of endeavours.

Lack of systematic evaluations is one reason for the wide discrepancy between scientific activities and the limited amount of their application, and contributes to the fact that the determination of seagoing qualities, except for roll, rested more on opinion than on knowledge.

As pointed out already, International Safety Regulations for passenger ships are restricted to elementary considerations only on longitudinal and transverse stability, especially in damaged condition—an insufficient approach.

It is utterly futile to search for coherent scientific ideas underlying present cargo vessel *freeboard regulations*, upon which the vessel's safety depends to a large extent. *Faute de mieux*, one would expect that there exists at least proportionality between freeboard f and length L of similar ships; but actually the freeboard ratio f/L for smaller ships, following tables, can be lower than for bigger vessels, although the former are much more imperilled by short waves which possess higher steepness ratio than longer ones.

Worse than that, safety considerations were sacrificed to arbitrary requirements presented by tonnage regulations. Open spaces with tonnage openings (and in earlier times scuttles) were an affront to common sense.

Under such circumstances it becomes understandable that the distinguished work of experimenters like R. E. Froude, [59] Kent, and Kempf, although aimed at immediate application, did not find the deserved response in practice, especially commercial practice. At the VII International Towing Tank Conference even a loss of experience in the experimental field was stated.

The lack of interest in theoretical findings is clearly demonstrated by the fact that apparently nobody objected over a long time to the inconsistency of period formulas for heave and pitch calculated without the added mass effect.

The narrow concern with the calm water resistance once led to the development of extremely U-shaped forebody sections with flat bottom parts; a design which was disproved by continuous damages due to slamming.

We have here purposely assumed an aggressive tone when dealing with the conditions in a not too distant past, because in our opinion an appeasing attitude towards irrational regulations has seriously hampered progress.

Obviously, there are inherent difficulties in dealing with our problem from the designer's point of view. It is already a task to appraise the importance of seakindly performance as compared with other postulates in design. The viewpoint of *enlightened practice* has been presented in discussions to Ref. 5. Researchers should consult a paper by Kent [7] which gives an interesting aspect of the problems.

In what follows, frequent reference will be made to work by E. Lewis [5], [60], [61]. We can slightly amplify a list of topics enumerated by him, some of which may be considered as criteria of danger:

1. Shipping green water,
2. Large displacements, especially angular displacements, due amongst other reasons to loss of stability in waves,
3. Impacts, especially slamming,
4. Large accelerations,
5. Conditions leading to high stresses, especially girder stresses,
6. Resistance increase,
7. Loss of steering properties.

To determine ship characteristics which guarantee a good behavior in a seaway, we need a complete survey of all ship motions. This exceeds by far the scope of the present paper. Our aim is rather to indicate simple methods which, duly extended, will enable us to meet requirements of present and future practice. We shall restrict ourselves to some remarks on roll, heave and pitch. Future extension in knowledge will modify the procedure and results here indicated.

2. *Transverse Stability and Roll*

Froude's work on roll has been widely used in design. The importance of the tuning factor Δ and of the factor of magnification μ with its dependency upon damping has been clearly understood. However, because of limitations of the linear theory more advanced considerations must be introduced to cope with the safety problem. A most urgent task is to evaluate the influence of the shape of Reed's diagram on the behavior of a vessel. So far this question has been discussed on a hydrostatic basis only.

Even such an elementary approach shows that there may be an appreciable loss of "stability" (restoring moment in roll) in waves under certain conditions [62]. Thus *minimum* requirements for the metacentric height and the curve of righting arms (Reed's diagram) derived from calm water conditions and from some statistical information must be raised, especially in the case of smaller cargo ships.

A standardization of the minimum values of stability parameters will rest on future research dealing with the actual pressure distribution around a rolling ship and the non-linear character of roll prevailing at large angles.

Valuable information is now being collected on damping as well as exciting forces. There is a tendency at present to underestimate the bearing of such scientific activities on the design of ship hulls because of the application of antirolling devices. It should not be overlooked, however, that an adequate understanding of the hydrodynamic phenomena involved may be a prerequisite for a successful development of stabilization. Reference is made to various effects mentioned before, especially the sway-roll effect; earlier lack of knowledge in this field has caused serious setbacks.

The present state of roll stabilization can be described by quotations from a paper by Chadwick [17]:

1. "the technical feasibility of stabilization is an established fact"
2. "the control system no longer should prove a limitation on system performance." Ideas are being developed at present to recover some of the seaway energy for stabilizing purposes.
3. "At least some feedback control should always be used. Arguments for feed-ahead control, predictors, etc. have largely been negated by advances in control technology." This statement, if corroborated, would dispose of the attempts of a school of thought which preached the superiority of feedahead control.
4. "For stabilization at speeds of 15 knots and above activated fins provide the most efficient system. Tanks and gyros are still worthy of consideration in low speed application." This assertion again practically means the victory of fins since there is not too much interest in low speed application.

We refer further to two principles of stabilization which we consider as important notwithstanding some doubts expressed [Ref. 17, discussion]:

1. with respect to the true horizon,
 2. with respect to the effective wave surface;
- the first one being the natural solution for men of war, the second presenting advantages for passenger vessels. Recently these ideas have been generalized [63].

3. *Heave and Pitch*

After a brighter outlook has been gained in the case of roll the primary interest is now concentrated on obtaining favorable "symmetric" motions (pitch and heave).

3.1. Basic Consideration

It is much to the credit of E. Lewis [61] that he had the courage to recommend rules for design derived from elementary theoretical findings including considerations yielded by the modern irregular seaway theory. His paper contributed numerous results of immediate value for practice.

Although we accept E. Lewis' work as basic, there are some reasons to reconsider the heave and pitch problem.

1. E. Lewis intentionally does not consider the influence of the longitudinal displacement distribution (more correctly, the shape of the load waterline) on the exciting forces for given main dimensions of the ship. This is, however, one of the few dependencies which can be investigated with some confidence within the range of validity of the Froude-Krylov hypothesis.

2. The present knowledge of damping properties admits of more definite statements although we are still far away from an exhaustive solution of the problem.

3. Objections have been raised against the choice of Taylor's displacement-length ratio by several writers including the present one. Let us consider the question once more.

The use of the parameter D/L^3 , consecrated by resistance research, has undoubtedly led to a preliminary orientation in the new field, since for fixed displacement, this main form parameter successfully shows the importance of length. For detailed work, however, one should avoid merging several characteristic form parameters into a single one.

The present survey is based on the simple Equations (1) and (2) and their solution for forced motions. Exciting terms can be corrected using the relative motion concept.

For our present discussion, which remains within the linear range, the wave length λ is the most interesting characteristic, although height (steepness or slope) obviously becomes equally important when limiting values or accelerations have to be considered.

Let us first collect some basic, useful formulae.

$$\text{Frequencies of encounter:} \quad \omega^* = \omega_0^*(1 - F\omega_0^* \cos \chi) \quad (17)$$

$$\text{Tuning factor:} \quad \Lambda = \frac{\omega_0^*}{\nu^*} (1 - F\omega_0^* \cos \chi) \quad (18)$$

$$\text{Amplitude ratios:} \quad z_m^* = \frac{z_m}{r_m} = E_z(\lambda^*, y^*) \mu_z(\kappa_z, \Lambda_z) \quad (19)$$

$$\psi_m^* = \frac{\psi_m}{\theta_m} = E_\psi(\lambda^*, y^*) \mu_\psi(\kappa_\psi, \Lambda_\psi)$$

$$\text{Phase lag:} \quad \tan \xi = \frac{\kappa \Lambda}{1 - \Lambda^2} \quad (20)$$

$$\text{Damping:} \quad \kappa_z = \frac{N_z}{\rho \sqrt{V'_z A_w g}} \quad (21)$$

$$\text{Magnification factor:} \quad \mu = \frac{1}{\sqrt{(1 - \Lambda^2)^2 + \kappa^2 \Lambda^2}} \quad (22)$$

In what follows $\cos \chi$ will be put equal to -1 , since the essential procedures can be demonstrated by considering the ship heading directly into waves.

Following traditions in naval architecture wide use can be made of graphical representation.

As first item we consider the tuning factor. Although the discussion of the latter represents a large part of the "theory" of roll, quite a bit remains to be done in the present case. There is occasionally some confusion about the supercritical range of ship operation due to the fact that for a given T the tuning factor can become greater than unity either because

$$\omega_0/\nu > 1$$

or because

$$1 + \frac{\mu}{c} \left(= 1 + F_{\omega_0^*} \right) \text{ is large.} \quad (23)$$

Obviously, when advancing in short waves the supercritical condition is the natural one for larger ships. This case is, however, in general not interesting because of the insignificant effects involved. The supercritical state becomes a problem in long waves only.

To answer our problems we need:

1. The equation of the hull $y(x, z)$, L , B , H , and the mass distribution. Call $y^* = y/L$.

2. Natural periods T_z , T_ψ or frequencies. For simplicity we neglect here the dependence of T upon ω^* (and a fortiori upon F), so that $T = \text{const}$. However, when exploring new conditions (high speeds) it may become necessary to consider $T(\omega^*)$ as variable. In any case reasonable data are available to calculate T by a strip method and applying corrections.

An experimental determination of period and damping is recommended in model research as a necessary prerequisite, notwithstanding the difficulty due to the evaluation of pertinent extinction curves.

3. Curves of damping in heave and pitch.

We can assume $\kappa = \kappa(\omega^*, y^*)$ neglecting the dependency upon F . κ is best calculated by a strip method introducing corrections when ω^* is small. As in the case of period calculations the appropriate frequency parameter in general is $\omega_B^* = \omega \sqrt{B/g}$. The conversion to $\omega^* = \omega \sqrt{L/g}$ requires care. For the present purpose we use curves in Fig. 1 and some averaged results. It appears fortunate that in the range of resonance $\omega^* = \nu^* = 2.5 \rightarrow 5^{(5)}$ we are not too far from the peak of the damping curve when the lower values of ν^* apply. This may be the case, for example, when large mass moments of inertia (in the supercritical range) are admitted. For large ν^* (ν_B^*) the decline in damping can become noteworthy.

To avoid confusion the *dimensionless* coefficient κ should be used for discussions of this kind.

The interval in which damping represents a *decisive* factor, say

$$0.6 \leq \Lambda \leq 1.2 \qquad 1.5 \leq \omega^* \leq 6 \qquad 0.5 \leq \omega_B^* \leq 2$$

is large but nonetheless not excessive. There has been, naturally, a concern about a possible loss of damping power when one departs strongly from present practice. It turns out now that this loss can be safely estimated and does not become prohibitive when ν^* is properly chosen.

Particularly, the range of frequencies where three-dimensional effects are pronounced, $\omega^* \leq 2.5$ for heave and $\omega^* \leq 3$ for pitch, lies normally at the border or outside of the synchronism zone. The most important task at present appears to be

to obtain consistent functions $\kappa(\omega^*, y^*)$ which allow us to calculate the influence of all important form variations.

4. We assume the curves of exciting force coefficients E_z , E_ψ as known, although explicit data for the diffraction effect are not yet available. For the present purpose the Smith effect and the relative motion correction can be included; in principle the second correction requires the use of Equation (2).

The functions E allow us to deal with any wave component constituting the irregular seaway, hence by superposition with the total seaway.

The curves E_z , E_ψ show that it is possible to consider $\lambda^* = 1$ as a significant point for determining the *character* of heave and pitch for two reasons:

1. firstly, the point $\lambda^* = 1$ lies in the final region of large wave effects;
2. secondly, the maximum steepness of the E_ψ curve occurs actually in the range $0.9 \leq \lambda^* \leq 2 \rightarrow 2.5$, although magnitude and slope strongly depend upon the shape of the waterline (the coefficient α). However, *quantitative* data so obtained cannot be widely generalized.

The analysis of ship motions can be conducted as follows [51, 5]:

We start with the relation $\Lambda = \frac{\omega_o^*}{v^*} (1 + F\omega_o^{*2})$ and plot a diagram of the function $\omega_o^* + F\omega_o^{*2}$.

Resonance conditions are found from

$$v^* = \omega_o^* + F\omega_o^{*2}, \quad (24)$$

yielding the critical value for ω_o^* or

$$\lambda_{*R}^*(F) = \frac{2\pi}{\omega_o^{*2}}. \quad (25)$$

For $F = 0$

$$v^* = \omega_o^*, \quad (26)$$

so that

$$\lambda_{*R}^*(0) = \frac{2\pi}{v^{*2}} = \frac{T^{*2}}{2\pi}. \quad (27)$$

$\lambda_{*R}^*(0) = \lambda_{*R}^*$ is a characteristic value for a given ship, $v^* = \text{constant}$, indicating the wave length ratio at which synchronism occurs in the hove-to condition [51].

Assuming

$$1.2 \leq T^* \leq 2.5, \quad \text{or} \quad 5.2 \geq v^* \geq 2.5, \quad (28)$$

one obtains

$$0.23 \leq \lambda_{*R}^* \leq 1. \quad (29)$$

The reasoning can be easily extended to cope with the general case. Resolving (21) with respect to F one obtains the critical Froude number

$$F_R = \frac{v^*\lambda^*}{2\pi} - \sqrt{\frac{\lambda^*}{2\pi}} \quad (30)$$

Curves can be drafted to show the dependence of the critical Froude number F_R upon λ^* with v^* as parameter. The strong variability of F_R with v^* is obvious.

The curves $F_R(\lambda^*, v^*)$ divide the subcritical from the supercritical range.

To investigate the behavior of a *given* ship, one proceeds as follows:

1. $\kappa(\omega^*)$ and μ can be plotted versus ω^* and Λ .
2. Exciting functions E_z and E_ψ are calculated and represented in the form $E\{\gamma\} = E(\lambda^*) = E[\omega^*]$.

Figure 3. Schematic diagram. Dimensionless pitching amplitude $\psi_m^* = \psi_m^*(\lambda^*; F; \Delta)$, $\nu^* = \pi$.

3. Three forms of amplitude ratio diagrams can be derived.

a. $z_m^* = z_m^*(\lambda^*; F, \Delta)$,

i.e., z_m^* (or ψ_m^*) as function of the wave length with F and Δ as parameters [51] (see Fig. 3),

b. $z_m^* = z^*(\Delta; F, \lambda^*)$.

Such a complete diagram has been first established by Gawn using experimental results [51]. See Fig. 4.

c. $z_m^* = z_m^*(F; \Delta, \lambda^*)$. See Fig. 5.

Figure 4. Schematic diagram. Dimensionless pitching amplitude $\psi_m^* = \psi_m^*(\Delta; F; \lambda^*)$, $\nu^* = \pi$.

Figure 5. Schematic diagram. Dimensionless pitching amplitude $\psi_m^* = \psi_m^*(F; \lambda^*; \Lambda)$, $\nu^* = \pi$.

It is now customary to plot model results in the form

$$z_m^*(\Lambda; \lambda^*) \text{ or } z_m^*(F; \lambda^*),$$

i.e., to investigate the ship motion in a given seaway λ^* , as function of Λ or F .

This settles the question for a given ship.

4. Using such diagrams, the influence of variations in mass distribution and form can be investigated.

In particular, if the hull form is kept constant, the influence of changing mass distribution is seen in these diagrams as the result of changing ν^* .

The chosen dimensionless form of representation lends itself when L remains constant. Another approach is more advantageous when L is varied.

3.2. Influence of Form Variations

As example we discuss shortly the influence of variation of some basic ship parameters on heave and pitch motions. The procedure is similar to that used in an investigation on ship resistance [64]. It has the advantage of showing the need for systematic experimental information in this field notwithstanding the large amount of model work completed. Only large values of λ^* are considered. We treat several cases:

1. *Principal dimensions constant.* $J_y = \text{const.}$ "V" versus "U" sections (increase of α).

Figure 6. Dimensionless amplitude of pitch, heave, and motion of the stem, $z_m f / r_m$, for models with differing α . $\alpha H = \text{constant}$, V constant, $\lambda^* = 0.815$.

The clear advantage of "V" sections established several times can be explained by 1) the *increase* of κ_z and κ_ψ which vary roughly as α following some approximate calculations, and by 2) the *decrease* of the *exciting functions* E_z , E_ψ . On the other hand, differences in natural periods T_z , T_ψ may be unimportant.

2. $L = \text{const.}$, $B = \text{const.}$, $J_y = \text{const.}$, $\alpha H = \text{const.}$

We discuss this somewhat artificial assumption since it has been tested using 3 wall-sided, flat-bottomed models with $\alpha = 0.533$, $\frac{2}{3}$, 0.8 to check Haskind's damping formulas [65]. See Fig. 6. Models with higher α (larger A_w and still larger I_y) experience much less motion at $\lambda^* = 0.815$ as well as $\lambda^* = 1.33$.

The explanation again is obvious, exciting functions and damping working in the same direction.

Natural periods disclose a somewhat erratic behavior which seems to be of no great moment; unfortunately, however, nothing has been said about the *wetness* of models.

Slamming, which will undoubtedly affect the very shallow full models, limits the practical applicability of results beyond the statements made with respect to damping and exciting forces.

3. Affine *distortions* of the *set of lines*, first so that $L = \text{const.}$; $BH = \text{const.}$ An *increase* of B and, therefore, decrease in H leads to a slight reduction in natural periods, but the most important change is the strong *increase* in *damping* coefficients.

The exciting forces are slightly *increased* due to a smaller Smith effect, but *decreased* due to a larger relative-motion (diffraction) effect, the net balance probably being close to zero. Thus, motions are reduced; however, the advantage may be destroyed by adverse impact effects.

4. Affine distortions of lines, with variation of length, such that $LB = \text{const.}$, $H = \text{const.}$ Let us consider two cases for which L is *increased* by decrease in beam L/B being strongly increased:

4a. $J_y = \text{constant}$.

Again, this artificial case is considered because systematic experiments have been made on damping properties of three affine models C_3 , C_2 , C_1 with L/B ratios 5, 7.5, 10.

Heaving periods T_z are not too much affected by the variations (decrease in added mass), pitching periods vary approximately as $\sqrt{I_y}$ where I_y is the moment of inertia of the waterline.

Keeping $\lambda = \text{const}$ such that $\lambda_3^* = 1.15$, $\lambda_2^* = 0.94$, $\lambda_1^* = 0.815$. z_m^* and ψ_m^* decrease heavily with increased length because of the decline of the exciting functions $E(\lambda^*)$. Beside, an increase in phase lag indicates stronger damping in pitch for C_2 and C_1 . See Fig. 7a.

Fig. 7b shows a more interesting comparison at a constant $\lambda^* = 1.33$.

The reduction of pitching motion for the longer models is due to an *increase* in the *dimensionless* damping coefficient in agreement with theory.

The heaving diagram cannot be satisfactorily explained by present theoretical reasoning.

4b. J_y variable, $\frac{J_y}{I_y} \approx \text{constant}$.

Natural periods are slightly reduced.

As under 4a. the *dimensionless* heave damping may be reduced and the same applies now to *pitching* although the absolute damping coefficient increases strongly with length.

The advantage of the form variation under consideration consists in the fact that, keeping the speed of advance constant, resonance for a longer ship occurs almost in the same seaway since T_ψ remains almost constant. This means a reduction in λ_R^* and thus amplitude ratios z_m^* , ψ_m^* .

Figure 7. Dimensionless amplitude of pitch, heave, and motion of the stem for models of differing length and waterline moments of inertia. $V = \text{constant}$, $J_y = \text{constant}$.
(a) $\lambda = \text{constant}$. (b) $\lambda^* = \text{constant}$.

5. A length increase with $LH = \text{constant}$ and low, $B = \text{const.}$, while the longitudinal mass distribution remains affine, involves

1. a moderate decrease in T_z and T_ψ , and
2. a slight increase in dimensionless damping κ_z and κ_ψ .

Resonance occurs at somewhat lower wave length λ , and λ_R^* is still further decreased as compared with 4.

The discussion can be, obviously, conducted in such a way that the advantages of lengthening with respect to higher speeds of advance are demonstrated.

6. Finally, it can be considered as well established that the *bulb* has no detrimental effect on motion performance [66], [31], [61].

These findings throw some light on the merits of the analysis by E. Lewis based on the parameter D/L^3 . Although effects are lumped together which follow different laws corresponding to B/L , H/L , α and δ variations, the overwhelming importance of the length dimension is clearly borne out by plotting results versus Taylor's parameter. To this extent, E. Lewis' synopsis, intended to cover the whole range of normal ship types, is justified. On the other hand, investigations on the sea performance of a given type require a more detailed approach.

We have seen that a ship with normal proportions and mass distribution will meet resonance in the hove-to conditions at λ_R^* well below unity.

At finite speeds the supercritical range is extended to larger λ_R^* . Theory and experiment show that absolute motions can be small in the supercritical range; phase relationship and accelerations require a special investigation. When sufficient power

is available to cross the resonance barrier in waves of larger λ_R^* , it can be advantageous to extend the supercritical range as far as possible by increasing natural periods. The means leading to this goal are large $\frac{H}{L}$, large $\sigma = \frac{\delta}{\alpha}$, large mass factors. The penalty which one has to pay consists in a pronounced deterioration of qualities in the resonance condition [10].

3.3. Wetness

Relative motions of the ship with respect to the water surface rather than the absolute motions, determine wetness. Naturally, the corresponding equations should be used, although this so far has not been done in general.

A lot of experimenting has been carried out on wetness. Using the standard steepness ratio $\frac{h}{\lambda} = \frac{1}{20}$, very unfavorable conditions are obtained in larger waves in the neighborhood of synchronism.

For a better understanding of the matter it is advisable to test models at lower steepness ratios where the assumption of linearity still holds and comparison with the existing theory is justified.

Again, only a small amount of *systematic* work based on such comparison has been performed. Szebehely [52] found that the present theory yields a useful guide for slamming investigations but that quantitative agreement between results of measurements and calculations is frequently lacking. Vladimirov [67] improved the prediction of wetness phenomena by adding wave effects, calculated for the speed of advance in calm water, to the surface of the sea. The theoretical determination of *phase angles* which is decisive in this kind of research is not too reliable. Further, the distortion of waves caused by the advancing and oscillating ship cannot yet be calculated by our present elementary approach. Earlier inspiring attempts by Kreitner, are based on rather arbitrary assumptions.

In the subcritical range the relative motion of the ship with respect to the corrugated water surface improves at large λ^* . The question arises as to what criteria should be used for activated pitch stabilization—a problem which becomes urgent at present after reasonable suggestions have been made some twenty-five years ago to use fins.

An increase in damping tends to tie up the ship with the wave surface in larger waves which appears to be a reasonable answer in the subcritical zone. In fact, for a phase angle $\varepsilon < 90^\circ$ it can be shown that the most favorable results with respect to wetness can be obtained at the bow when the linear motion amplitude $Z_{mbow} \approx r_m \cos \varepsilon$. When $\varepsilon > 90^\circ$ the best answer is "horizontalization" i.e., $\psi_m = 0$. This seems to settle the question.

4. Hydrodynamic Impacts

At present, this topic has been so frequently discussed [4], [5], [52] that we can restrict ourselves to few remarks only.

The following form characteristics appear desirable to avoid hydrodynamic impacts:

1. high rise of floor at the ends, especially at the bow,
2. sufficient draft, again primarily at the bow,
3. moderate angles of entrance at and above the waterline,
4. moderate, not excessive, flare in the forebody (St. Vincent),
5. to combine conditions 3. and 4. a clipper stern is advantageous,
6. high freeboard and sheer, including a forecastle to keep the deck free from impacts by green water. The forecastle should be designed with respect to horizontal impacts (for example, whale-back).

The difficulties due to slamming experienced in ballast condition have been emphasized by several writers; Akita has pointed out at the detrimental influence of a flat stern in this case [53].

Heavily raked stems appear advantageous only if the wedge angle of the sections at the centre plane is not too high.

The proper design of superstructures, hatches, wave breakers etc. in the forebody must be kept in mind.

To investigate impact properties of the hull above the waterline at high sustained sea speeds, especially in unfavorable phase conditions, it is suggested to develop a model, the lines and superstructures of which are designed primarily from impact considerations, and to compare such a model with a normal form. A similar proposal intended to reduce the excessive air resistance of ship superstructures have been made some thirty years ago. All big "roughnesses" were stripped off the deck and the superstructures "streamlined". This exaggerated simplification inspired the designer of the "Bremen" to try a new look which was successful. The grave error committed by dropping the forecastle has not been suggested by the designer.

5. Conclusions

Since considerable efforts are being made at present (contrary to the earlier state) to elucidate our problems, it is appropriate to try to answer in a summary way the question: what will come out of present and future scientific endeavors in our field for shipbuilding and shipping practice?

1. It can be claimed that an understanding of general phenomena has been reached already by the applications of the extended Froude-Krylov theory. The overwhelming importance of the ratio $\lambda^* = \lambda/L$ is established.

2. The influence of principal variations of hull (dimensions, proportions and form), and of mass distribution on motions in regular waves can be investigated to a first approximation by such elementary means. A lot of actual work remains to be done.

3. The introduction of the modern irregular seaway concept has an important bearing on application in practice. Because of the presence of long waves in a fully developed heavy irregular seaway the appraisal of the length dimension undergoes some changes. Earlier suggestions to increase the length of our largest ships in such a way

that the ratio $\lambda_{\max}^* = \frac{\lambda_{\max}}{L}$ remains noticeably smaller than unity are superseded.

In general, the advantage of long ships as compared with shorter ones established by the regular seaway concept may be slightly reduced; however, this problem can be settled only when a satisfactory standard representation of irregular seaways will have been reached. There is no need, however, to prove the truism that on the average larger ships are more seaworthy than smaller ones and that from our present point of view lengthening constitutes an important means to improve motions and speed qualities.

4. Conditions are similar to those in ship resistance research. By trial and error good or reasonable solutions to numerous problems have been found. It is, therefore, probable that work directed to improve normal ship types will yield slow progress. It should be different when new solutions which depart from the established routine are sought for, as shown by E. Lewis when investigating effects of lengthening ships.

5. The Froude-Krylov theory is an inadequate tool to reconsider the problem of ship behavior from principle, with the purpose to develop ship types possessing optimum qualities, though its possibilities as a guide for design were not exhausted as shown recently by the discussion on the supercritical range of ship operation.

The present state of knowledge admits, however, the definite hope that the problem of optimization may become reasonable within the near future and may yield new basic results for design, though a damped optimism is the appropriate attitude when

speculating in the field of an old profession such as ours. The fact that rather awkward phenomena have been explained and found by more advanced theories is promising.

The paper by Kent [7] indicates some practical problems which should be investigated. It will be quite a task to prove or to disprove his assertions as to optimum form parameters and characteristics, and to extend the research to similar problems suggested by practice.

6. It is expected that the present freeboard regulations will crumble under the impact of the present research impetus and that ship theory will contribute its share in shaping more reasonable ones. This alone will have a most beneficial effect by increasing the safety of small ships, the freeboard being a most decisive parameter. In the same way, it is hoped that the safety regulations for passenger ships will profit from results of research.

7. By the introduction of the irregular seaway concept reliable information will be obtained on the average performance of ships with respect to motions and resistance.

The development of new instruments for measuring the seaway promises to increase decisively the value of statistical investigations.

The problem of service speed can be brought nearer to a solution, and the possibility of appraising ship behavior in relation to other fundamental properties will be promoted. For example, data will become available which permit one to decide when stabilization is desirable.

Statistics will furnish data on permissible acceleration and contribute to a better understanding of safety limits, though, as mentioned earlier, knowledge from other sources is indispensable to solve this problem.

8. The overwhelming importance of damping has been frequently emphasized. Already the present solution of the problem enables the designer to act on a rational base when assigning proportions and forms, and further development will help us avoid presently unknown pitfalls similar to those indicated by Ursell. There is not too much hope for unexpected favorable solutions.

9. We made the same statement earlier, especially with respect to exciting forces. However, in the case of the roll motion where small changes in the decisive ship parameters can have large consequences, conditions may be different. Reference is made to Grim's finding on the possibility of reducing the exciting roll moment by reasonable form variations in the range of synchronism.

10. The feasibility and efficiency of fins having been established, a wide field is opened for their applications as means of control around all three axes.

11. A better understanding of conditions leading to improved maneuverability in a seaway will furnish design data to reach this goal and to increase thereby the safety of ships.

12. Although roll stabilization is developing successfully, the danger of capsizing, or rather of reaching large angles of heel, still requires further investigations. Proposals in this direction have been made over a considerable time but the work so far invested does not correspond in any way to the importance of the task.

13. The study of hydrodynamic impacts has now reached the degree of intensity which corresponds to its fundamental importance. The continuation of research on slamming and its extension to phenomena in a horizontal plane may decisively influence the design of high-speed vessels; it may lead to considerable progress in designing important details like the forebody, superstructures, rudders, etc.

14. Means to reduce motions will frequently lead to a reduction of resistance and improvement of propulsion conditions. Furthermore, when reflection effects are better understood suitable changes of form for influencing resistance directly may be found. Old dreams of inventors to influence ship wave generation by fins at the bow appear to be substantiated by recent investigations on performance in a seaway.

15. Although possibly a refined determination of extraneous forces (pressures) acting on the ship will not change basically the assumptions underlying the contemporary strength calculations dealing with the ship as a girder the whole problem will be

put on a safer foundation. The importance of the recent model work aimed at the determination of bending moments and shearing force cannot be overestimated from a point of view of practice.

More basic and revolutionary can be the contribution of ship theory when a large increase in ship speed is postulated. It is difficult to predict how strong a pressure will be exerted in this direction by navies, shipping and shipbuilding practice. It is, however, the obligation of science to appraise the possibilities and to prepare the necessary solutions.

16. The future of the development of hydrofoil vessels depends upon their seaworthiness. Much more work must be invested in this field. Beside its immediate significance it may have a considerable stimulating effect on advanced design of displacement ships especially with respect to fin action.

III FUTURE RESEARCH

1. Facilities

When describing a new large wind tunnel Betz denounced some twenty years ago the trend to increase the size of facilities as uninspiring, and asked (without success) for suggestions to stop this race. The development did not bring any relief in aerodynamics, and, generally speaking, in ship research.

After Havelock had disclosed that wave effects are the principal source of damping in heave and pitch the use of small models for seaway research, notable after the war, appeared justified. Minimum size models become especially desirable when arbitrary courses in a seaway are investigated. Unfortunately, the situation is less simple with self-propelled models which obviously present definite advantages.

It is the author's contention that the establishment of large wave tanks is justified provided the number (and stature!) of scientists connected with such a tank work corresponds to the size of these facilities. If only a limited amount of funds is available it is preferable to invest the money in researchers rather than in facilities.

Beside large tanks, however, there will be an increasing need for facilities intended to tackle special problems.

The scientific foundation of methods used in "General Wave Tank Work" will require a large number of special investigations.

From the synopsis presented it seems at first easy to make proposals for the future research by analytical methods, model experiments and full scale investigations. Certainly it is a more pleasant task to speculate about things which should be done by others than to do the work oneself; nonetheless the task presents difficulties.

Roughly speaking, research in our field is proceeding following two lines:

1. Development of ship mechanics.
2. "Rigorous" solution of the boundary problem.

The first way leads to a program of research. Problems presented in such a way can be solved experimentally as well as analytically, whereby more rigorous methods become more and more popular.

We enumerate now a list of concrete tasks which are being tackled or should be attacked; they embrace surface and submerged displacement vessels as well as hydrodynamic craft.

1. The establishment of general equations of motion of displacement ships in a seaway.

- 1.1. Complete equations.

- 1.2. Linearized equations.

As mentioned above this work constitutes a necessary prerequisite for systematic work in our field yielding a comprehensive program of research. Simplified problems which can be treated with success can be derived from this general task.

2. Determination of forces experienced by an oscillating body moving in calm water as substitute for the actual seaway problem, one and several degrees of freedom.

Added masses, damping, coupling terms and resistance as functions of the frequency parameter and the Froude number.

3. Forces experienced by wholly submerged bodies moving arbitrarily in a non-uniform and non-steady field (Cummins' problem).

4. Rectilinear motions of bodies moving at or beneath the corrugated water surface. Pressure distributions. Diffraction effects. Exciting forces.

5. The two-dimensional problem of a ship moving in a regular seaway.

6. Ship moving on a straight course in a regular seaway.

Transverse stability in waves.

Directional stability in waves.

Appraisal of the method mentioned under 2.

7. Theory of resistance augmentation of a ship moving as under 6.

Appraisal of the method mentioned under 2.

8. Propulsion in a regular seaway.

9. Interaction between propeller and ship.

10. Nonlinear problems of roll.

11. Other nonlinear problems.

12. Study of severe transient conditions.

13. Irregular seaway. Correlation of model and ocean waves.

14. Full-scale investigations (tests) on ships. Limiting data for accelerations etc.

15. Reductions of motion by artificial means. Roll and pitch stabilization.

16. The general problem of speed increase in a seaway. Cavitation.

17. Manoeuvrability in a seaway.

18. Development of ship forms with high damping qualities.

The influence of viscosity upon motions.

19. Development of ship forms which experience low exciting forces in a seaway. The investigation and appraisal of nonlinear effects.

20. Development of the theory of impacts following various assumptions.

Horizontal impacts.

21. Development of ship forms under water, above the surface, on deck with respect to impacts. Theory of the behavior of hydrofoil craft in a seaway.

ACKNOWLEDGMENT

It was my aim to indicate the satisfactory scientific progress in our field and to emphasize that application of its methods and results to problems presented by practice can be made nowadays with much more confidence than before. The scientific problems involved are both interesting and fruitful. Notwithstanding the age of our profession we have by no means come to a blank wall, and there is no danger of the status of diminishing return in research work which may decisively influence the future development of the ship.

A wide gap has been left in this report with respect to problems presented by *resistance* and *propulsion* in a seaway due to reasons mentioned in the text which are partly of an occasional, partly of a substantial character. It must be repeated that special efforts are needed to promote solutions of these problems, which recently have been somewhat neglected except by Japanese scientists. With respect to problems presented by the irregularity of the seaway, reference is made to numerous excellent papers published in this country.

It is my opinion that because of the ample scope of the subject and the fast rate of development, it is rather difficult for a single person to cope adequately with the problem. The kind support given by the initiators of this symposium and by colleagues especially from this country encouraged me to present this lecture. It is fortunate that thorough synopsis of the field is being prepared by Professor Korvin-

Kroukovsky which will serve as a source of references, and it is gratifying to state that the year 1957 has brought us many valuable publications part of which can be found in the Proceedings.

My special gratitude is due to the University of California, which gave me the opportunity to complete my report during a short stay at Berkeley as visiting professor.

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W. H. Warnsinck, "Schip en Werf" (1954).

The present list contains only a selection of papers appropriate to the limited scope of this report.

The following abbreviations are used:

DTMB — David W. Taylor Model Basin, Washington 7, D. C.

JAN — Izvestiia, Akademiia Nauk SSSR, otdelenie tekhnicheskikh nauk.

JSTG — Jahrbuch der Schiffbautechnischen Gesellschaft, Hamburg.

PMM — Prikladnaja, Matematika i Mekhanika, Moscow and Leningrad.

QJMAM — Quarterly Journal of Mechanics and Applied Mathematics, Oxford.

SNAME — Transactions of the Society of Naval Architects and Marine Engineers, New York.

TINA — Transactions of the Institution of Naval Architects, London.

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SYMBOLS

A_w	load waterline area
B	beam
C	restoring coefficient in pitch (eq. [7])
D	displacement

E	exciting function, e.g., $E_x = E_{33}$, $E\psi = E_{55}$
F	Froude number
G	coupling coefficient in eq. [7]
H	draft
I	area moment of inertia
J	mass moment of inertia
K	inertia coefficients
K_x	function in eq. [3b]
L	length
M	moment
N	damping coefficient, e.f., $N_x = N_{33}$, $N\psi = N_{55}$
$N(\xi)$	damping coefficient per unit length.
$\bar{N}^{(d)}$	diffractional damping per unit length
$\bar{N}^{(d)}$	diffractional damping coefficient
T	natural period, $T^* = T\sqrt{g/L}$
U	uniform speed of advance;
X Y Z	forces
V	
c	wave phase velocity; restoring coefficient in eq. [6]
f	freeboard
g	gravity acceleration; coupling coefficient in eq. [6]
h	wave height
i	$\sqrt{-1}$
m	mass; m_{ij} — inertia values; $m_x = m_{33}$; $J_{yy} = m_{55}$
r	wave surface elevation; r_m wave amplitude
t	time
x y z	coordinates, $y^* = y/L$
z_m	
z_m	heaving amplitude, $z_m^* = z_m/r_m$
α	waterline area coefficient
β	midship section coefficient
$\gamma = \frac{\pi L}{\lambda} = \frac{\pi}{\lambda^*}$	
δ	block coefficient
ε	phase angle
θ	angle of yaw
κ	dimensionless damping coefficient
λ	wave length, $\lambda^* = \lambda/L$
μ	resonance factor, amplification factor
ν	natural frequency $\nu^* = \nu\sqrt{L/g}$
ρ	density

ϕ	rolling angle
χ	heading angle
ψ	pitching angle; ψ_m = pitching amplitude; $\psi_m^* = \frac{\psi_m}{\Theta_m}$
ω	exciting circular frequency (frequency of encounter)
$\omega_o = \sqrt{\frac{2 \pi g}{\lambda}}$	wave frequency $\omega^* = \omega_o \sqrt{L/g}$
$\sigma = \delta/\alpha$	vertical prismatic coefficient
Θ	wave slope, Θ_m = maximum wave slope
Λ	tuning ratio or factor
Φ	velocity potential

NOTES

1. To denote dimensionless values ample use is made of an asterisk. Where necessary the length of reference is indicated by subscripts, for example,

$$\omega_{*b}^* = \omega \sqrt{B/g}.$$

The subscript L is omitted in expressions like

$$\omega^* = \omega \sqrt{L/g}.$$

2. Two subscript notations are used to designate values pertaining to motions:

2.1 $x, y, z, \Phi, \psi, \Theta$, when a small number of motions is investigated;

2.2 numbers 1, 2, 3, 4, 5, 6 in more general expressions.

3. In the latter case added moments of inertia are denoted by m_{ij} .

Virtual generalized masses are denoted by a prime, e.g.,

$$m'_{33} = m + m_{33}, \quad m'_{44} = J_x + m_{44}, \text{ etc.}$$

4. Two-dimensional mass and damping values are distinguished by a bar, e.g. $\bar{m}_{33}, \bar{N}_{33}$.

DISCUSSION

Paul Kaplan

Since this talk was only a synopsis, and Dr. Weinblum expects the discussion to amplify and expand many of his statements, I will attempt to consider some of the work in hydrofoil craft and its extensions in greater detail.

With regard to the analysis of motions of hydrofoil craft in a seaway, many works of linearized theory exist with the use of quasi-steady forces in their derivation; among them, one due to Dr. Weinblum. Comparison of the theoretical motions with experiment, however, doesn't show good agreement, and presently, two alternate developments, making use of the forces determined from the theory of unsteady motions, are under investigation. Other work is being done on a study of scaling of the performance of hydrofoil craft in waves. This concerns the change in wave sizes that a hydrofoil craft can safely operate in when a change in size of the craft is made, with the speed maintained relatively constant. These studies may contribute to a better knowledge of the seaworthiness of hydrofoil craft, on which their future utilization greatly depends.

Another point about hydrofoil craft that makes them attractive for research

purposes, is that, aside from the fact that they are a particular class of seagoing vehicles of importance, they are relatively simpler to analyze theoretically than a displacement ship. The forces are relatively localized and simple experiments can also be used to determine various coefficients in the motion equation. The techniques of aircraft motion analyses are easily applied to *this* case, and these may now be extended, in the broad sense, to the displacement ship problem.

The field of ship mechanics, as Dr. Weinblum refers to it, should develop along the lines of aircraft studies. The development of aircraft dynamics, from simple stability analyses to response to arbitrary gusts, even those of random nature, should be followed by the natural extension of ship maneuvering and directional stability studies in smooth water to studies of similar problems in waves. The general equations of ship motion can be easily constructed (in a rough, not too rigorous manner) and the evaluation of the pertinent coefficients is then the main problem. One of the techniques that may be used for this purpose is a relatively recent development in the aeronautical literature known as slender-body theory. This theory allows a greater fullness to the cross-sectional area of the body and may be more realistic than the classical Michell ship. Some work along these lines has been done by Dr. Cummins of DTMB for the case of wave resistance in smooth water, and has been indicated in a general way recently by Professor Fay of M.I.T., for the case of symmetric motions of a ship in waves, i.e. heave and pitch in pure head and following seas. The main difference between the aeronautical application of slender-body theory and application to the ship motion problem lies in the boundary condition for the potential, i.e., the free surface boundary condition, but alas, that is the core of the problem. These ideas should be taken up soon, in order to extend our meager knowledge of ship motions in the horizontal plane in oblique waves. More rigorous techniques may be developed in the future, but a working tool is now necessary and may be obtained in this manner.

I may have departed somewhat from my original discussion on hydrofoils, but I think I have maintained my general philosophy of applying the ideas of aircraft motion analysis to the seaworthiness problem.

R. W. L. Gawn

Prof. Weinblum's survey is admirable and comprehensive and his proposals for further research are in my view essential in all respects.

One or two aspects may be usefully emphasized. The simple theory of rolling, now nearly a century old, has pointed the way in association with model experiments to the best arrangement of bilge keels to damp the motion. Much greater damping has been effected in recent years by activated fins. Theory played a full part in the development of this device although the engineering aspects are a major feature.

Increasing attention is being given to the development of the linear theory of pitch and of heave and in association with model experiments this has confirmed that the scope for increasing damping by passive fins or change of hull shape is very much greater than formerly supposed. This is therefore a real contribution from theory.

So far research both by theory and experiment has been largely pre-occupied with regular seas but there is a natural urge to extend the development to complex seas. The additional variables and complications involved are many and the resources of staff and facilities will be overstrained and useful results unduly protracted unless the work is confined to essential channels. It is suggested to the Author that theory could help by defining a wave system of a standard complexity which would serve for most tests and so avoid, or at least considerably curtail, the multiplicity of experiments that would otherwise be necessary in different seas of random complexity.

The important contribution of adequate freeboard to the sea-keeping qualities of a ship cannot be too strongly stressed and it was a pleasure to hear Prof. Weinblum's insistence on this. This is a matter on which theory has drawn a blank and the designer must perforce rely on previous experience and model tests in order to

arrive at a decision as to the amount of freeboard. My personal experience favours $6\frac{1}{2}$ per cent of the length as a good standard, although it is realized that some ocean-going ships have a smaller ratio. Exceptional standards were aimed at for H. M. Yacht BRITANNIA and the freeboard coefficient was finally determined at about 8 per cent following a series of model tests. Perhaps the Author would comment on these figures and say whether he is aware of any theoretical or empirical rule to guide the designer in this vital problem. One empirical approach which has been attempted at Haslar is represented by the diagram herewith, which is self-explanatory. Experience to date indicates that curves as shown on the diagram fairly well differentiate between dry and wet ships. (See Fig. 1.)

P. Golovato

Dr. Weinblum has mentioned recent work at the Taylor Model Basin on the forces and moments on oscillating ship forms. I would like to elaborate on his remarks and summarize the principle results of this continuing investigation.

We at the model Basin are indebted to Dr. Weinblum for the foundations he laid for this work while he was with us several years ago. The very test facility and model we have utilized were originally built for Dr. Weinblum's use.

We have constrained a ship model with fore-and-aft symmetry to oscillate

harmonically in heave in calm water and have measured the heaving forces and pitching moments. This enabled us to determine the frequency, amplitude and speed dependence of one-half of the coefficients in the commonly used pair of linear differential equations of motion. Subsequent pitching experiments later this year will determine the missing coefficients.

The principal results concerning the added mass and damping are as follows:

1. The added mass and damping coefficients are independent of the amplitude of motion.

2. The added mass in heave shows a strong frequency dependence which closely resembles in form the theoretical results of Ursell and Grim.

3. The added mass is essentially independent of forward speed.

4. The variation of damping coefficient with frequency shows the characteristic peak at the frequency predicted by the Distributed Source technique and the Grim solutions but the former greatly overestimates its magnitude while the Grim solutions are an improvement. The Haskind-Roman experiments at zero speed were re-analyzed and the same superiority of the Grim solutions was demonstrated.

5. The speed effects on damping are not very great. The maximum value is about the same for all speeds but the peak occurs at slightly different frequencies.

6. The existence of quadratic damping is indicated by harmonic content in the measured forces. It was then demonstrated that the neglect of these non-linear terms would lead to a serious overestimate of the motions in the vicinity of resonance.

In addition, significant cross-coupling moments were obtained for the symmetrical model which increased with increasing speed.

1. The pitching moment due to heave velocity showed a very strong frequency dependence that is not indicated by the theoretical work of Haskind and Havelock.

2. A moment due to heaving acceleration was obtained which is sharply peaked at low frequencies. This coupling moment is not indicated previously by theory or experiment for a symmetrical model.

E. V. Lewis

I have enjoyed Professor Weinblum's interesting survey on the seaworthiness problem very much, and appreciate the honor of being asked to comment on both it and some of our experimental work at Stevens.

It is of interest to note that Professor Weinblum now rates the importance of the irregular seaway concept very high. But I certainly agree with him that the study of the theory of motions in regular waves is no less important than before. In fact, it is probably even more so, and the many interesting graphs and data that Professor Weinblum has assembled from sources rather difficult to obtain in this country will be of great value to us in this work.

Professor Weinblum records definite progress in the overall problem of ships in a seaway, but he does not expect revolutionary improvements, within practical limits of ship form and proportions. My own feelings are somewhat more optimistic, because besides the possible use of fins, it does seem that the possibility of gains in speed by careful consideration of ship proportions are quite large, even though perhaps not "revolutionary."

For example, within the scope of the Series 60 models that are now being extended by Dr. Todd, there is the likelihood of an appreciable increase of sea speed. In considering this matter, it is convenient to use the condition for synchronism with a wave of ship length as a rough criterion of relative speed attainable in rough irregular head seas. This is a useful reference because although the oceanographers feel that most seas have a very wide range of different frequencies present, motion studies show that wave components appreciably shorter than the ship's length are not serious, even if synchronous conditions exist. So if we consider the speed for synchronism with the shortest wave components that is apt to be significant, i.e. a wave of about the length

of the ship, we have a relative measure of what we might expect to be the upper limit of speed in a rough head sea condition.

On this basis, it is found that a 500-foot ship built to the lines of the parent Series 60, 0.60 block model, might be expected to be in trouble in a particular sea at around 12 knots. Above this speed the motion would be too violent, and the ship would have to slow down. A ship of this same displacement, with the length over breadth increased from $7\frac{1}{2}$ to $8\frac{1}{2}$, and beam over draft increased from $2\frac{1}{2}$ to 3, which is one of the new series, would be a 580-foot ship, and on the basis I have outlined we might expect it to run into storm difficulties at a speed of $17\frac{1}{2}$ knots, instead of 12, on a comparative basis. This means a 15% increase in sea speed with a 15% increase in length.

This advantage of increasing length in irregular head seas has been shown by model tests to be more than theory. In a project at Stevens, sponsored by ONR, a wide variety of different models is now being studied. For example, a trawler model was tested in its original form, and then lengthened out and made more slender. When compared at the same displacement, the greater potentiality for speed for this lengthened ship could be clearly seen.

Unfortunately, length is, as we well know, the most difficult dimension to increase, not only from the point of view of cost, but from practical reasons, such as limitations of harbors and ports. Still, in many cases there may be a possibility of some increase, and whatever can be done will be to our advantage for speed in rough weather.

Finally, I should like to emphasize again Professor Weinblum's point about the need for ship data on permissible limits of motions and accelerations, both for comfort and for safety. Without more information of this sort new theoretical and experimental developments cannot be effectively applied. For example, even though model techniques, irregular-wave theory and new oceanographical data permit us to predict for a given condition the average or maximum amplitudes of motion, accelerations, etc. with some degree of confidence for a new design over a range of speeds, we still are unable to predict what would be the limiting speed for the actual ship.

This and the other problems outlined by Professor Weinblum will keep us all busy for some time to come.

V. G. Szebehely

It is a pleasure to make these few comments on Professor Weinblum's lecture at his request.

In order to save time, only a systematic listing of projects connected with the topics mentioned in the lecture will be given with a very few remarks.

1. In connection with Dr. Weinblum's reference to fins, attention is called to an investigation of the effect of anti-pitching bow fins on the motion of aircraft carriers and merchant vessels. Before the feasibility of applying fins was considered, the effects of bulbous bows on the motion were investigated by theoretical and experimental means. No significant motion reduction was found with normal size bulbs. Fixed fins at the bow have two beneficiary effects; firstly they reduce pitching, secondly they reduce speed losses in waves. Various planforms, profiles, area, combinations, speeds and waves were investigated and it was found that 40-60 percent pitch reduction can be obtained in certain practically important conditions with bow fins whose area was approximately 2 percent of the water plane area [ref. 1, 2].

2. A study of speed reduction in waves is given in ref. 3. The work evaluates the speed maintaining characteristics of a destroyer, of a fast and of a slow cargo ship. From simple theoretical consideration, formulas (which are very sensitive to phase relations) were derived and at the present time are being compared with the experimental findings.

3. Motion predictions were performed in connection with model tests. Phenomena in which phase relations play significant part, seem to be hard to predict. Con-

sequently, there is room for considerable improvement in connection with prediction of slamming, propeller emergence, speed reduction, water over the bow, pitching axis location, etc.

4. The effect of transients in irregular seas was investigated considering a trend of waves of constant length and variable height. The steady state response was compared, cycle by cycle with the measured response in ref. 1.

5. The influence of the block coefficient on the time and space dependence of the impact pressure was investigated and pressure mappings were computed [ref. 4]. The computed values are presently being compared with experimental results. Correlation between slamming pressure and acceleration are given in ref. 5.

6. It is felt that the new slamming model proposed by Dr. Weinblum has merits for so-called flat impacts. The practical importance and the usefulness of the "enclosed cavity" approach will have to be shown before this model can take over the presently established and practically useful methods based on Wagner's idea [ref. 6].

7. Head-on collision type impacts were observed and reported in ref. 7. Longitudinal accelerations amounting to 0.1 to 0.2g were recorded in heavy head seas on destroyers. Such values can be arrived at using Wagner's method to compute forces developed during water entry.

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J. F. Allan

I think it is interesting to note that since the classical analysis and presentation of Krylov, there has been very little real development of the subject until the last ten years or so. In that recent period the interest in the behavior of ships at sea, and the interest in the possibility of analyzing that against the mathematical theory has grown enormously.

I wish to pay a tribute to the enormous amount of work which has been done in this direction in the United States, both at the David Taylor Model Basin, and the Stevens Institute. This work has been presented in a number of papers, and it really places a very useful tool in our hands.

One should mention in passing, the work that Kent did at Teddington in the 1920's and 1930's, the chief point of which might be considered to be the indication of the importance of the relationship between the downward motion of the bow and the actual position of the bow wave. The recent work has shown how the theory explains a number of these characteristics, but, unfortunately, the accuracy of the theory and our knowledge of the characteristics is such that it is not reasonable to come to detailed conclusions on the comparisons. The broad picture is clear, but one cannot pursue it in too much detail.

Probably the most significant indication, from the recent developments, is to show, or to suggest that the only really effective way of improving the seagoing quality of ships is to increase the damping factors.

Previous speakers have suggested that varying the proportions of the ship can achieve an improvement, but from the practical point of view the possibility of pursuing that line is distinctly limited.

As regards rolling, the natural damping factor is small, and while it would be possible to increase this factor greatly by giving the ship, for example, large bilge protuberances, from the practical point of view this has never been pursued, because it would interfere both with the propulsive characteristics and the handling of the ship.

The rolling problem has been substantially solved by the introduction in Great Britain of the controlled fin stabilizer, which increases the damping factor to such an extent that rolling is substantially eliminated.

Having successfully controlled rolling, one's ideas naturally turn to the question of pitch and heave reduction. It is an open question whether the shape of the ship can be so modified to increase the damping factor in pitch and heave, without paying a large penalty from the propulsion point of view.

If one applies controlled damping, in the form of controlled fins or other means, the question of the time element is of quite considerable significance. In other words, one must operate the control and produce the variation of the force on a time plan which matches the pitch frequency. As you are aware, this frequency is about twice that of roll and the problem is correspondingly more difficult.

A question, perhaps more properly discussed tomorrow morning, is that of driving ships very hard in bad weather conditions, which is of considerable interest in view of the possibility of largely increased driving power. If one equips a ship with a controlled mechanism which stops rolling and damps pitching, it is quite probable that one will be able to drive it successfully through very bad seas.

But there is a very serious practical point which arises, and that is, what happens if the control mechanism fails? The ship might meet catastrophe before the position could be rectified.

It may be of interest to note that the "Queen Elizabeth," which is equipped with roll stabilization, can be driven at full speed in conditions which would require a reduced speed if there was no stabilizer, but the captain is worried as to what would happen if the machine suddenly ceased to function in such circumstances. Reliability is of paramount importance.

M. C. Eames

Regarding Dr. Kaplan's amplification of Dr. Weinblum's reference to the study of the seaworthiness of hydrofoil craft, it is felt that one point should be clarified. Dr. Kaplan has quite rightly pointed out that a linearized quasi-steady theory of the pitching and heaving response of hydrofoil has proven inadequate for predicting the motions of such craft in regular seas.

The experimental work on which this conclusion is based, however, covered a necessarily limited range of wave conditions and craft speed. There is reason to believe that once the craft is well clear of conditions of resonant response, the linearized quasi-steady theory proposed by Dr. Weinblum suffices to give a very reasonable approximation to the true motion.

This in no way detracts from the remarks of Dr. Kaplan and his very fine work on the effects of unsteady motions. The most important case does, of course, correspond with resonant conditions and the correct prediction of the motions resulting from such conditions is vital to the safe performance of hydrofoil craft. Unless the writer misunderstood the speaker, however, there was a hint that the more elementary approach should be discarded altogether, and it is felt that insufficient experimental evidence has been obtained to date to justify this.

Possibly this represents the wishful thinking of a simple-minded hydrofoil craft designer, but the feeling nevertheless remains.

In this valuable paper, Dr. Weinblum has presented a very comprehensive review of the entire subject of the analysis of ship motions. It is not feasible to present a discussion of such a broad field; a discussor must single out a few aspects in his particular sphere of interest. This discussor appreciates the clear definition of two basic directions of ship theory development as given by the author: a. to state the practical conditions as completely as possible and to look for an approximate solution (the vigorous approach), and b. to simplify to the utmost the problem presented by practice (i.e., substitute a simple mechanical model), and to apply rigor in the mathematical solution.

The family tree of the first method is easily traced from the initial work of Kriloff (1896) through the work of Weinblum and St. Denis (1950) and St. Denis (1951) to Korvin-Kroukovsky (1955). During the past year it has received important additional development (as yet unpublished) by Prof. M. Abkowitz and Prof. Fay of M.I.T. and by Miss W. R. Jacobs (this discussor's associate). Valuable contributions for the elements of this theory appeared also in the more mathematical works of Havelock (1955, 1956). Were all these developments taken into account, the general tone of the author's outline would appear to be too reserved. The successful comparison of calculations with experimental data for several models in several wave lengths could well call for more enthusiasm. Unfortunately, a certain amount of time will have to elapse before these latest developments can be published.

The author points correctly to the fact that calculation of the exciting forces for a submerged body are simpler and appeared earlier than those for a surface ship. This discussor cannot agree with the author, however, that the calculation of the latter was made by "analogy". The technique was first developed for a submerged body, and then was applied to a surface ship, but this latter application was made independently without resort to analogy. The striving for simplicity required that the basic solution be developed for a semi-circular ship section; the results were subsequently generalized to apply to any form of cross section by introducing the coefficient of accession to inertia in view of the basic work of G. I. Taylor (1928) as well as Havelock's (1954) work on submerged spheroids. The word "analogy" is therefore partly applicable only to this final step. The computed exciting forces for a ship model were found to agree very well with the experimental data.

The cross-coupling damping terms which appear in the work of Haskind and Havelock (1955) deserve additional comments. It should be observed that these "damping" terms result from the calculations based on the inviscid fluid, and in Havelock's case, without wave formation. They do not involve, therefore, any energy dissipation, and can be termed "dynamic" damping, in distinction to the wavemaking damping defined by the works of Holstein, Havelock, Ursell and Grim. This wavemaking damping must be added as a separate step to any calculation in which the surface wave formation is not included in the initial setup of the problem, as, for instance, that of Korvin-Kroukovsky (1955).

"Dynamic" damping terms of the general form

$$Ak_2mV\dot{z} \quad \text{and} \quad Bk_2mV\dot{\theta} \quad (31)$$

appear in the work of Korvin-Kroukovsky (1955), as well as in the yet unpublished work of Prof. Abkowitz and Prof. Fay. It is gratifying to see that with regard to these terms the simplified work of these authors is in agreement with the more rigorous mathematical work of Haskind and Havelock. While there is an agreement in regard to the form, the coefficients A and B have different values with each author, and apparently depend on the assumptions made as to the nature of the free boundary and of the ship surface. In particular, comparison of the work of Korvin-Kroukovsky (1955) and of Fay (not yet published) brings out the difference between the assumptions of cylindrical or conical form for an element of ship length.

In connection with the significance of the cross coupling, the work of Prof. Fay brings out the fact that heaving motion is strongly affected by pitching, while pitching is relatively little affected by heaving. This explains why in the publications in which the pitching only is considered, as Cartwright and Rydill (1956) for example, the effect of cross coupling does not appear to be pronounced. This effect becomes immediately evident when the heaving is considered, and in particular it leads to a large increase of heaving motions. This is important in practice because large heaving motion of a ship leads to the occurrence of slamming.

G. P. Weinblum

As expected, the discussers have succeeded in supplementing my sketchy exposé in such a way that a more adequate representation of the ample subject has been reached. May I express my sincere thanks for their contributions. Obviously, it was a hard task for them to prepare their remarks without knowledge of my manuscript which I was able to make available in a very rudimentary state only to Prof. Korvin and Prof. Lewis. As mentioned at the beginning of the lecture, my wish to dwell at some length on foreign work less known to the audience and to assume other basic publications as familiar, has distorted the oral representation of the subject.

Dr. P. Kaplan is to be congratulated on behalf of his theory on non-steady hydrofoil performance. The present writer had earlier underestimated these effects. It is interesting to note, however, that Mr. Eames is supporting the rudimentary quasi-steady approach. I concur with him that it should not be wholly discarded for the time being. In fact, work is going on following these lines under the present writer's guidance dealing with the general case of hydrofoil motions on arbitrary courses in regular waves.

Mr. Golovato's contributions have clarified important problems and promise to promote knowledge in our field in the future. His success is especially gratifying to me after I was afraid to lose my reputation completely because of my responsibility for suggesting the oscillator used by him. There appears to exist a slight contradiction between point 1. and 6. of the discussion with respect to nonlinear effects.

Thanks to Dr. Szebehely for his competent remarks on a subject which he has successfully developed for several years. With respect to point 6. I wish to emphasize that in my opinion Bagnold's ideas on hydrodynamic impacts may lead to a better understanding of some pertinent effects but by no means can take over Wagner's theory.

In his oral reply Dr. Gawn has justly pointed out an understatement with respect to Froude's work on roll. I agree wholeheartedly with Dr. Gawn that theory has drawn a blank by neglecting severely the freeboard problem. The condensed experience contained in the figure presented by him should be used when freeboard regulation will be reconsidered. Theory may contribute useful results to the determination of freeboard, sheer and size of superstructure especially as a guide to experimental work, provided extensive numerical evaluations will have been performed. The lack of the latter precludes at present to discuss the figures communicated by Dr. Gawn.

Dr. Allan refers to Kent's work to whom the profession should feel deeply indebted as well as to Kempf. Dr. Allan's remarks on damping and especially on controlled damping are highly appreciated as those of a pioneer in the latter field. The postulate "reliability is of paramount importance" is to the point especially at a meeting of the present character. Some doubts expressed as to the effectiveness of varying proportions lead to the discussion by Professor E. Lewis, the protagonist of the problem at stake.

To avoid any apparent disagreement with him, I have somewhat mitigated in my final text the slightly nebulous statement concerning revolutionary improvements within normal limits of ship form and proportions. It has been my aim for many years to emphasize an almost trivial fact—the importance of the ratio $\lambda^* = \lambda/L$

when dealing with most motions. Further, it has been pointed out, that synchronism loses its danger when occurring at low values of λ^* . Due acknowledgement to Prof. E. Lewis' achievements has been paid in the text and some hints have been given as to how to extend the pertinent analysis. A numerical investigation of conditions on arbitrary courses appears to be urgent especially in the case of supercritical running.

My earlier reserve with respect to the solution of the irregular seaway referred to the formal representation only. Some doubts still persist as to the final character of the Pierson-Neumann approach.

Professor Korvin-Kroukovsky points out quite justly that the word "analogy" is applicable to one part only of his research on the behavior of surface vessels—the investigation on exciting forces. His investigation on coupling terms is farther reaching. It is to be expected that further work in this direction will go beyond Haskind's bold attempts which by no means have so far led to final conclusions.

V

WAVE RESISTANCE OF THIN SHIPS

John V. Wehausen

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The theory of the interaction between a ship or similar body and the undulating sea through which it moves has been developed by various approximations. In particular, in developing the theory of ship waves and wave resistance one neglects viscosity in the general analytical formulation, later trying to take account of it in a patchwork fashion. Even with this simplification, mathematical progress has been made only in cases where one may linearize the boundary conditions on the water surface. The resulting theory is valid only when the disturbance of the surface is small. In situations where one is dealing with waves generated by a moving body, this means that some aspect of the body or its motion must be restricted in a suitable way. A dimensional analysis will usually show one or more parameters whose vanishing leads automatically to vanishing of the surface disturbance. In the case of steadily moving ships and related submerged bodies, the restrictions inherent in the problems treated analytically seem to fall into one of the following classes: 1. "thin" ships, approximating a vertical disc; 2. "flat" ships, approximating a horizontal disc on the free surface; 3. bodies submerged so deeply that their influence on the surface is small; 4. submerged "thin" wings of small angle of attack (in this case approach to the surface leads to a further complication). In the case of oscillating bodies some statement about smallness of amplitude is necessary.

The theory of perturbations has been extended in recent years to surface-wave problems by F. John [1], Stoker and Peters [1], Sekerzh-Zenkovich [1] and others. It was our original intention to make the appropriate expansions in each of the cases listed above, as a means of formulating the proper linearized problem as well as approximations of higher order, and then to survey the existing status of theory and experiment in each case. In the present paper, however, only the first part of this program has been attempted, the rest being left for a later time. Even so, much of the material has been gone over lightly because of the existence of several available surveys. We mention specifically papers by Hogner [2], Wigley [1, 2, 3], Weinblum [10], Havelock [10, 11], and particularly the comprehensive surveys by Lunde [1, 2]. Inui [1] has recently outlined the achievements of Japanese scientists in this field. It seems appropriate to remark that since Michell's original contribution [1], the development of the "thin"-ship theory has been carried on until recent times almost exclusively by T. H. Havelock and L. Sretenskii, the bridging of theory with experiment and design almost exclusively by W. C. S. Wigley and G. Weinblum.

I. LINEARIZATION BY PERTURBATION

Let us suppose that the y -axis is directed vertically upwards, the x -axis to the right and the z -axis coming out of the plane of the paper. We assume that the fluid is bounded partly by a free surface and possibly also by solid boundaries. When the fluid is at rest, the plane $y = 0$ will be taken to coincide with the free surface. Let the equation of the free surface be represented by $y = \eta(x, z, t)$ and the surfaces of

solid boundaries by suitable equations. We assume that a velocity potential $\varphi(x, y, z, t)$ exists; here $\vec{v} = \text{grad } \varphi$. Then φ must satisfy certain well-known conditions:

$$\Delta\varphi = 0 \text{ in the region occupied by fluid} \quad (1)$$

$$\eta_x(x, z, t)\varphi_x(x, \eta(x, z, t), z, t) - \varphi_y + \eta_z\varphi_z + \eta_t = 0. \quad (2)$$

$$g\eta(x, z, t) + \varphi_t(x, \eta(x, z, t), z, t) + \frac{1}{2}(\text{grad } \varphi)^2 = 0. \quad (3)$$

If the equation of a solid surface is given by $F(x, y, z, t) = 0$, then on such a surface

$$F_x\varphi_x + F_y\varphi_y + F_z\varphi_z + F_t = 0, \quad (4)$$

i.e. The normal velocity of the fluid equals the normal velocity of the body. If the body is fixed, $F_t = 0$ and $\partial\varphi/\partial n = 0$.

The conditions at infinity are somewhat more troublesome; they should at the same time be physically reasonable and guarantee a unique solution. There are, however, few uniqueness theorems for the general problem. If the body starts from rest in still water, one can assume that

$$\lim_{x^2+z^2 \rightarrow \infty} \eta(x, z, t) = 0. \quad (5)$$

If one wishes to consider a steady-state problem, with, say, $\eta = \eta(x-ct, z)$, or even a more general motion where the body has moved in one direction for a long time, one requires that the motion vanish far ahead of the body:

$$\lim_{x \rightarrow \infty} \eta(x, z, t) = 0. \quad (6)$$

In the case of a steadily oscillating body a "radiation" condition is usually imposed at infinity, but after linearization; this is not, however, always sufficient for a unique solution. At infinite depth one imposes:

$$\lim_{y \rightarrow -\infty} \text{grad } \varphi = 0. \quad (7)$$

Let us now suppose that some dimensionless parameter ϵ may be associated with the motion in such a way that when $\epsilon \rightarrow 0$ the motion approaches a state of rest. Some examples of the choice of ϵ will be given later. We now assume that φ and other functions entering into the description of the motion may be expanded in power series in ϵ , where φ itself starts with the first power. For the present we shall be concerned only with φ and η :

$$\varphi(x, y, z, t) = \epsilon\varphi_1(x, y, z, t) + \epsilon^2\varphi_2(x, y, z, t) + \dots,$$

$$\eta(x, z, t) = \eta_0(x, z, t) + \epsilon\eta_1(x, z, t) + \epsilon^2\eta_2(x, z, t) + \dots$$

If these expansions are substituted in conditions (1) — (3) above and coefficients of the same power of ϵ grouped together and equated separately to zero, one obtains conditions to be satisfied by each of the φ_i and η_i . One finds immediately: $\eta_0 \equiv 0$. Then from the coefficients of ϵ one gets the usual equations for a potential function satisfying the linearized free-surface boundary condition:

$$\Delta\varphi_1 = 0 \text{ for } y \leq 0. \quad (8)$$

$$- \varphi_{1y}(x, 0, z, t) + \eta_{1t}(x, z, t) = 0. \quad (9)$$

$$g\eta_1(x, z, t) + \varphi_{1t}(x, 0, z, t) = 0. \quad (10)$$

The last two conditions combine in the usual way to give:

$$g\varphi_{1y}(x, 0, z, t) + \varphi_{1tt}(x, 0, z, t) = 0. \quad (11)$$

One notes that it is a result of the method that the boundary condition on φ_i is imposed on the plane $y = 0$ and not on the actual free surface.

The corresponding equations for the coefficients of ϵ^2 are still not too unwieldy to write down:

$$\Delta\varphi_2 = 0. \quad (12)$$

$$\eta_{1x}(x, z, t)\varphi_{1x}(x, 0, z, t) + \eta_{1z}\varphi_{1z} - \eta_1\varphi_{1yy} - \varphi_{2y} + \eta_{2t} = 0. \quad (13)$$

$$g\eta_2(x, z, t) + \varphi_{2t}(x, 0, z, t) + \eta_1\varphi_{1ty} + \frac{1}{2}(\text{grad } \varphi_1)^2 = 0 \quad (14)$$

After eliminating η_1 and η_2 from the last two equations and performing some manipulations one obtains a counterpart to the condition for φ_1 on $y = 0$:

$$g\varphi_{2y}(x, 0, z, t) + \varphi_{2tt} = - \frac{\partial}{\partial t} (\text{grad } \varphi_1)^2 + \varphi_{1t} (\varphi_{1yy} + g^{-1}\varphi_{1ty}). \quad (15)$$

Conditions for φ_i , $i > 2$, will not be given explicitly. However, inspection of the procedure shows that

$$\Delta\varphi_i = 0, \quad (16)$$

$$g\eta_i(x, z, t) + \varphi_{it}(x, 0, z, t) = A_i\{\varphi_1(x, 0, z, t), \dots, \varphi_{i-1}(x, 0, z, t)\}, \quad (17)$$

$$g\varphi_{iy}(x, 0, z, t) + \varphi_{itt}(x, 0, z, t) = B_i\{\varphi_1(x, 0, z, t), \dots, \varphi_{i-1}(x, 0, z, t)\}, \quad (18)$$

where A_i and B_i are certain functionals of the functions in braces. It may be noted that after the $\varphi_1, \dots, \varphi_{i-1}$ have been found, the equations determining φ_i are the same mathematically as those for the linearized problem of wave motion when a prescribed surface pressure $p_o(x, z, t) = \epsilon p_{o1} + \epsilon^2 p_{o2} + \dots$ is given. In this case the boundary condition at $y = 0$ is $\varphi_{1tt} + g\varphi_{1y} + g^{-1}p_{o1t} = 0$, and η_1 is given by $g\eta_1 = -\varphi_{1t}(x, 0, z, t) - g^{-1}p_{o1}(x, z, t)$. For the case of steady motion, this problem has been investigated by Hogner [1] and others. However, this observation is not of great practical value since the expressions for A_i and B_i become rapidly more complicated as i goes beyond 2.

If this method is used to obtain the form of plane progressive periodic waves with $\epsilon = a/\lambda$ ($=$ the ratio of amplitude to wave length) and with constant velocity $c = c_o + \epsilon c_1 + \epsilon^2 c_2 + \dots$, one finds

$$\eta = A[\epsilon \cos kx + \frac{1}{2}A\epsilon^2 \cos 2kx + \dots], \quad k = 2\pi/\lambda, \quad (19)$$

where A can be determined so that the amplitude is a . One also finds, as expected, $c_o = (g\lambda/2\pi)^{1/2}$ and $c_1 = 0$. A further step in the approximation would show $c_2 \neq 0$.

II. RECTILINEAR MOTION OF A THIN SHIP

Thin Ships. Let us now suppose that the fluid contains a ship which can be described in a coordinate system fixed in the ship by the equation $z = \pm \zeta(x, y)$. We shall assume that the ship is moving in the positive x -direction with velocity $c(t)$. The condition (4) of part I becomes

$$\zeta_x(x - \int c(t)dt, y)\varphi_x(x, y, \zeta(x - \int c(t)dt, y), t) + \zeta_y\varphi_y - \varphi_z - c\zeta_x = 0. \quad (20)$$

We must now select a parameter ϵ . Let $2b$ be the beam and l the length of the ship. Take $\epsilon = b/l$ and write $\zeta(x, y)$ in the form $\zeta = \epsilon \zeta_1(x, y)$. the function $\zeta_1(x, y)$ (with l for both length and beam!) is fixed and we consider the family of forms $\epsilon \zeta_1(x, y)$. Clearly, as $\epsilon \rightarrow 0$, one approaches the problem of a flat plate of the form of the center-plane section moving lengthwise through the water. This will cause no disturbance of the water, so that the choice of ϵ fulfills the requirements of part I. This choice of ϵ also defines a "thin" ship, sometimes known as a Michell ship, namely, one which can be represented in the form $\zeta = \epsilon \zeta_1(x, y)$ with small values of ϵ .

Equation (4) may now be expanded in powers of ϵ , just as for (1), (2), and

(3), and the coefficients of the various powers of ϵ equated to zero. From the coefficient of ϵ one gets the linearized boundary condition on the ship:

$$\varphi_{1z}(x, y, 0^+, t) = -c\zeta_{1x}(x - \int^t c dt, y). \quad (21)$$

From the coefficient of ϵ^2 one gets

$$\varphi_{2z}(x, y, 0^+, t) = \zeta_{1x}(x - \int^t c dt, y)\varphi_{1x}(x, y, 0^+, t) + \zeta_{1y}\varphi_{1y} - \zeta_{1z}\varphi_{1xz} \quad (22)$$

In the part of the plane $z = 0$ outside of the centerplane section of the ship both φ_{1x} and φ_{2z} must be zero, for we have assumed a symmetric ship and φ_z must be zero there. The boundary condition for φ_i , $i > 2$, will be of the form

$$\varphi_{iz}(x, y, 0^+, t) = C_i\{\zeta_{1, \varphi_1}, \dots, \varphi_{i-1}\}, \quad (23)$$

where C_i represents some functional of the functions in braces.

Several problems immediately occur to one, and, it seems to the author should be considered if the procedure outlined above is to be regarded as more than formal. The first concerns the class of functions to which one must restrict ζ_1 . Can, for example, infinite slopes be allowed at the bow and stern? Although it does not interfere with the formal procedure for linearization, it would seem to conflict with the requirement of small surface disturbance if the infinite slope is near the water-line; on the other hand, exclusion of infinite slopes below the waterline would exclude treatment of bulbous bows. Such questions are presumably connected with the convergence of the perturbation series. As far as the author is aware, the only cases where convergence has been proved do not involve solid boundaries penetrating the surface. A further question concerns specification of the proper region of the (x, y) -plane in which condition (23) is to be satisfied. The simplest thing to do would be to take the part of the centerplane section below the undisturbed water plane. However, this would neglect the effect of the part of the hull above the waterplane on the wave-making, and this must, of course, eventually be taken into account. If the region for φ_i is to depend upon the wave profile determined from φ_{i-1} in the preceding step, as seems reasonable, then (23) shows that one must first be prepared to extend one's knowledge of φ_k , $k < i$, beyond the regions in which they are originally defined. This problem does not occur, of course, for a completely submerged "thin" body, for it arises from the meeting of two boundaries with different boundary conditions.

A point worth noting is that it results from the procedure of linearization that the boundary conditions for φ_1 , insofar as they concern the ship hull, are imposed on the centerplane of the ship. Indeed, the same remark applies to φ_i for $i > 1$. Thus recent attempts to improve the linearized theory by altering the boundary conditions on the centerplane or by adding additional source distributions off the centerplane do not seem to the author to be well founded as an attempt to find a better approximation to the exact solution. Such attempts usually have as goal to satisfy the exact boundary conditions on the ship hull and the linearized boundary conditions on the free surface. But this is contrary to the physical assumptions inherent in the use of the linearized free-boundary conditions. The perturbation procedure described above does give, subject to the noted difficulties, a systematic procedure for improving simultaneously and step by step the accuracy with which the boundary conditions on both the hull and the free surface are satisfied. On the other hand, one should remain aware that even if one finds φ_2 (or further φ_i) in the preceding formulation, one is still dealing with a "thin" ship and the method is not likely to be suitable for a form which varies radically from a thin ship. For a form such as a barge a different method of linearization would be used, i.e., a different choice of ϵ would be made. Here one should consult remarks of Stoker and Peters [1, pp. 14, 15].

The force on the ship may now be computed by integrating the pressure over the wetted hull:

$$\begin{aligned} R &= \iint_{\text{hull}} p \cos(n, x) d\sigma = 2 \iint_S p(x, y, \zeta(x, y)) \zeta_x(x, y) dx dy \\ &= -2\rho \iint_S [\varphi_t(x, y, \zeta(x, y)) + \frac{1}{2}(\text{grad } \varphi)^2 + gy] \zeta_x(x, y) dx dy \end{aligned} \quad (24)$$

from Bernoulli's equation, where S is the projection of the wetted hull on the centerplane. We now substitute for φ and ζ their expansions in power of ε :

$$\begin{aligned} R &= -2\rho \iint_S [\varepsilon \varphi_{1t}(x, y, 0, t) + \varepsilon^2 \zeta_{1t} \varphi_{1tz} + \dots + \varepsilon^2 \varphi_{2t} + \dots + \frac{1}{2} \varepsilon^2 (\text{grad } \varphi_1)^2 \\ &\quad + \dots + gy] \varepsilon \zeta_{1x} dx dy \\ &= -2\rho \varepsilon \iint_S gy \zeta_{1x} dx dy - 2\rho \varepsilon^2 \iint_S \varphi_{1t} \zeta_{1x} dx dy - 2\rho \varepsilon^3 \iint_S [\zeta_{1t} \varphi_{1tz} + \varphi_{2t} \\ &\quad + \frac{1}{2} (\text{grad } \varphi_1)^2] \zeta_{1x} dx dy \\ &\quad + \dots \end{aligned} \quad (25)$$

If one is going to approximate R by computing only as far as φ_i , then one may carry out the integrals only over S_{i-1} , the portion of the centerplane section below $y = \eta_0 + \varepsilon \eta_1 + \dots + \varepsilon^{i-1} \eta_{i-1}$. Thus, in the first approximation one need integrate only over S_0 , i.e. the part of the centerplane section below $y = 0$. In this case, the first integral representing the hydrostatic force vanishes. Since this term gives the impression of being the most important, it should be noted that it really starts to contribute to R only with ε^3 , for by the mean-value theorem

$$\begin{aligned} \iint_S y \zeta_x dx dy &= \int dx \int_0^\eta dy y \zeta_x = \int dx \zeta_x(x, y_0) \int_0^\eta dy y = \frac{1}{2} \int dx \eta^2 \zeta_x(x, y_0) \\ &= \frac{1}{2} \varepsilon^3 \int dx \eta_1^2 \zeta_{1x}(x, y_0(x)) + \dots \end{aligned} \quad (26)$$

In the important case of linearized boundary conditions the resistance R may be written:

$$R = -2\rho \iint_S \varphi_t(x, 0, y, t) \zeta_x(x - \int^t c dt, y) dx dy. \quad (27)$$

Before passing on to special cases, we observe that the method for finding the linearized boundary conditions for a thin ship generalizes to the case when several thin ships are moving through the water. In case the ships are moving on parallel paths, the linearized boundary condition to be satisfied by φ on each centerplane section $S_0^{(i)}$ will be:

$$\varphi_z(x, y, z_i^\pm, t) = \mp c_i \zeta_x^{(i)}(x - \int^t c_i dt, y), \quad (x, y) \text{ in } S_0^{(i)}. \quad (28)$$

Since we shall generally be dealing with linearized problems, it will be convenient hereafter to drop the subscript 1, as we have done just above.

Potential and Resistance for Steady Motion. If a thin ship moves with constant velocity in the x -direction, then $\varphi(x, y, z, t) = \bar{\varphi}(x - ct, y, z) = \bar{\varphi}(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x}, \bar{y}, \bar{z}$ are coordinates moving with the ship. Then $\varphi_t = -c \bar{\varphi}_{\bar{x}}$, $\varphi_{tt} = c^2 \bar{\varphi}_{\bar{x}\bar{x}}$, and the linearized boundary conditions for $\bar{\varphi}$ become, after dropping the bars over the letters:

$$(1) \quad g\varphi_y(x, 0, z) + c^2 \varphi_{xz}(x, 0, z) = 0, \quad (29)$$

$$(2) \quad \varphi_z(x, y, 0^\pm) = \pm c \zeta_x(x, y) \quad \text{for } (x, y) \text{ on } S_0, \quad (30)$$

$$(3) \quad \lim_{x \rightarrow \infty} \varphi_x(x, 0, z) = 0, \quad (31)$$

$$(4) \quad \lim_{y \rightarrow -\infty} \text{grad } \varphi = 0. \quad (32)$$

One can clearly satisfy (2) by a distribution of sources over S_0 with density $-c\zeta_x/2\pi$. However, (1), (3), and (4) must also be satisfied. This can be done if one can find a function $H(x, y, z; \xi, \eta, \zeta) = r^{-1} + h$, $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, where H satisfies (1), (3), and (4) and h is harmonic for $y < 0$. This function is occasionally called a Havelock source of strength -1 . A distribution of Havelock sources over S_0 with density $-c\zeta_x/2\pi$ will satisfy all conditions.

The determination of H may be made in several ways. Timman and Vossers [1, 2] use double Fourier transforms, and indeed most methods seem to use them in some way. Perhaps the greatest difference in methods comes from the way in which (3) is satisfied. Havelock [6] and Sretenskii [3] introduce a "fictitious viscosity", a device originally used by Rayleigh [v. Lamb, *Hydrodynamics*, p 399]. Although Lamb's statement [loc. cit.], "This law of friction does not profess to be altogether a natural one: but it serves to represent in a rough way the effect of dissipative forces", is misleading in that it seems to imply that the fictitious viscosity has some connection with a physical force, its use does require one to interpret certain improper integrals properly, so as to obtain the desired solution. T. Y. Wu and C. R. DePrima in recent unpublished work have shown why this gives the correct solution, and indeed give a physical interpretation to the fictitious viscosity. Timman and Vossers [loc. cit.] and Kochin [1] confront (3) directly. Havelock in several papers [e.g., 11] has derived the function H by letting the source start from rest at time $t = 0$ and move with constant velocity; he then finds the limit as $t \rightarrow \infty$. This method has the virtue that (3) is automatically satisfied. In any case, one obtains finally the result:

$$H(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} - \frac{1}{r'} - 4\nu \int_0^{\pi/2} \frac{e^{\nu(y+\eta) \sec^2 \theta} \sin [\nu(x - \xi) \sec \theta]}{\cos [\nu(z - \zeta) \sin \theta \sec^2 \theta] \sec^2 \theta} d\theta \quad (33)$$

$$- \frac{4\nu}{\pi} \int_0^{\pi/2} d\theta \int_0^\infty \frac{e^{\mu(y+\eta)} \cos [\mu(x - \xi) \cos \theta] \cos [\mu(z - \zeta) \sin \theta]}{\mu \cos^2 \theta - \nu} d\mu,$$

where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2, \quad r'^2 = (x - \xi)^2 + (y + \eta)^2 + (z - \zeta)^2, \quad \nu = g/c^2, \quad (34)$$

and the principal value is to be taken in the second integral. The velocity potential is then given by:

$$\varphi(x, y, z) = \iint_{S_0} \frac{c}{2\pi} \zeta_x(\xi, \eta) H(x, y, z; \xi, \eta, 0) d\xi d\eta. \quad (35)$$

In substituting in the formula for the wave resistance, with $\varphi_t = -c\varphi_x$, only the first integral in the expression for H leads to a nonzero term. It gives the well-known Michell's integral:

$$R = \frac{4g^2\rho}{\pi c^2} \iint_{S_0} dx dy \iint_{S_0} d\xi d\eta \zeta_x(x, y) \zeta_x(\xi, \eta) \int_0^{\pi/2} d\theta \sec^3 \theta e^{\nu(y+\eta) \sec^2 \theta} \cos [\nu(x - \xi) \sec \theta]. \quad (36)$$

For convenience let us denote the last integral with respect to θ by $K(\nu(x-\xi), \nu(y+\eta))$. The function K occurs in the literature in various forms. For example, letting $\lambda = \sec \theta$, one gets

$$K(x, y) = \int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} e^{\lambda^2 y} \cos \lambda x. \quad (37)$$

If one lets $\mu = \nu\lambda$, one gets

$$K(\nu x, \nu y) = \frac{1}{\nu^2} \int_{\nu}^{\infty} d\mu \frac{\mu^2}{\sqrt{\mu^2 - \nu^2}} e^{\mu^2 y / \nu^2} \cos \mu x \quad (38)$$

Both forms occur frequently. Birkhoff and Kotik [1, 2] introduce new variables: $u = x - \xi$, $v = y + \eta$, resulting in

$$R = \frac{4g^2\rho}{\pi c^2} \int_{-L}^L du \int_{-2H}^0 dv M(u, v) K(\nu u, \nu v), \quad (39)$$

where

$$M(u, v) = \int_{\max(-\frac{1}{2}L, -\frac{1}{2}L-u)}^{\min(\frac{1}{2}L, \frac{1}{2}L-u)} dx \int_{\max(-H, v)}^{\min(H+v, 0)} dy \zeta_x(u+x, v-y) \zeta_x(x, y); \quad (40)$$

here the hull extends from $-\frac{1}{2}L$ to $\frac{1}{2}L$ and the draft is H . The obvious advantage of this form is that the data concerning the hull are concentrated in the one function M which Birkhoff and Kotik call the "hull function". Finally, by changing the order of integration and using simple properties of the exponential and cosine functions, one may obtain

$$R = \frac{4g^2\rho}{\pi c^2} \int_1^{\infty} d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P^2 + Q^2], \quad (41)$$

where

$$P = \iint_S dx dy \zeta_x(x, y) e^{\nu\lambda^2 y} \cos \nu\lambda x, \quad Q = \iint_S dx dy \zeta_x(x, y) e^{\nu\lambda^2 y} \sin \nu\lambda x. \quad (42)$$

If $|\int dx \zeta_x(x, y)| \leq A$, a constant, it is easy to show that the integral for R is absolutely convergent (one gets $R \leq 4\pi^{-1} \rho L^2 A^2 c^2$), so that the change of order of integration is allowable [cf. Birkhoff and Kotik, 2, pp. 6-7, 19-21].

If the function $\zeta(x, y)$ is of the simple form $\zeta(x, y) = X(x)Y(y)$, where, say, $Y(0) = 1$ and thus $X(0) = b =$ half the beam, the last expression for R takes on a somewhat simpler expression insofar as one has to deal only with single integrals in the expressions for P and Q :

$$P = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx X'(x) \cos \nu\lambda x \cdot \int_{-H}^0 dy Y(y) e^{\nu\lambda^2 y}, \quad (43)$$

and similarly for Q with sine replacing cosine. Ships which may be so represented are called "elementary ships" by Weinblum [10, p. 83] and have been treated extensively by him, Havelock, Wigley and others. In the special case when the ship is wall-sided, $Y(y) = 1$, $-H \leq y \leq 0$, and the second integral in P and Q becomes

$$(1 - \exp - \nu\lambda^2 H) / \nu\lambda^2.$$

If H is allowed to become infinite, one obtains the wave resistance for an infinitely long vertical strut:

$$R = \frac{4\rho c^2}{\pi} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} d\xi X'(x) X'(\xi) K(\nu(x - \xi)), \quad (44)$$

where

$$K(x) = \int_1^{\infty} d\lambda \frac{\cos \lambda x}{\lambda^2 \sqrt{\lambda^2 - 1}} \quad (45)$$

The function $K''(x) = \frac{1}{2}\pi Y_0(x)$, where $Y_0(x)$ is Weber's Bessel function of the second kind. Indeed, if in the expression for R , one integrates by parts, once with respect to x and once with respect to ξ , and makes use of $X(-\frac{1}{2}L) = X(\frac{1}{2}L) = 0$, one obtains for the strut (cf. Pavlenko [1]):

$$R = \frac{-2\rho g^2}{c^2} \int_{-L/2}^{L/2} dx d\xi X(x)X(\xi)Y_0(\nu(x - \xi)). \quad (46)$$

Birkhoff and Kotik [2] have succeeded in reducing the general Michell integral to a form involving $Y_0(x)$. Unfortunately, a statement of the precise result requires considerable preparation, so that we refer to the original paper.

Before passing on to other aspects of Michell's integral, we should like to mention an interesting result of Stoker and Peters [1]. They assume a thin ship, but instead of assuming it towed in a fixed position with constant velocity as we essentially have done, they assume a constant thrust T along the longitudinal direction of the ship and a fixed weight. In the linearized theory this leads to a relation between T and the resulting velocity which again is just Michell's integral, i.e., the linearized theory does not distinguish between thin ships held fixed and ones free to trim. In addition, they obtain expressions for the vertical force and the moment about a transverse axis, as integrals similar in nature, but more complicated than Michell's integral. Thus both the sinkage and trim can be computed also (see also Havelock [8] and Lunde [1, app. 5]). The result as formulated by Stoker and Peters is actually more general than this in that they consider a thin ship heading into periodic infinitesimal waves and find the transient forces and moments which are to be added to the equilibrium ones mentioned above.

Limitations of the Theory. Since ships as actually designed do not seem to have the appearance of what one might expect for a "thin" ship and since they move in water and not in perfect fluids, one may reasonably ask: What are the limitations of Michell's integral in use and what useful information can be obtained from it?

An estimate by purely mathematical means of the possible error in using the linearized theory for a conventional or even a simplified hull shape has not been attempted, as far as the author is aware. One might try to make such an estimate from the second-order theory in the perturbation series, but it would likely be difficult and crude. In any case, this would still be only for a perfect fluid. Comparison with experiment remains, and has frequently been made. This leads to another type of difficulty, how to separate and measure the resistance caused by waves. If L is the length and c the velocity of a towed ship-like form, one may write.

$$R = \frac{1}{2}\rho c^2 L^2 C(\text{Re}, F), \quad (47)$$

where $\text{Re} = cL/\nu$, the Reynolds number, and $F = c/(Lg)^{1/2}$, the Froude number. Thus, from tests of a hull model, or of a set of geometrically similar models, one can obtain information about $C(\text{Re}, F)$ for some range of values of these two variables (actually, for points on several lines in the (Re, F) -plane radiating from the origin), but no direct information concerning the wave resistance. Considerable ingenuity has been used in trying to make this step. All methods seem to rely on first separating the contributions of the tangential and normal components of the force on a surface-element of the hull, which gives after integration over the hull

$$C(\text{Re}, F) = C_t(\text{Re}, F) + C_n(\text{Re}, F), \quad (48)$$

and then assuming

$$C_t = C_t(\text{Re}), \quad C_n = C_n(F). \quad (49)$$

The coefficient C_t is usually estimated from data for the frictional resistance of a flat

plate of the same length and wetted area as the model at rest, although occasionally more exact methods taking account of hull curvature and wave profile are tried. C_t does of course depend upon F also since both the wave profile and the consequent local velocities near the hull will change with F . The so-called form resistance will be represented in the coefficient C_n , as well as the wave resistance, and is strongly dependent on the Reynolds number. Methods are available for estimating this part of C_n , but they will not be described here; let us call this estimated coefficient C_e . The wave resistance coefficient is then assumed to be given by

$$C_w = C - C_t - C_e. \tag{50}$$

Experimenters do not, of course, delude themselves concerning the reliability of C_w as a coefficient for comparison with a theoretically computed coefficient. However, it is sometimes difficult to know whether a theoretically predicted effect is really absent or is lost in the working-up of experimental data. One can find more comprehensive discussions of these difficulties in various places, for example, Birkhoff, Korvin-Kroukovsky and Kotik [1] and Apukhtin and Voitkenskii [1].

In view of the preceding remarks it may seem surprising how good the agreement sometimes is between the experimentally estimated and the theoretically computed wave resistance. Figure 1 from a report by Weinblum, Kendrick and M. A. Todd [1] reproduces some values of C_w computed from Michell's integral and a faired curve of the "residual resistance coefficient" $C_r = C - C_t$ derived from experiments on a towed "friction plane" (here the coefficients are formed by dividing R by $\frac{1}{2}\rho c^2 S$, S the wetted area; this changes only the vertical scale). One would expect good agreement here, if ever, for the towed body approaches in form what one would expect of a "thin ship" (here beam-length ratio = 0.0265), an accurate estimate of C_t is possible, and C_e should be negligible. And indeed the agreement is good. Comparison of measured and computed wave profiles in the same report shows fairly good agreement at Froude numbers 0.292 and 0.326, but progressively worse as the Froude number increases. Here, however, the comparison is somewhat inconclusive since the calculations were made for a form of infinite draft.

Figure 1.

Figure 3. (From Shearer [1], by permission of the North-East Coast Institution of Engineers and Ship-builders.) (Note that Shearer's f is our F , for Froude Number.)

to lie below measured ones for hulls of small prismatic coefficient, but to lie above them for higher prismatic coefficients and to have much more pronounced humps and (especially) hollows; these are shifted somewhat to the left of the observed ones. As the Froude number becomes smaller than about 0.20 it becomes questionable whether the residuary resistance should be compared with the calculated wave resistance.

The discrepancies are generally ascribed to neglect of the effect of viscosity on the wave formation, to the assumptions inherent in the linearization, and, of course, to the difficulty of determining experimentally something which may be called the wave resistance. There have been several attempts to take account of viscosity. An empirical method of Wigley's [8], based upon damping of the bow wave, has been used to correct the calculated curves in Figure 2. Havelock [9] has proposed computing the wave resistance for a hull enlarged at each point by the amount of the displacement thickness of the boundary layer at that point; a tail then follows the ship. This method has also been considered by Okabe and Jinnaka [1], and by V. M. Lavrent'ev [1]. Although it does damp out the oscillations at low Froude numbers in the calculated curve for C_{w0} , it does not shift the positions of the humps and hollows appreciably. Inui [1, 3] introduces an empirically determined shift in the stern wave to take account of this. The empirical methods are perhaps objectionable because of their ad hoc character. On the other hand, they do produce better agreement with experimental results and may possibly give some idea as to where to look for reasons. Finding the causes for discrepancies seems more important here than finding empirical curves to fit experimental data.

Attempts to go beyond the linearized theory have been made by Guilloton, Okabe and Jinnaka [1] and by Inui [2]. In general, these attempts aim at trying to satisfy exactly the boundary conditions (for a perfect fluid) on the hull while retaining the linearized boundary condition on the free surface. We have commented earlier on this in connection with the derivation of the linearized theory. It seems to the author that an attempt at a complete second-order theory should be postponed temporarily. The expressions involved will be so unwieldy for computation that it may be easier to deal directly with the exact equations in machine computation. In addition, a too refined perfect-fluid theory may be out of keeping with neglect of a more fundamental treatment of the effect of viscosity.

Proposals have been made for other methods of determining experimentally the wave resistance. We mention especially one by Tulin [1].

These few remarks can not, of course, cover adequately the relation between measured and calculated wave resistance. For a more comprehensive treatment one should refer to Weinblum's excellent report [10], to Wigley [3], and to Birkhoff, Korvin-Kroukovsky and Kotik [1]. However, the agreement seems good enough to give support to proposals of Weinblum, Guilloton and others that one should reverse the present method of treating towing-test data by computing the wave resistance and considering the difference as a Reynolds-number dependent frictional-plus-form resistance. The current emphasis on methods of computing Michell's integral might seem misplaced otherwise. However, the author is not aware of published comparisons of computation and test for any ship forms in actual use.

Relation Between Hull Form and Wave Resistance. Once it has been established that the linearized theory gives a reasonable prediction of the wave resistance, it is possible to use it to investigate in a qualitative way the effect of various changes in hull shape, and even to try to find shapes of minimum wave resistance under certain restrictions. This is perhaps an even more important use of the theory than calculations for special forms, and it has occupied a great part of the efforts of Havelock, Weinblum, Wigley and others.

There are a few immediate corollaries of Michell's integral: 1. R varies as the square of the beam for a family of hulls related by

$$\zeta(x,y;\beta) = \beta \zeta(x,y;1). \quad (51)$$

(In comparing this conclusion with experimental results given as resistance coefficients, one must take care that the coefficient is based on a length squared and not on wetted area or displacement %.) 2. The resistance is independent of the direction of motion. 3. For any given hull shape the symmetrized hull,

$$\zeta_s(x, y) = \frac{1}{2}[\zeta(x, y) + \zeta(-x, y)], \quad (52)$$

has less resistance than ζ if $\zeta_s \neq \zeta$. (In fact, the symmetrization can be carried out with respect to any plane perpendicular to the x -axis, so that even a pair of ships in tandem can be so produced.) The displacement remains, of course, the same. Wigley [4, 5] and Weinblum [3] have made experiments to test the validity of these conclusions. 1. is satisfied approximately; however, an empirical determination of n in a law $R \sim B^n$ would show n somewhat smaller than 2 (see Weinblum [10], p. 27ff.). As might be expected, conclusions 2. and 3. are not well satisfied. The reason lies in the neglect of viscosity (see also Weinblum, loc. cit., pp. 61-65).

The behavior of the wave resistance for small and large values of the Froude number $F = c/(gL)^{1/2}$ can be found as an asymptotic expansion. Wigley [9] has derived results of this sort; Inui [1, pp. 67-68] finds several terms in the expansion for small F and a special class of hulls and uses it in some computations; Kotik, in unpublished work, derives the expansion for a very general class of hulls and includes the effect of sharp angles in the hull. A method for obtaining such an expansion can be outlined fairly easily for an "elementary" ship. Let us introduce dimensionless variables by measuring lengths in terms of L and let $f = gL/c^2 = F^{-2}$. Then we may write

$$R = \frac{1}{2}\rho c^2 L^2 \cdot \frac{8}{\pi} f^2 \int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P^2 + Q^2], \quad (53)$$

$$P = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx X'(x) \cos f\lambda x \cdot \int_{-H/L}^0 dy Y(y) e^{-f\lambda^2 y} = C(f\lambda) \cdot W(f\lambda^2) \quad (54)$$

and similarly for $Q = S(f\lambda) \cdot W(f\lambda^2)$ with sine replacing cosine. If $X(x)$ has no corners, i.e. if X' has no discontinuities, integration of $C(f\lambda)$ by parts yields:

$$C(f\lambda) = \frac{1}{f\lambda} [X'(\frac{1}{2}) + X'(-\frac{1}{2})] \sin \frac{1}{2}f\lambda - \frac{1}{f\lambda} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx X''(x) \sin f\lambda x, \quad (55)$$

where the last term is $O(1/f^2\lambda^2)$ (i.e. $f\lambda$ times the integral remains bounded as $f\lambda \rightarrow \infty$) if X'' is of bounded variation (a standard result from the theory of Fourier coefficients). Similarly, one may obtain

$$S(f\lambda) = \frac{1}{f\lambda} [-X'(\frac{1}{2}) + X'(-\frac{1}{2})] \cos \frac{1}{2}f\lambda + \frac{1}{f\lambda} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx X''(x) \cos f\lambda x \quad (56)$$

and

$$W(f\lambda^2) = \frac{1}{f\lambda^2} [Y(0) - Y(-H/L)] e^{-f\lambda^2 H/L} - \frac{1}{f\lambda^2} \int_{-H/L}^0 dy Y'(y) e^{f\lambda^2 y}, \quad (57)$$

where the term in $W(f\lambda^2)$ is $O(1/f^2\lambda^4)$ if Y' is bounded. In order to carry the asymptotic expansion further, one performs further integrations by parts, assuming that the higher derivatives behave properly. Finally, we note that if $X(x)$ had corners at, say, $x_1, x_2, \dots, x_n$,

$$-\frac{1}{2} < x_1 < x_2 < \dots < x_{n-1} < \frac{1}{2}, \quad (58)$$

with jumps in X' given by

$$\Delta X'(x_i) = X'(x_i^+) - X'(x_i^-), \quad (59)$$

then, letting

$$x_0 = -\frac{1}{2}, \quad x_n = \frac{1}{2}, \quad \Delta X'(-\frac{1}{2}) = X'(-\frac{1}{2}), \quad \Delta X'(\frac{1}{2}) = -X'(\frac{1}{2}), \quad (60)$$

the first terms in the expansions for $C(f\lambda)$ and $S(f\lambda)$ become

$$-\frac{1}{f\lambda} \sum \Delta X'(x_k) \sin f\lambda x_k \quad \text{and} \quad \frac{1}{f\lambda} \sum \Delta X'(x_k) \cos f\lambda x_k. \quad (61)$$

The effect of corners in $Y(y)$ will not show up until another partial integration is performed and thus their effect on the wave resistance will be of a higher order. The term $Y(-H/L)$ in the expansion of $W(f\lambda^2)$ will ordinarily be zero, but would presumably be kept if one were dealing with a sharply cut off strut. Forming $P^2 + Q^2$ for the case where X is smooth and $Y(-H/L) = 0$, one finds

$$\frac{1}{f^4\lambda^6} \left\{ X'^2(\frac{1}{2}) + X'^2(-\frac{1}{2}) - 2X'(\frac{1}{2})X'(-\frac{1}{2}) \cos f\lambda + O\left(\frac{1}{f\lambda}\right) \right\} \cdot \left\{ Y(0) + O\left(\frac{1}{f\lambda^2}\right) \right\}. \quad (62)$$

Then the resistance coefficient $C_w = R/\frac{1}{2}\rho c^2 L^2$ is

$$C_w = \frac{8}{\pi} \frac{Y^2(0)}{f^2} \left\{ [X'^2(\frac{1}{2}) + X'^2(-\frac{1}{2})] \int_1^\infty \frac{d\lambda}{\lambda^4 \sqrt{\lambda^2 - 1}} - 2X'(\frac{1}{2})X'(-\frac{1}{2}) \int_1^\infty \frac{\cos f\lambda}{\lambda^4 \sqrt{\lambda^2 - 1}} d\lambda \right\} + O\left(\frac{1}{f^3}\right). \quad (63)$$

For large values of f , one has the asymptotic expansion (see, e.g., Erdélyi [1], pp. 46-51)

$$\int_1^\infty \frac{\cos f\lambda}{\lambda^4 \sqrt{\lambda^2 - 1}} d\lambda = \sqrt{\frac{\pi}{2f}} \cos(f + \frac{1}{4}\pi) + O(1/f); \quad (64)$$

the first integral in C_w has value $2/3$. Hence, and returning to the usual Froude number,

$$C_w = \frac{16}{3\pi} Y^2(0) [X'^2(\frac{1}{2}) + X'^2(-\frac{1}{2})] F^4 - 8 \sqrt{\frac{2}{\pi}} Y^2(0) X'(\frac{1}{2}) X'(-\frac{1}{2}) F^5 \cos(F^{-2} + \frac{1}{4}\pi) + O(F^{-6}). \quad (65)$$

In case $X(x)$ has corners, the expansion becomes

$$C_w = \frac{16}{3\pi} Y^2(0) \sum [\Delta X'(x_i)]^2 F^4 - 8 \sqrt{\frac{2}{\pi}} \sum_{i=j} \frac{\Delta X'(\frac{1}{2}) \Delta X'(-\frac{1}{2})}{|x_i - x_j|^{\frac{1}{2}}} F^5 \cos(F^{-2} |x_i - x_j| + \frac{1}{4}\pi) + O(F^{-6}) \quad (66)$$

The foregoing is only a caricature of Kotik's result, which does not assume an "elementary" ship and carries the expansion much further.

Unfortunately, the Froude-number region in which this expansion has its greatest validity is one in which the wave resistance is not of great practical importance and also one where the influence of viscosity on wave making may be presumed to be great. However, it does give information of the sort one hopes to get from a theoretical investigation, a statement concerning the dependence of wave resistance on some simple parameter of the form, in this case the tangents of the angles at the bow and stern at the waterline. If a sharp turn at the shoulder may be taken to approximate a corner, the second formula indicates that this also has an important wave-making effect at small Froude numbers.

At the other extreme of infinite Froude numbers, or small f , an asymptotic expansion in the general case does not seem to work out as neatly. However, by working with upper and lower approximating functions, one can see that the expansion starts out like

$$C_w = Af^2 + Bf^2 \log f + \dots, \quad (67)$$

which is also confirmed by Havelock in his discussion of Wigley [9]. The coefficient A appears to be proportional to the square of the volume divided by the distance between the surface and the centroid of the midship section. In view of the fact that the method used is awkward and can almost certainly be improved, it would be unpleasant to commit it to paper. In the case of the long vertical strut one may write out the asymptotic expansion immediately from the known expansion of $Y_0(x)$ for small x (see below). It is of the same form as for the ship of finite draft and begins with the term

$$\frac{8}{\pi} f^2 \{ \gamma + \log \frac{1}{2} f \} \{ \int dx \zeta(x) \}^2, \quad \gamma = \text{Euler's Constant} \quad (68)$$

Observation indicates that the higher the Froude number, the thinner a "thin" ship should be in order not to violate the condition of small disturbance of the water. Thus the applicability of an expansion at infinite Froude number to normal ship forms seems dubious.

In the Froude number range 0.2 to 0.6, important in practice, studies of influence of hull form have to be chiefly computational or experimental. Havelock [1-5] has made a systematic study of the effect of varying various aspects of the hull form; many of his results are summarized in [10] and in Wigley [1]. His computations have included varying the form of the cross-section of an infinitely long vertical strut while keeping its area constant, adding parallel middle-body to such a strut, and cutting the strut off at various depths. Wigley [4-10] and Weinblum [2-9] have made extensive investigations, both theoretical and experimental, of the interrelation of ship form and wave resistance, with the aim of both testing the predictions of Michell's integral and of explaining theoretically known facts about ship forms. The results don't lend themselves to a brief summary; one should consult the original papers and also summaries by the authors themselves, e.g., Wigley [1, 2, 3], Weinblum [3, 10]. The benefits to be gained from further studies of this sort have certainly not been exhausted. Newly developed computational methods and tables to facilitate them will make such studies much easier (see Guilloton [1] and Weinblum [11]).

It would be a gross exaggeration to state that these studies have had any substantial effect on ship design. However, most naval architects are aware of their existence and that they do give insight into the wave-making properties of ship forms. As shown by both Wigley [7] and Weinblum [7], the advantages to be obtained by use of a bulbous bow under certain conditions could have been predicted theoretically from Michell's integral. Unfortunately, the theory followed the discovery in this case, but one can hope that it will also happen the other way occasionally.

Ships of Minimum Wave Resistance. Once a theoretical expression for the wave resistance in terms of ship form is found, it is natural to try to use this expression

to deduce forms of minimum wave resistance in case such exist. One must, of course, take care in formulating such a problem in order that a solution will exist. For example, it is not sufficient to fix just the volume and ask for a form of minimum resistance, for by distributing the volume deeper and deeper the resistance becomes less and less without reaching a minimum. A solution, if it exists, will of course depend upon the Froude number. This problem has been treated extensively by Weinblum and also by Pavlenko [1] and Sretenskii [1].

If one fixes the centerplane section of the ship and its volume, and uses Michell's integral for the resistance, the mathematical problem is to minimize a quadratic form subject to a linear constraint. This leads to the following integral equation for $\zeta_x(x, y)$:

$$\iint_{S_0} d\xi d\eta \zeta_x(\xi, \eta) K(f(x - \xi), f(y + \eta)) = kx, (x, y) \text{ in } S_0. \quad (69)$$

A solution ζ_x is sought such that

$$\iint_{S_0} d\xi d\eta \xi \zeta_x(\xi, \eta) = V, \quad V = \text{fixed volume}. \quad (70)$$

The latter condition will determine the constant k in the integral equation. Other constraints could replace this one, or be added to it. For example, one could fix $\zeta_x(1/2, 0)$, $\zeta(0, y)$, etc. The integral equation is of the first kind, and these are notoriously difficult.

Sretenskii [1] deals with the analogous problem for the infinite vertical strut (although he states that his result is valid for "elementary" ship), and claims to prove

Figure 4.

Figure 5.

that no solution can exist among square-integrable functions. It seems to the author that what Sretenskii takes as eigenfunctions of the kernel are not really such and that his proof is not valid. Pavlenko also deals with the strut, side-steps the question of existence, integrates by parts with respect to x (remembering that $\zeta(\pm 1/2) = 0$) and treats numerically the resulting integral equation:

$$\int_{-1/2}^{1/2} d\xi \zeta(\xi) Y_0(f|x - \xi|) = k, \quad |x| \leq 1/2, \quad (71)$$

where

$$\int_{-1/2}^{1/2} d\xi \zeta(\xi) = A = \text{fixed cross-section area}, \quad \zeta(\pm 1/2) = 0. \quad (72)$$

Pavlenko replaces the integral equation by a set of linear algebraic equations and solves these. Figure 4 shows the resulting strut cross-sections for several Froude numbers. It is somewhat amusing that for Froude number less than 0.325 Pavlenko got negative ordinates near the ends. Weinblum [1] considered "elementary" ships represented by the product of two polynomials, so that his computed minimal forms are for a very restricted class of functions. However, they show the same general behavior as functions of Froude number as do Pavlenko's. Figure 5 shows the water-plane sections for various Froude numbers.

According to a recent paper by Dörr [1], the solution of the strut integral equation, if it exists, can be expressed as a series of Mathieu functions ce_{2n} . In fact, Dörr has shown that

$$ce_n(\beta; f^2) = \lambda_n \int_0^\pi d\alpha ce_n(\alpha) Y_0(f|\cos \alpha - \cos \beta|). \quad (73)$$

Then, if one expands the constant k in a series in ce_n (actually, only ce_{2n} are needed)

$$k = \sum_{n=0}^{\infty} a_n ce_n(\beta; f^2) \quad (74)$$

and lets

$$G(\beta; f) = \sum_{n=0}^{\infty} a_n \lambda_n \operatorname{ce}_n(\beta; f^2), \quad (75)$$

the solution of the integral equation is given by

$$\zeta(x; f) = \frac{2}{\sqrt{1-4x^2}} G(\arccos -2x; f) \quad (76)$$

Unless G has compensating behavior at $x = \pm 1/2$, the solution has a singularity at these points which deprives it of physical meaning; and, of course, the condition $\zeta(\pm 1/2) = 0$ is not satisfied. This seems indeed to be the case for sufficiently small f (large Froude number), either from a study of the asymptotic expansions of λ_n and ce_n or from the development of Y_0 about the origin:

$$Y_0(x) = \frac{2}{\pi} \log \frac{x}{2} \left[1 - \left(\frac{x}{2} \right)^2 + \dots \right] + \frac{1}{\pi} \left[2\gamma - \left(\frac{x}{2} \right)^2 (2\gamma - 2) + \dots \right]. \quad (77)$$

The latter leads to the simpler integral equation for small f

$$\frac{2}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi \zeta(\xi) \log \frac{1}{2} f \delta |x - \xi| = b, \quad |x| \leq \frac{1}{2}, \quad \log \delta = \gamma = \text{Euler's Constant}, \quad (78)$$

and eventually to the solution

$$\zeta(x) = \frac{A}{\pi \sqrt{1-4x^2}} \quad (79)$$

independent of f . It is not known at present whether this behavior persists for small Froude number. The matter should be investigated. The asymptotic expansion of Michell's integral for small Froude number which was given earlier indicates that an optimum form should have cusps at bow and stern and elsewhere avoid discontinuities in slope and curvature.

Establishing the existence of a solution in a physical problem may seem like a luxury. However, unless a physically acceptable solution exists, Weinblum's investigations of optimal forms among restricted classes of functions such as polynomials lose much of their significance; for the underlying assumption is that these are approximations to an exact solution. This applies even more strongly to Pavlenko's approach. One must, of course, remain aware of the fact that ships are not necessarily "thin" and that they do not sail in perfect fluids, so that the practical implications of such studies are not necessarily important. However, they may put one in possession of the techniques for finding, say, optimal forms for the forward half of a ship when the after half is given.

Several Ships and Ships Near Walls or in Canals. The wave-resistance integral applies equally well to two or more ships in a row if they all travel at the same speed. For let

$$\zeta(x, y) = \zeta_1(x, y) + \zeta_2(x, y), \quad (80)$$

where $\zeta_1(x, y) = 0$ except on S_1 and $\zeta_2(x, y) = 0$ except on S_2 , where the centerplane

sections S_1 and S_2 are at a distance d between centers. Treating ζ as a single ship, one may compute, say,

$$\begin{aligned} P(\lambda; d) &= \iint_{S_1} dx dy \zeta_{1x}(x, y) e^{\nu \lambda^2 y} \cos \nu \lambda x + \iint_{S_2} dx dy \zeta_{2x}(x, y) e^{\nu \lambda^2 y} \cos \nu \lambda x \\ &= \iint_{S_1} dx dy \zeta_{1x}(x, y) e^{\nu \lambda^2 y} \cos \nu \lambda x + \iint_{S_2} d\bar{x} dy \zeta_{2x}(\bar{x} - d, y) e^{\nu \lambda^2 y} \cos \nu \lambda (\bar{x} - d), \end{aligned} \quad (81)$$

where in the second integral we have shifted to local coordinates centered in S_2 . Expansion of $\cos \nu \lambda (\bar{x} - d)$ leads to the following:

$$P(\lambda; d) = P_1(\lambda) + P_2(\lambda) \cos \nu \lambda d - Q_2(\lambda) \sin \nu \lambda d. \quad (82)$$

Similarly,

$$Q(\lambda; d) = Q_1(\lambda) + Q_2(\lambda) \cos \nu \lambda d + Q_2(\lambda) \sin \nu \lambda d. \quad (83)$$

The subscripts refer, of course, to the individual ship. Substitution in the resistance integral leads to

$$R = R_1 + R_2 + 2 \frac{4\rho g^2}{\pi c^2} \int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} \quad (84)$$

$$[(P_1 P_2 + Q_1 Q_2) \cos \nu d \lambda - (P_1 Q_2 - Q_1 P_2) \sin \nu d \lambda] = R_1 + R_2 + R_{\text{int}}(d).$$

If the ships are identical, $P_1 Q_2 - Q_1 P_2 = 0$ and

$$R = 2R_1 + 2 \frac{4\rho g^2}{\pi c^2} \int_1^\infty d\lambda \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} [P_1^2 + Q_1^2] \cos \nu d \lambda \leq 4R_1. \quad (85)$$

It follows from theorems on Fourier transforms that $R_{\text{int}} \rightarrow 0$ as $d \rightarrow \infty$. However, one can also find the asymptotic expansion for large d :

$$R_{\text{int}}(d) \approx \frac{4\rho g^2}{\pi c^2} \sqrt{\frac{2\pi\nu}{d}} \cos(\nu d + \frac{1}{4}\pi) [P_1^2(1) + Q_1^2(1)]. \quad (86)$$

Thus, at some separations d , the total resistance is less than that of the ships separately. A similar expansion holds for the case when the ships are not identical. At the other extreme, $d = 0$, the expression for R verifies the quadratic dependence on beam (a statement which has more mathematical than physical meaning). Kostyukov [1] has also treated the problem of a caravan of n ships, but without the above asymptotic expansion.

The case of two ships moving abreast, or of one parallel to a wall, is somewhat more complicated. The boundary condition which should be fulfilled on the center-plane sections of each ship is, according to the linearized theory, that $\varphi_n = \mp c \zeta_{ix}$, $i = 1, 2$. However, a source distribution of density $-c \zeta_{ix}/2\pi$ over S_i will not satisfy this exactly but will approximate it at small values of the ratio beam/separation (which can presumably be used as a further expansion parameter). The same difficulty does not arise in the case of a ship moving down the center of a deep canal. Both problems are treated by Lunde [1] and the details will not be repeated here. The second one was considered first by Sretenskii [2 and 3, chap. II]. Numerical computations are scanty. However, Sretenskii has computed the resistance for a vertical strut of cross-section $\zeta(x) = B(1 - x^2/L^2)$, $|x| \leq L$, for a number of values of gb/c^2 and $2L/b$, b the canal width; he gives both tables and a graph of $\pi R/64\rho g B^2 L$. For the case of the strut Keldysh and Sedov [1] obtain the theoretically interesting result

$$\lim_{c \rightarrow \infty} R = 2\rho g A^2/b, \quad (87)$$

where A is the cross-sectional area.

Water of Finite Depth. The theory of "thin" ships has been extended by Sretenskii [3, Chap. IV] to motion in water of finite depth h , and he has obtained resistance integrals in terms of ship form which are analogous to Michell's. In [4] he has made numerical computations for the form

$$\zeta(x, y) = B(1 - x^2/L^2)(1 + y/H) \quad (88)$$

for $h = L$ and several values of H/L ranging from 0.025 to 0.3. For comparison the calculations were repeated for $h = \infty$ and the same values of H/L . The resulting

Figure 6.

graphs are shown in Figures 6 and 7. For the case of canals of finite depth the problem was solved independently by Keldysh and Sedov [1] and by Sretenskii [3, Chap. IV]. The author is not aware of any published computations for special shapes; they would obviously be important in estimating corrections to towing-tank test data. Wigley [3] has computed the resistance of the submerged body generated by a source and sink separated by $2l$ and located at a depth $0.0375h$ in a canal of width $b = 2.5h$; graphs are given of the ratio of the resistance to the resistance in unrestricted water for $c/\sqrt{gh} = \sqrt{2}$ and $\frac{1}{2}\sqrt{2}$ and for values of $2l/b$ between 0 and 1.8. Lunde has informed the author that he has partially completed computations for an "elementary" ship form. The theory for finite depth is given in detail in Lunde [1] and will not be

repeated here. The resistance integral for the case of an infinite ocean of finite depth h is given by

$$R = \frac{2\rho g c}{\pi} \int_{\mu_h}^{\infty} d\mu \sqrt{\frac{\mu}{\mu - \nu \tanh \mu h}} [P^2 + Q^2], \quad (89)$$

where

$$P = \iint_S dx dy \zeta_x(x, y) \frac{\cosh \mu(y + h)}{\cosh \mu h} \cos x \sqrt{\nu \mu \tanh \mu h} \quad (90)$$

and Q is a similar function where sine replaces cosine; here μ_h is the nonzero solution of $\mu = \nu \tanh \mu h$ if such exists, otherwise zero. The latter case occurs if $c^2/g h \geq 1$. As $h \rightarrow \infty$, $\tanh \mu h \rightarrow 1$, $\mu_h \rightarrow \nu$, and R approaches Michell's integral in the form it takes if one makes the substitution $\mu = \nu \lambda^2$.

Figure 7.

Accelerated Motion. The velocity potential for a submerged source moving with velocity $c(t)$ has been given Sretenskii [3], and by a different method by Have-lock [e.g., 11]. From this one can find the wave resistance of a thin ship as before. A detailed exposition is given in Sretenskii [3] and Lunde [1]. The result is:

$$R = \frac{\rho c'(t)}{\pi} \iint_S dx dy \iint_S d\xi d\eta \zeta_x(x, y) \zeta_x(\xi, \eta) \{[(x - \xi)^2 + (y - \eta)^2]^{-1/2} - [(x - \xi)^2 + (y + \eta)^2]^{-1/2}\} + \frac{2\rho g}{\pi} \iint_S dx dy \iint_S d\xi d\eta \zeta_x(x, y) \zeta_x(\xi, \eta) \cdot \int_0^\infty dk k e^{k(y+\eta)} \int_0^t d\tau c(\tau) \cos \sqrt{gk} (t - \tau) J_0(k(x - \xi) + k \int_\tau^t d\tau' c(\tau')). \quad (91)$$

Lunde [1, 3] has extended this to the case of accelerated motion in a canal of finite depth. The first summand in the formula above is an added-mass term. If one takes the case of a ship starting suddenly from rest at constant velocity c , this term vanishes and the second summand approaches Michell's integral as $t \rightarrow \infty$. Since towing-tank tests approximate this situation, it would be of interest to have computations of the transient part of this for a ship-like form. Lunde informs the author that such are in the case of a ship starting suddenly from rest at constant velocity c , this term vanishes progress. The author has obtained asymptotic expansions for large t .

Other unsteady-state problems which would be of interest but which have not been attacked as far as the author is aware are motion over a stepped bottom or approaching a beach. The steady-state problem of motion parallel to a beach or step has also not been treated. Hansen [1] has considered the case of a pressure point moving parallel to a beach, but, although this gives a qualitative notion of the form of the waves, it doesn't relate them to the geometry of the ship or to the forces on it.

Ship Waves. In conclusion we should like to call the reader's attention to the fact that we have seldom discussed the actual waves produced by a ship. There seems to have been little investigation of the form of the free surface away from the ship as it relates to the form of the ship. The studies of Hogner [2] and others on waves generated by moving pressure distributions give qualitative information about ship waves, but not in relation to a specific hull geometry.

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JSTG — Jahrbuch der Schiffbautechnischen Gesellschaft (Berlin).

PRSL — Proceedings of the Royal Society of London. Series A.

Trans. INA — Transactions of the Institution of Naval Architects (London).

Trans. NECIES — Transactions of the North-East Coast Institute of Engineers and Shipbuilders (Newcastle-upon-Tyne).

Trans. SNAME — Transactions of the Society of Naval Architects and Marine Engineers (New York).

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DISCUSSION

T. B. Benjamin

I should like to mention briefly some aspects of wave resistance which I have examined recently, both theoretically and by way of some experiments with the water channel belonging to the Engineering Department of Cambridge University. An account of this work has already appeared¹, but so recently that I may be justified in outlining some of it now. It seems to be appropriate to the present discussion, although I fear it is rather far removed from Professor Wehausen's topic.

The study was confined to two-dimensional flow along a straight horizontal channel, and was concerned with the effects of placing various rigid obstacles in the

¹ *J. Fluid Mech.* 1, 227. (1956)

stream. In particular, I attempted to clarify the different circumstances under which on the one hand, the development of a drag on the obstacle is accompanied by the formation of a train of periodic waves on the downstream side and, on the other hand, the receding stream converges towards a uniform state. As Professor Lighthill explained this morning, a great many different flow phenomena possible in a straight channel can be summarized very neatly in the energy/momentum diagram * which was put on the blackboard this morning. It is still there, and we may conveniently refer to it now.

The diagram shows that whenever a slightly subcritical stream (i.e., one for which the Froude number is slightly less than unity) is obstructed, the reduction of momentum associated with the drag on the obstacle gives rise to a train of waves on the downstream side. Increasing the drag, by deeper immersion of the obstacle, causes the amplitude and length of the waves to become larger; and eventually the wave train resembles a succession of 'solitary waves.'

However, a different state of affairs arises when the undisturbed stream has a Froude number less than about 0.8, the value at which the solitary wave of maximum height (i.e. the sharp-crested form of the wave) is formed by reducing the momentum of the stream as much as is possible. In this lower range of Froude numbers, the resistance experienced by an obstacle cannot, except for very small values, be ascribed to wave formation alone. The effect of gradually increasing the resistance is first to produce periodic waves of maximum height, and then to produce breaking waves. In other words, the reduction in the momentum of the stream must eventually be accompanied by a loss of energy. Nevertheless, by increasing the resistance sufficiently it is possible to bring about once again a steady lossless flow on the downstream side. In this extreme case, the receding stream converges steadily towards a uniform supercritical state, and waves are entirely absent; this is exemplified by the flow under a vertical sluice-gate, although such a flow is possible under an obstacle of any shape which is adequately immersed.

These effects were clearly demonstrated in some experiments in a horizontal water channel with glass sides. A metal plate was supported from a horizontal axle, and spanned the whole width of the channel. The plate could be swung down broadside to the stream, and was balanced in such a way that the drag due to contact with the water could be fixed at any desired value. The observed effects of gradually lowering the plate into a subcritical stream with Froude number less than 0.8 was as follows. First, a smooth train of periodic waves was formed, whose amplitude and length increased as the drag was increased. Then the leading wave broke at its crest, accompanied by a reduction in the amplitude of the waves in the rear, so that the front of the wave train resembled a turbulent 'hydraulic jump.' On a further increase of the drag, the wave train was swept downstream out of the channel, and the flow at a distance behind the plate became approximately uniform; thus, the maximum drag occurred for a uniform supercritical flow downstream, as indicated by the theoretical considerations summarized above.

To account theoretically for the case of maximum drag, where waves are absent, it is convenient to invert the problem; that is, the depth and Froude number far downstream are specified, so that the drag is determined by a simple momentum relation between the uniform flows up and downstream, and the position of the obstacle remains to be found. In my paper I showed that the profile of the converging flow in the rear of the obstacle is determined uniquely by the ultimate conditions reached far downstream; that is to say, the flow pattern is independent of the shape of the obstacle. However, this conclusion does not apply to the region where the free surface first springs clear of the solid boundary, and where its curvature is large. This general wake profile has a simple approximate analytic expression in which the Froude number is a parameter, and it may be seen that this expression is the same as a well-known equation for

* Figure 9 of Prof. Lighthill's paper in this volume.

the solitary wave, although this wave cannot itself occur under the circumstances considered.

To find the shape of the free surface very close to the separation point, other methods have to be used. In the example of a vertical sluice-gate, I was able to apply a fairly straight-forward method based on conformal transformation whereby the free-surface profile in the hodograph plane is approximated by an arc of an ellipse. A similar method is also adequate when the obstacle is an inclined plane. It appears that these methods will provide estimates of the drag with an accuracy considerably better than 1%. The results of these calculations compare favourably with the solutions which Southwell and Vaisey² obtained by relaxation methods.

Although this work has no direct bearing on the problem of the resistance of ships in deep water, it may possibly have applications to other problems within the sphere of this symposium. For instance, it seems to be relevant to the performance of hydroplanes and seaplane floats in shallow water.

J. J. Stoker

Both of the lecturers this afternoon have made reference to the work of Arthur Peters and myself about the motion of ships in a seaway. I thought I might try to tell you what we had in mind to do, although it is impossible to give any details.

For this audience the best way to put it would probably be to say that what we did was to formulate a theory which is a thoroughgoing generalization of the classical theory of Michell. We make the same basic assumptions as in that theory. Those are principally that we have a perfect fluid, so that the turbulent wake is ignored, for example; and then on top of that we make an assumption that is necessary to linearize the problem, namely that the ship is slender in some sense so that it is possible for it to have a forward motion with finite velocity, and still create waves of small amplitude. The result is a theory which is a generalization of the Michell-Havelock theory, in the sense that the latter results when we make the restrictive assumptions that are necessary for that case. What we have done, in effect, is to give up the strong assumption that the ship is held fixed in space, while the water streams past. That is what Michell assumed, and the one quantity he computed is the wave resistance, the force necessary to maintain the body in that position.

We assume that the body floats freely, calculating its motion from the pressure force of the water, treating the combined system of ship and sea without making any *a priori* assumptions with respect to the coupling of the various degrees of freedom of the ship or the coupling of the motion of the ship and the motion of the water.

In developing the theory our procedure was to carry out a formal development of all quantities with respect to a small parameter, which was a thickness-length ratio of the ship's hull. Depending upon the manner in which this parameter is chosen, various theories arise. If, for example, the ship is regarded as small in the beam-length ratio, our generalization of Michell's theory arises. If it is the draft-length ratio that is assumed small, still a different theory arises which might be called a theory of planing ships. For a yacht-shaped hull, which is a combination of the previous two cases, still a third theory is derived.

In the first case, i.e. the case of a ship with a thin vertical section—a ship of Michell's type—it turns out that a more general problem than that of Michell can be solved explicitly. That is the problem of motion of the ship when it is assumed that it moves only in the vertical longitudinal plane through waves which are symmetrical with respect to that plane. The amplitude of the oscillations, the trim, and the wave resistance can be calculated by explicitly given integrals.

However, it is an inescapable consequence of this generalization of Michell's theory that the pitching and heaving modes of oscillation are not damped. The reason for that is simple: the ship is like a knife-blade placed vertically in the water, and small

² Phil. Trans. A, 240, 117.

vertical oscillations can create only waves with amplitudes which are of higher than first order in the thickness of the blade. Since the only mechanism for energy dissipation is the creation of waves which carry off energy to infinity, it is clear that no damping of such oscillations is to be expected. Since these modes of oscillation seem actually to be damped when observations are made for actual ship's hulls, it is indicated that Michell's assumptions are not always the appropriate ones, even though they do lead to fairly reasonable results for the wave resistance. However, two other forms of our theory are available, both of which would lead to damping in the pitching and heaving modes of oscillation.

Unfortunately, only for ships of Michell's type are there explicit solutions available for the complicated boundary problems in potential theory posed by our basic theory—and only then for the restricted type of motions mentioned above. We have formulated the problems in terms of integral equations over the hull of the ship, but unfortunately it turns out that the integral equations have singular kernels, and are in general of unconventional types so that even their numerical solution with modern computing equipment presents problems of some difficulty.

R. F. Chisnell

I wish to describe an unsuccessful attempt to use a Ferranti Mark I computer at Manchester on the problem of wave resistance of non-thin ships. As this problem is still one of the unsolved problems urgently requiring attention, I mention this attempt as firstly, there may well be others contemplating a similar program who would like to hear of the difficulties encountered; and secondly, with the expected arrival in Manchester of a Ferranti Mark II machine the attack on this problem may be continued, and we would be pleased to hear from people interested in this work.

A ship of arbitrary hull shape is assumed to move with constant velocity into an undisturbed ideal fluid. The free surface condition is linearized, and may be expressed in terms of a velocity potential function as,

$$\phi_{xx} + \frac{1}{F^2} \phi_y = 0, \quad \text{on} \quad y = 0, \quad (92)$$

where x is the direction of motion, y normal to the plane of the free surface and F the Froude number of the motion. On the hull the normal velocity ϕ_n must satisfy

$$\phi_n = F \cos(n, x) \quad (93)$$

n being normal to the hull.

It is the existence of these two separate boundary conditions that makes the problem of finding the appropriate solution of Laplace's equation so difficult.

The method employed was to represent the ship as a distribution of Kelvin sources over its wetted area. A Kelvin source has a source-type singularity and also satisfies the free surface condition (1). The second boundary condition leads to an integral equation to be satisfied by the strength of the Kelvin source distribution.

For numerical work the continuous distribution is replaced by a sufficiently fine Kelvin point source distribution, the integral equation becoming a set of linear equations for the strengths of the point sources. The majority of terms in these equations are velocities due to Kelvin sources at the location of another of the sources. Part of the modification necessary to convert the velocity potential of a source into that of a Kelvin source is a double integration over a semi-infinite rectangle. The integrand is fierce, containing a curve of singularities. One to two hours computing time was envisaged for each integral and of the order of a hundred integrals. The reliability of the machine, which was the first production model, was not great enough to perform these integrations. The possibility of performing the integration in several smaller stages was considered, but as it would result in an appreciable increase in running time it was decided to postpone the work till the arrival of the Mark II machine.

M. C. Eames

While much attention has been given in recent years to refinement of the linearized theory of wave resistance as initially formulated by Michell and developed by Havelock and others, and now there is evidence of some success in Russia and Japan of handling the formidable mathematics of non-linearized solutions, the fact remains that comparatively little has been done in the opposite direction. By this is meant the simplification of solution to the point where first approximations to the wave resistance of an actual ship form may be carried out in the preliminary design stages, calculated by naval architects as opposed to mathematicians (or complex computing machines).

It is, of course, unlikely that an accurate wave resistance calculation for a practical ship form will ever be reduced to the simplicity of, for example, a stability calculation, but the present writer believes that an approximation which is sufficiently close to yield a useful result in a comparative sense, is possible.

An approach which was developed by the present writer in 1952 will be briefly outlined by way of illustration. It should be clearly pointed out however, that the technique remains largely unproven, since other commitments have so far prevented the writer from concluding the work.

The ship is treated in a number of horizontal layers, the layer including the waterline being by far the most significant. By direct substitution of ordinates from the lines plan as coefficients in simple polynomials, these layers are then represented mathematically with a fair degree of accuracy. A refinement consists of adding to the ordinates in the after body so as to take into account the growth of the boundary layer according to the ideas expressed by Havelock in his 1948 paper to the Institution of Naval Architects.

The linearized Michell solution, in one of the forms developed by Havelock can be written directly in terms of relatively simple integrals involving these layer polynomials, as has been shown in a particular case by Weinblum, who has also tabulated values for one of the resulting integrals.

The other integral has been tabulated by the present writer and the Weinblum table extended to cover the range of practical interest. (The polynomials used by the writer involve higher powers than those considered by Weinblum, thereby increasing the accuracy of representation of the ship form.)

The procedure is then purely automatic and can be conveniently arranged in a tabular form of calculation well within the capabilities of a calculations-draughtsman.

Consideration of some work by Emerson applying Havelock's well known source and sink compartmentisation of the ship has shown that very few vertical subdivisions of the ship are necessary to give a satisfactory result, provided the layer containing the waterline is kept narrow. Thus the amount of arithmetic involved in the method proposed by the writer need not be excessive, and one obvious advantage of the technique over that of the source and sink calculation is that systematic variations of hull lines can be more readily performed and traced through the calculation so that their significance on the final result is more clearly understood.

Finally it might be remarked that if, for initial estimates, a two layer horizontal subdivision is adopted, it is possible to modify the method so that the only ordinates required to perform the calculation are those of the waterline, and those of the curve of section areas—the two basic curves from which line plans are usually derived. This immediately suggests a technique for rapidly comparing fundamental differences of hull form from the wave resistance point of view at the very outset of a design, which might be of particular value where other requirements dictated a rather radical departure from "normal" ship forms.

It is not suggested that calculations of the form outlined above will ever reach a state whereby they would replace the techniques of the model experiment, but it is felt that they could constitute a useful control on such work, and in many cases serve to reduce the number of models required by suggesting the most promising hull forms thereby reducing the range of possibilities.

VI

HYDRODYNAMIC BARRIERS IN SHIP DESIGN

J. C. Niedermair

U. S. Navy Bureau of Ships

A discussion of some of the related unsolved problems and the continued necessity for the application of empirical methods to the design of ships.

The naval architect is surrounded with barriers of varying degree of density in practically every phase of his work in ship design that is related to hydrodynamics. Barriers exist in the theory and mathematics, the experimental work in the laboratories, the full-scale performance of the ship, the weather and the sea its affects, and in a large measure in custom and tradition. Certain hydrodynamic barriers that exist for the surface ship vanish in the case of the submarine.

Aside from the barriers associated with hydrodynamics there is an economic barrier. The very physical size of a ship and its cost and time required for construction, constitute a most formidable barrier to progress. The aircraft designer is seldom expected to produce a successful plane in his first try. Several experimental planes are built and tested before a prototype is reached. Not so with a ship, every piece of equipment and machinery must be tried and true so that the very first prototype will be successful or very nearly so in meeting all the essential requirements.

The demands for the attainment of more advanced properties in our ship designs are becoming more and more urgent due to the changing technology of the times in which we are living. The fact that the art of shipbuilding is among the oldest known to man has in a large measure handicapped the scientific side of our work. In a period when newer industries were making rapid progress through a more scientific approach the need for theoretical research was not considered an essential part of the naval architect's work because great success had been attained by evolution and watchful attention to the performance of full-scale ships. Fortunately, or possibly unfortunately, our predecessors were so very successful in meeting their requirements on the basis of good judgment that the need for more extensive theoretical work was not readily apparent. Ship hydrodynamics was, in a sense, a plaything for the theoretical minded but of little import to the practicing naval architect. In the meantime the competitive aircraft industry has come along in giant steps largely on a basis of theoretical and applied research. Because of this, and the many advances in the fields of physics, we too feel compelled to encourage the scientific approach.

In this brief discussion reference will be made to a few of the important barriers we have been dealing with in the past and to some of the more important ones which may limit our progress now and in the future. The specific areas of importance are speed, frictional and wave-making resistance, rough water performance, maneuvering, flow noise, and cavitation.

To obtain speed at the least cost has been one of our ever present problems and it is in this area that our experimental towing tanks have made their greatest contribution to our work in hull form and propeller design. The price of ship speed is often not fully realized, especially when we note the tremendous progress that has been made in the speed of airplanes. Because of this progress in aviation it is often wondered why ships seem to be standing still in this matter of speed increase. The answer is quite

simple. For example, to double the speed of a 50,000 ton displacement ship from 30 knots to 60 knots would require an increase of power by a factor of about ten. Which in this case would mean increasing the power from 140,000 to 1,400,000. In this example the displacement was assumed constant. This, of course, could not be so unless the increased power and fuel requirements could be met without increase in weight and space. To some degree this has been the case in surface ships because of the improvements, over the years, in the reduction of the machinery fuel rate and weight per shaft horsepower. Much of our increase in the speed of surface ships can be attributed to the improvements in the machinery plant rather than to any recently gained knowledge of ship hydrodynamics. On the other hand, the speeds of submarines have been increased because of recent work concerning hydrodynamics of submarine hull forms.

Frictional or viscous resistance has been under constant study since 1872 [1], [2], [3], [4], and we are still in doubt as to the exact nature of this phenomenon. While the correct understanding of frictional resistance cannot be considered a serious barrier, its importance should be recognized. This kind of resistance [3] accounts for some 50 per cent of the total resistance of many of our surface warships. Ship trials have indicated that the effect of different paints may cause as much as 15 per cent variance in total power, or some 30 per cent in skin friction. When the amount of power being developed in such ships to overcome frictional resistance is considered, the necessity for an understanding of the mechanism of skin friction should be obvious. Not only does the nature of the surface affect the drag but the wake is also changed, which is reflected in differences in the propeller performance. In the case of submarines practically all of the submerged resistance is frictional or viscous drag. To reduce this drag has intrigued many research minds and some thought has been given to the possibility of maintaining laminar flow by boundary layer control.

It is in the area of wave-making resistance where the surface ship meets some of the most formidable barriers, and it has been one of our great aims in life to avoid or reduce this type of resistance. The determination of the wave-making resistance has been one of our most active fields for theoretical and applied research. We have had to rely entirely on the results of model tests, singly and in groups or series, for the specific values of the effects of wave making on resistance. The basic problem results from the failure of the residual resistance coefficients to follow a simple law for different speeds and hull forms because of the change in the flow pattern with speed. Remarkable results have been obtained from the application of mathematical methods to the investigation of wave-making resistance of well-defined ship hull forms. The results of these investigations have reached a stage in which they have been correlated by experiment; but direct application to problems of actual surface ships have not been realized as yet [5], [6], [7].

An important property which has been well established is that maximum values of the residual resistance occur in Froude number (speed-length ratios) ranges of practical interest. The critical $V/\sqrt{L}$ values at which the maxima or humps occur are close to 0.8, 1.0, 1.5. Most of our large ships are limited in speed because of the great increase in power required to drive them beyond the speed-length range of 1 to 1.25. At the present time we see little likelihood of breaking down this barrier to high speed in large ships. As a point of interest I would like to mention that the fast ocean liners operate at speeds close to $V/\sqrt{L}$ of 1.0 in smooth water.

The two most obvious motions of a ship in a seaway are pitching and rolling. Rolling can be the more violent of the two and in a heavy sea a captain will change course to reduce this violent motion. Such a course change will result in increased pitching but this can be controlled to some extent by speed reduction. In rough water a surface ship may be forced to slow down for several reasons and pitching is probably the principal one. Pitching exaggerates nearly all causes of speed loss. It leads to shipping water over the bow, slamming, and increased resistance. The point of greatest

loss of speed and the greatest amplitude of pitching occurs as the wave lengths approach the ship's length, [8], [9], [10].

On the basis of our present knowledge there is little we can do to greatly improve the rough water performance of a conventionally designed ship. The evidence, though meager, seems to indicate that minor changes in hull form, based on optimum performance in smooth water, will not materially improve the rough water performance. Obviously we must consider major changes in concept in order to attain, for example, the same smooth performance for a surface ship in rough water as for a submerged submarine. Much thought is being given to this problem, as evidenced by recent papers on ship speeds and motions in regular and irregular seas [11], [12]. It is possible that we may be at the threshold of some practical progress in this area [9], [13], [14]. Successful control of rolling, to a reasonable degree, may be accomplished by means of controlled fins [15], [16], [17]. Pitch damping is also under study and some success seems indicated by the use of hydrofoils at the bow. These and major changes in hull form above and below the waterline may produce a surface ship with considerably improved performance characteristics in rough weather.

The manuevering of ships has always received much practical attention. For many years what little we knew came from full scale tactical trials which were quite common with naval vessels. From the designer's standpoint little theoretical advance was made in the way of making it possible to calculate the turning and steering properties of a ship. At the present time it is still necessary to rely upon an empirical approach to the calculation of tactical diameter and directional stability [18], which is an important element of course keeping. Model testing techniques have been developed to a point where we can determine, with a high degree of confidence, the full scale performance of a ship or submarine. However, it should be noted that here again, as in wave making resistance we depend on the model test results rather than on a theoretical solution of the turning problem.

It is still practically impossible to determine, in the absence of specific model tests, the force and moment characteristics of a ship [19]. The problem is complicated by the fact that ships operate at the interface of two different fluid media and also by the complex geometry of ship forms. Complete theoretical treatment is therefore rendered very difficult.

Great advances in the field of electronics have also affected our outlook and trends in design. Electronics has made possible completely new evolution in missile design, provided long range electronic eyes and ears which have affected the topside as well as the under water parts of ship designs. The influence of radar is readily apparent by comparing the topside structure of any modern naval surface ship with one of 20 years ago. Sonar likewise is revolutionizing the design of naval vessels. The capabilities of electronics have influenced the need for control of the motions of surface ships and below the sea surface they have brought about the demand for reduced noise, from any source, which may be transmitted through the water. While much noise is associated with machinery and other components in the ship it is the hydrodynamic noise that is one of our main concerns. This noise is caused principally by flow conditions which produce cavitation, separation, and vortices. This is one hydrodynamic barrier which is particularly important to the submarine.

The exact mechanism associated with the generation of noise by flow over the hull of a ship is not too well understood. Flow into the propeller, and propeller location relative to the hull and other structures, are known to be important elements in the noise problem. The problem is so complex that thorough theoretical research is needed to guide our progress in noise reduction.

The determination of the physical quantities which influence cavitation inception is another one of our important needs [20], [21], [22]. The development of hydrofoil shapes in which cavitation is delayed or avoided entirely has shown much progress due to the work done by the aeronautical laboratories and by our model towing basins. Cavitation is an important area for continued study particularly in connection with

noise problems, our efforts to reduce cavitation erosion, the improved performance of control surfaces and the design of flight foils of hydrofoil craft.

We have always had a great deal of help from many contributors on the theoretical side of naval architecture. Their analytical efforts have increased our understanding and confidence concerning the empirical methods we are accustomed to use by necessity. Unfortunately, however, the laboriousness of the calculations and the inability to readily fit the calculations to our actual hull forms has been a great barrier to the advance of theoretical and mathematical solutions to our problems. It does appear that the burdens of laborious calculations can now be lifted from the human computer or "humac," as we delight in classifying him, to the more rapid electronic computers such as "Univac," "Busac," etc. Now this may appear as being a case of not using what we have and know already. This is not so. For example, even where we know the definite theory and mathematics, as we do in the hydrostatics of ship design, it has been found that electronic computers at present are of but little help to the naval architect.

In the Bureau of Ships we have designed an electronic computer to help solve some of our problems in hydrostatics but at present only in a manner similar to mechanical integration. What appears to be required in the case of hull form design is a completely mathematical method of designing.

A mathematical method of defining hull form would undoubtedly tie in with developments in the hydrodynamic theory and, by a process of optimization of essential elements, the hull form parameters could be directly selected by the naval architect to meet his requirements. Progress in this direction has been slow considering the many years during which effort has been expended on this problem. However, with the many newly developed mathematical tools, particularly in the realm of advanced statistical analyses, and with the development of electronic computers, analogues, and allied devices, it at last begins to appear that we may yet have an essentially mathematical approach to hull form and the solution of problems concerning speed, ship motions and predictions of other performance features.

The lack of basically accurate information has not handicapped the naval architect in the past because there was always enough approximate data available for him to carry on in a reasonably efficient fashion. The barriers he was faced with were concerned with modest speed and other performance requirements.

An important point which should be stressed is that theory alone does not help the naval architect in creating a basic design. He must have, in addition, some basic numbers that he can latch onto in formulating the basic design concept of a ship. These numbers must be related to each other by a set of parameters which are basic to the phenomena which are to be emphasized in the design.

It is realized that the steps from theory, to basic parameters, to numbers are probably the most difficult and monotonous jobs in research work. However, once the proper parameters and numbers have been established, it is the most rewarding part of research work insofar as the naval architect is concerned. The Froudes, Wm. and his son, R. E., and D. W. Taylor had the genius to express their findings in terms of parameters which made it possible for the naval architect to detect trends and to correlate his own meager data collected from smooth water tests of full scale ships and models. To this day much of our work in our experimental laboratories has been a strengthening of their original concepts.

At the present time there is under preparation a complete review by Captain H. E. Saunders of hydrodynamics in ship design [23]. These texts should provide us with the most complete assembly of reference material for the naval architect that is available anywhere. It is believed, however, that while some additional light will be given on the barriers referred to in this paper, the areas of uncertainties and need for further study will still be largely as has been indicated in this discussion.

In closing let me suggest that some of our hydrodynamic barriers might be reduced by developments outside the field of hydrodynamics. Take the case of the limited

endurance which has long plagued the designer of surface ships and submarines. With the advent of nuclear propulsion, this barrier has been largely circumvented. We have only begun the exploitation of nuclear propulsion in submarines, but just this one development has given us a type of ship relatively free from hydrodynamic barriers.

From the foregoing discussion it is apparent that the several barriers to progress in naval architecture are largely interrelated and quite complex. This is true of most scientific work when it once passes the rudimentary stages. It almost seems that 90% of the progress is made in the very first crude step. The man who evolved the formula

for the flexural stress in a beam, for example, $S = \frac{My}{I}$ provided a theoretical tool which

permitted the creation of many structural wonders, and all the modern research of the past 50 years in this field has refined our design approach but little. The parallel with hydrodynamics is obscure because the latter science has proven more complex and less tractable to simplifying assumptions, but it is real nevertheless, and the initial strides in understanding made by Froude, Taylor and others are not likely to be equalled by all our high powered modern techniques. We have only the 10% which they left us to play with. The challenge nevertheless is great and the 10% looms large in the highly competitive world of today.

As a final thrust for the future I would like to present a few questions which were asked by Admiral E. L. Cochrane in 1954, as a means of emphasizing the problems we face in this matter of barriers and the work that needs to be done.

What are the forces acting on a ship at sea as compared to those acting upon a model in an experimental towing tank?

What does happen to a ship's speed in rough weather?

What happens to a naval vessel's weapons platform or flight deck in a storm?

What does happen to a ship in a really rough storm?

These are not new thoughts, in fact they are very old, probably as old as when man first ventured out to sea. Questions similar to these must have been in the mind of Brunel, the designer of the "Great Eastern," when he requested William Froude in 1860 [24], to investigate the probable sea-going characteristics of that ship.

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Abbreviations

INA	Transactions of the Institution of Naval Architects, London, England.
SNAME	Transactions of the Society of Naval Architects and Marine Engineers, New York.
NEIES	North-East Institute of Engineers and Shipbuilders in Scotland.

DISCUSSION

H. E. Saunders

If I may seem a bit critical of some of the author's remarks it is not because we have a different basic thought on these items, but because we look at them a little differently.

Mr. Niedermair said that certain hydrodynamic barriers which exist for the surface ship vanish in the case of the submarine. This is a pure submarine, entirely submerged.

He mentioned hydrodynamic noise, which I think all of you can visualize. However, if you think that just going down in a pure submarine, and getting below the surface, eliminates some of these problems, you might try calculating the submergence depth at which separation at some point disappears, if it does disappear. I am talking now of a separation which occurs at some place on the submarine when it is at or near the surface. Just try this for breaking one of the hydrodynamic barriers.

It is true that in the old days naval architects got along with very little in the way of an analytic approach, relying on practical experience and intuition. That was possible, because if the ship didn't behave the way you wanted it to, well, you just stopped and tried to figure out what to do next. But unfortunately, the airplane designer, or pilot, if he gets up in an airplane, can't quite do that. He has to keep on flying and to figure out, if he can, what is going to happen until he gets back down to earth again.

Mr. Niedermair made the comment that it was the development and the use of the analytic approach in the aircraft industry, and all of the things that went with it, which stimulated us to do likewise in naval architecture. I would like to think that this is so, but there is a long period of relative inaction between what the airplane people were doing back in the 1910's and what we are doing now in the 1950's.

What really poked us from behind, to use a frank term, was the realization of the fact that the empirical approach was just no longer adequate. We had problems to be solved which could not be solved by the engineering knowledge then in our posses-

sion. Nor could we solve them by intuition, or from the seat of our pants, if you want to put it that way. We had to find some better way of getting answers.

These problems faced us very critically during the past war. Perhaps because of this situation a great many of us realized that we had to start *thinking* about the problems; we had to apply the analytic approach. And that is what we are doing now, thanks to the efforts of a great many people and to conferences of this kind.

What Mr. Niedermair said about rough-water performance is true, namely that minor changes in the form of a ship do not appreciably affect its behavior in a seaway. Nevertheless, we have in this country, in the last few years, produced a ship, a big ship, which has phenomenally good rough-water performance, particularly in regard to maintaining speed in heavy weather. This ship not only can go faster than any other large ship in smooth water, but she can pretty much match that performance in anything that the North Atlantic has so far been able to put in front of her.

Not many of you remember or realize that over forty years ago we did the same thing with a pair of little ships called the *Great Northern* and *Northern Pacific*. They were built before the first World War. These ships were not large ships; they did not have the length that was mentioned here yesterday. Their water line length was only of the order of 520 feet. Nevertheless, these ships in two wars, on the West Coast and the East Coast, in some of the worst weather in the world, were able to maintain schedule in good weather and bad. As a matter of fact, they were able to do almost as well as the *Leviathan* in the first World War, our yardstick in those days. There was definitely something about these little ships that made them good.

One might say that in maintaining speed, dryness, and so on we have surmounted this hydrodynamic barrier. But we haven't, because we don't know what it is that makes these few ships so good. We can't put our finger on it. We have ideas, but not very many of us agree on anyone idea. And we won't produce the better ships until we know why the ships already built were so good.

The *Albacore* was mentioned here yesterday. She is a remarkable pure submarine, no question about it. But the problem of designing that ship for low resistance was not particularly acute, because we had a great deal of previous experimentation, previous knowledge back of us in the way of work on airships, in the earlier years of this century. We knew more or less what an optimum pure submarine would look like. The *Albacore*, incidentally, isn't too different in shape from the submarines that were built over 50 years ago by J. P. Holland.

Now the thing we don't know, and the thing we must know, or must find out, is what does the optimum wavegoing ship look like? Suppose we had no considerations of cost, no considerations of displacement, no considerations of this, that, and the other thing, what is the optimum form of ship for traveling in waves? That we do not know, and that we must know some time or other.

There has been a lot of talk about electronic computers, but don't let us forget that, in general, in order to work an electronic computer we have to have some mathematical formulas. In order to derive the mathematical formulas we must know the physical processes. In many cases I am afraid that we have not known the physical processes too well before we ventured out on very thin ice with the mathematics. So we had to go back and try over again.

One last item has to do with this matter of hydrodynamic barriers on the new type of submarines. I am talking about the ones that are not atomic-powered, but atomic-fueled only. Propulsion of these vessels is still hydrodynamic.

Perhaps I can best explain what hydrodynamic barriers we are up against by pointing out to you that we work in a very limited depth of water with any of these submarines.

Consider that you are a pilot of a big new fighter plane in the next national emergency. This plane is as large as an existing transport plane. You and your crew are supposed to carry out some attacks, in the course of which you have to dodge shell fire, and to dodge all sorts of missiles down close to the earth. All this time you have

the certain knowledge that if you climb, with that airplane, more than say 500 feet above the earth, it will surely explode. One such climb, to too high an altitude, will mean the end of you and your crew, and of the airplane.

This is the sort of problem which we still face with the submarine, regardless of the type of propulsion, or the type of fuel, or the type of power. I am sure that the aerodynamicists and aeronautical engineers, if they were faced with a problem like this, would be very much busier than they are today.

J. F. Allan

I congratulate Mr. Niedermair on his paper, and particularly on the title, which I found stimulating and speculative.

There is an old saying that an engineer can solve any problem in the technical sense, provided he is given time, but time, of course, costs money, and that brings us up against the economic aspect of the matter.

There are two approaches to the situation, one from the service point of view, and the other from the commercial point of view. These two approaches tend to be very different, as from the defense point of view economy is not the first consideration even in times of peace.

When we come to the commercial approach, at least in my country, the ship owner is not interested in technical developments unless he can see that they are going to lead to a clear commercial benefit in what is a very highly competitive industry. It is against that background that I tend to view this question of hydrodynamic barriers. These barriers may also be looked at from two angles. One is concerned with the improvement of existing techniques, and this is of particular interest to the commercial user of ships, and also, of course, to the service user of ships. The other is the more spectacular and revolutionary approach, where quite new techniques may be proposed, and successfully introduced, and these are of primary importance from the service point of view. In many cases they lead to developments which are of great value from the commercial point of view.

The author stated that the improvements which had been provided in propulsion of ships by the marine engineer were much more spectacular than those provided by the naval architect. One must agree with that statement, but we should not forget that the naval architect has been responsible for improvements of the order of 30 per cent in the last 25 or 30 years, and that is no mean achievement.

I mention the friction barrier, which is a very important contributor to the total resistance of the ship, and although a great deal has been done in recent years to reach a better understanding and appreciation of the importance of a smooth surface, I think we still have quite a lot to learn, perhaps not so much in producing an initially smooth ship, but in preserving that smoothness over the life of the ship. It may even be possible that spectacular developments will come along to reduce friction much further. We do not fully understand the detailed mechanism of fluid friction. As regards wave-making resistance that, I am afraid, will always be with us on surface ships, and the only thing one can do is either go down below or get up above.

We have had quite a lot of comments today and yesterday about the remarkable performance of the latest submarine down below, and it may be that there is a future in that direction for cargo carrying, but it would be difficult to persuade passengers to travel in a high speed submarine. The alternative is to lift the vessel above the water on hydrofoils, and considerable success has been achieved in that direction. There may be service applications for such vessels, but I doubt if there will be much big scale commercial application. If you want to lift the boat out of the water, it seems to me the obvious thing to do is to get right up in the air and fly properly.

As regards sea speed, there may be some very interesting developments in that direction in the next few years. Captain Saunders mentioned several American ships

which he considered have remarkable sea-going performances, but he admitted that nobody understood why these ships were outstanding.

Roll stabilization has been successfully achieved in the United Kingdom, and the system is now being applied to American ships. Pitch stabilization is more difficult, but a useful degree of pitch damping is possible. With these things accomplished it will probably be reasonable and economic to drive at full speed in bad weather conditions.

I mention again the point I made yesterday, that if a ship is being driven hard in such conditions, it is of the very greatest importance that the control mechanism on roll and pitch stabilization does not fail, otherwise the ship could be in serious difficulty.

May I conclude by recalling a speech which was made at a dinner here five years ago in connection with the International Towing Tank Conference. The then Secretary of the Navy cited the remarkable developments in aircraft, and rather threw down the gauntlet in regard to ships. At the time, the proposals he was putting forward seemed to me rather fantastic, but it may be that in the course of time we will get some way towards achieving them.

F. H. Todd

Mr. Niedermair has given us an excellent review of some of the problems facing the naval architect today.

As many of you here today are working in other fields than ship design, I would like to focus your thoughts on the ship problem by considering the design of a large passenger vessel. The civil engineer has to face many problems in the design of modern bridges, the architect and the civil engineer have to combine their skills in the erection of many modern buildings, whilst the mechanical engineer is concerned with all forms of power plants and other machinery. In the design of a large passenger vessel, the naval architect is faced with all these problems at once, and generally in a more acute form. The ship hull is a much more complicated structure than a bridge because of the variation which is necessary along its length to accommodate various machinery, boiler and other spaces, the accommodation has to include all the amenities of a first-class hotel whilst circumscribed between the sides of the ship's hull, and the machinery plant has to have power equivalent to that necessary to supply a small city with electricity. Add to these the fact that this whole structure has to cross rough and stormy seas at a speed of 30 knots or more, safely, economically and to time table, and you begin to get a picture of the problems which the naval architect has to face and, indeed, has overcome with considerable success in the past.

At one place in his paper, Mr. Niedermair refers to Ship Hydrodynamics as being "a plaything for the theoretical minded." When asked what I do, my wife is prone to reply "research" and when asked to describe this, she describes it as "organized play for adults." As a player in this game, I would like to comment on a few of the points which Mr. Niedermair has raised and on which research work is presently being carried out.

The author has drawn attention to the great effect which frictional resistance has on a ship's performance. For a great number of years now, much effort has been devoted to obtaining surfaces with as low a resistance as possible. It would seem that as long as a surface is very smooth and hard, its exact nature has little, if any, effect upon the resistance. With a hard, smooth paint of this kind, most of the excess frictional resistance on a new, clean ship has in the past been due to seams, laps and rivet points. With the advent of welding, this "structural roughness" resistance has been greatly reduced, particularly where care has been taken to grind off the welds and make them as smooth as possible. On clean, new ships with this type of surface, the additional structural roughness resistance appears to be very small and the resistance approaches that of a smooth plane surface. Two things can spoil this effect. Anti-fouling paints if carelessly applied can increase the resistance materially, as noted by

Mr. Niedermair. Also, for vessels which spend a considerable portion of their lives at anchor, the resultant fouling of the surfaces by marine organisms can increase the resistance to a very great extent. Our aeronautical friends have achieved considerable success in aircraft by the use of laminar flow wings and the question is often asked whether advantage can be taken of this idea in ships. The degree of smoothness required to maintain laminar flow is exceedingly high and it is doubtful if it could ever be achieved on a ship's hull. Even if it were, excessive care would have to be taken to prevent fouling, as this would rapidly eliminate any chances of maintaining such a flow. Moreover, in the case of surface ships it is probable that the sea near the surface is so turbulent that the achievement of laminar flow is physically impossible anyway. Perhaps the best hope for such success would be in submarines or torpedoes running very deep, where the level of turbulence in the ocean might be very low. There have been many other suggestions for reducing frictional resistance, including the use of soft surfaces similar to those of fish, of various liquids which would be slowly shed and of blankets of air bubbles. It has yet to be demonstrated that any of these can produce lower drags than a hard, clean, smooth paint.

The other important component of resistance in a surface ship is the wave-making. The amount of this is very small at low speeds, but increases very rapidly at higher speeds and makes the achievement of these speeds very costly. There are two possible ways to overcome this barrier: to go below the surface or to lift the main body of the ship clear of the water by means of hydrofoils. The submerged cargo ship has been considered many times, and there are certain cases where such a craft would be invaluable for military use. A commercial, completely-submersible ship would have many handicaps to overcome in comparison with a surface craft. The hull would be much heavier and more complicated, and for the same available deadweight would have to be considerably larger than the corresponding surface ship. Because of this and the fact that the ship would be completely submerged, the wetted surface would be much greater and therefore at low speeds the resistance would be considerably higher. The advantage of complete submersion on the wavemaking resistance would therefore not appear until relatively high speeds were reached; then the saving in wave-making would be such as to offset the initial increase in skin friction. There would also be considerable difficulties in the handling of cargo in such a craft, and the most promising developments in this respect would be for oil-carrying vessels.

Lifting the body of the craft out of the water with hydrofoils also introduces many problems in anything but small size ships. In larger vessels capable of carrying any appreciable deadweight, the problem of the size and weight of the foils and their supports becomes a serious one, as does the excessive draft in port when not foil-borne. As such craft, to be of any value, would have to be high speed vessels, the problem of power and its transmission to propellers located deep below the craft also becomes extremely difficult.

In the realm of ship motions, great efforts have been devoted in the past to the reduction of rolling. This is very desirable for military vessels today in view of the many delicate instruments and weapons which they carry. As regards comfort in passenger vessels, it has always been my personal experience that pitching is far more uncomfortable than rolling. Some years ago the Model Basin collaborated with the Naval Medical Services in an effort to determine the cause of seasickness. We supplied instruments and observers to record the pitch, roll, heave and accelerations on certain Navy transports running between New York and Europe. The Medical Corps provided doctors who administered different anti-seasickness drugs to the troops in the various compartments on the ship. There were many confusing features in the results, but one fact certainly did seem to be quite clear. As the weather deteriorated, the first cases of sickness appeared at the ends of the ship; as the weather got worse, the degree of sickness in the end compartments increased and the incidence spread inwards towards midships. An analysis of the results appeared to give very good correlation between the incidence of seasickness and the vertical acceleration. Any attempt

to reduce pitching motion by means of fins or other devices located towards the ends of the ship introduces much more severe problems than the reduction of rolling because of the much greater forces and leverages involved. It is interesting to note that despite these problems serious consideration is now being given to the fitting of such fins, and we have carried out experiments at the Model Basin which indicate that reductions in pitching can be achieved by such means and that the forces acting on the fins, while large, can still probably be handled. Of course, the achievement of substantial reductions in pitch by such fins will contribute not only to greater comfort on the ship but will also minimize the reduction of speed, the damage due to slamming, and if the shipping of green seas can be avoided, then damage to superstructures should also be greatly reduced.

The recent emphasis on rough water performance has demonstrated that reduction in speed is normally dictated by the necessity of avoiding excessive ship motion rather than by lack of power. The choice of a ship's lines should no longer be made solely on the basis of good smooth water performance. It may well be that some of the latter will have to be sacrificed in order to achieve the best performance under average sea conditions. In the relatively slow speed cargo ship, for example, the best smooth water performance can generally be achieved using full bow lines with a very fine stern. If this tendency is overdone, however, the ship may give disappointing results at sea and lose speed quite rapidly. Research into the seakeeping qualities of ships has been greatly handicapped through the lack of proper test facilities but fortunately this is now being remedied.

Although Mr. Niedermair's paper is concerned with hydrodynamic barriers of ship design, I think that the overcoming of such barriers is partly dependent on overcoming some economic barriers. Mr. Niedermair comments on the rapid strides made by aeronautical engineers and this is in large measure due to the immense sums of money that have been spent in research and development in the field of aeronautics. In comparison, the money devoted to research in hydrodynamics is almost negligible. In the matter of experimental craft, too, the naval architect has been much too conservative in the past. The aircraft industry thinks nothing of building experimental craft at great cost to try out new ideas, but most new ships, whether of naval or merchant type, are usually expected to pay their way when completed. In recent years, the value of a ship designed primarily as an experimental craft has been well shown by the success achieved with the submarine ALBACORE. It is to be hoped that in the future more experimental ships will be built in order that the naval architect may have opportunities similar to those of his aeronautical colleague to try out new ideas without the burden of having to make the ship pay its way in the military or commercial sense.

H. de Luce

As Mr. Niedermair brings out in his paper, we Naval Architects actively engaged in designing ships have relied almost entirely on full scale experience and on empirical research. On this basis, over the years, we have succeeded reasonably well in producing war ships and merchant vessels of increasing size, speed, and fitness for their intended services.

I have no doubt but that empiricism will continue to provide the solutions to our problems for some time in the future. On the other hand, the interpretation and evaluation of empirical data and experience may certainly be misleading without theory to establish the significant parameters and their relationships. And, looking to the future, it seems to me to be only reasonable to expect that the configuration of ships will undergo as radical a change as has occurred in aircraft with the advent of supersonic speed. In developing this ship of the future, we are certainly going to need all the guidance we can get from fundamental theory and basic research.

I think all of us interested in the design of ships can feel great satisfaction in

this symposium. To me its significance lies not only in its coordinating force, in terms of hydrodynamic research, but in stimulating such research and its practical application by bringing together physicists, mathematicians, experimenters and engineers. We must improve the communication between these groups if we are to properly define our problems, plan the appropriate analytical and experimental research, and then apply, in engineering terms, the results of this research. Our several talents and interests are all needed if we are to produce ships to best serve the future military and economic needs of the Navy and the Merchant Marine.

May I say that I feel it particularly appropriate that a naval architect of Mr. Niedermair's stature should present a paper before this symposium. May I also say I am very happy to have this opportunity to make these brief remarks.

K. S. M. Davidson

I am afraid I must take issue, not so much with the statements in this paper, as with one or two of the implications. I dare say John Niedermair suspected I might.

In the first place, I do not like the word "barrier." I do not think the aeronautical people were very smart when they coined that word, and then very soon pierced the barrier they were talking about. The naval architects are not confronted with anything like a real barrier to increasing the speeds of ships. A good deal of power is needed, but we have in fairly advanced states of development various means of getting much more power without materially increasing the machinery weights. One hears people these days talking soberly of building inter-continental missiles and satellites that will circle the earth. Yet when one talks of increasing ship speeds by 25 or 50 per cent, the same people seem apt to think that the world is coming to an end.

Personally, I do not believe that there is a very great difference between commercial and military requirements for speed. It is a matter of second-order differences. I think it is high time that the Western Allied Powers began to give ships enough speed to permit them to run away from submarines. We nearly lost two wars because of submarines and everybody knows that submarine speeds are going up. I say that it is quite feasible to give cargo ships very much greater speeds.

I promise to push as hard as I can to get things going, if somebody will set to work to design a really fast cargo ship, instead of talking about how difficult it is to do.

Mr. Niedermair said that ten times the power is needed if speed is increased from 30 to 60 knots. That is true, provided however that one insists upon building the same ship. But the knowledge has been available for a long time that reducing the fullness of the hull reduces the wave-making resistance enormously. I do not have exact figures here, but if one does not insist upon 60 knots and is satisfied with say 45 (which is, after all, a fairly respectable speed) and if one makes the hull a good deal skinnier than is usual (which is not impossible to do), less than five times the power is required.

I submit that there is a considerable incentive to go this far, first from the military point of view, and second from the commercial point of view. There are various indications from recent studies to show that the cost will be less than many people are inclined to think.

I would like to see work started now on the design of a cargo (not a passenger) ship of 45 knots, and I would like to see plans laid to build such a ship very soon, as an experimental ship looking to the future. It would make a challenging research target.

R. W. L. Gawn

We are fortunate in having such a searching paper from Mr. Niedermair which reflects his outstanding ability, ripe experience and fine achievement in ship design.

Much research by theory and experiment has been devoted to the improvement of the operational qualities of ships. Emphasis is strongly veering towards the development of the best shape of hull and distribution of load for seagoing conditions rather than for calm water, to which much attention has been devoted in the past. This is evident from the new facilities being constructed in America and Great Britain and that recently completed in the Netherlands for realistic investigations of seaworthiness and ship motion. There is also a rapidly expanding interest in cavitation and additional facilities for investigating this phenomenon are being provided in various countries. The boundaries of research are being thus extended in order that ships may clear the hydrodynamic barriers as far as possible. We may look forward to faster, steadier, handier and more economical ships, although advance will continue to be circumscribed by the physical limits imposed by the density and viscosity of water and of air and of the vapour tension, gas content and surface tension of water.

Much fuel is squandered by ships in overcoming the skin friction resistance of the hull and to a smaller extent the propeller. There has been an improvement in this respect of about 10 per cent following the advent of welded hulls. If a ship could be built as smooth say as bonded plastic asbestos, research informs us that the frictional resistance of a clean hull might be further reduced by say 10 per cent. Luxurious animal and vegetable life thrives on a ship's hull to such an extent that the frictional resistance may be increased by 25 per cent when six months out of dock. This is about one half the increase that obtained some years ago, which is one indication of the achievement of research. Thus the door to the barrier of viscosity is being pushed slowly open.

The inventor seeks to break through by clothing the ship with air but is defeated by the excessive power of the necessary air compressors. Laminar flow and boundary layer control offer entrancing theoretical prospects but formidable problems are presented including a revolution in shipyard practice to obtain the necessary smoothness of hull finish and the annihilation of the barnacle and its fellow travellers.

Rough weather can severely limit the overall operational performance of a ship due to the consequential oscillations in the six degrees of freedom, shipping heaving seas over the bow and exposed operational positions, and by reducing speed.

It may be necessary to ease the speed below that obtainable from the propelling machinery to reduce the ship motion and prevent structural damage. Generally conditions are most severe in seas of a length somewhat greater than that of the ship, for example, studies on a model of a carrier about 700 feet long indicated that the speed would be a minimum in steep seas about 800 to 900 feet long and about 40 per cent of the full speed. Ninety-five per cent of the full speed could be obtained in waves about half the critical length. By and large the severity of the motion changes similarly with the length of wave. Detailed investigation indicated that the probability of the carrier meeting conditions limiting flying operations was on a conservative basis 18.5 per cent in the North Atlantic and 12 per cent as an average of all the oceans of the world.

If the ship were twice the size, namely 1400 feet length the probability would be expected to be about halved, although this has not been examined in detail. Thus a definite if not very striking gain in sea-keeping qualities would be expected if the size of ship could be increased well above the present maximum length of about 1000 feet.

Ships generally are much smaller than the few giants of about 1000 feet length. The probability of objectionable motion is greatly enhanced as critical seas are more frequently encountered. Model experiments have shown that the pitch and heave amplitudes can be reduced by a worthwhile amount by suitable shape of hull ('V' sections forward and flat sections aft are one example) and by about 50 per cent if longish passive fins are fitted. The damping varied with speed of model and size of wave. The relative gain was less in short waves but the motion was then small. Thus there could be a considerable advance if the fin device were considered practicable,

although this would be well short of the remarkable achievement of active fins in reducing roll. Comparable improvement in pitch and heave could, in principle, be obtained from active stabilization but at the expense of excessive size and weight of the equipment. Thus for the present it is suggested that the search should be for limited stabilization of surface ships rather than the ideal target of complete stabilization. The wave barrier is of course completely avoided by the submarine when operating at deep submersion.

Man in the urge for speed has succeeded in travelling nearly 200 knots on calm water but only as a record sprint. Thirty to 35 knots is about the present limit for large ships and the speed of intermediate and small vessels is commensurate with 40 knots. Confronted by ocean waves, speed is greatly reduced and very few ships have succeeded in crossing the Atlantic at 30 knots. The price of speed is high, the real physical barrier being the large density of water. None will deny the scope for improvement both in steadiness and speed. The Author's paper and the wide net cast by this Symposium encourages the hope that achievement will not fail for lack of skillful effort.

M. C. Eames

Mr. Niedermair, and some previous speakers, have made reference to the hydrofoil craft. As is likely to be better known by readers of popular picture magazines than by students of technical transactions, the Defense Research Board of Canada is sponsoring a research project on the development of a particular type of hydrofoil craft at its Naval Research Establishment at Halifax. Since in the design of the craft, an entirely new form of hydrodynamic "barrier" has to be considered, it is felt that the following notes may be of interest.

While the most ardent supporter of the hydrofoil craft would agree with Dr. Allan and others, that in general if one wishes to attain high speeds, one should take to the air completely, there nevertheless exists a regime in the speed-size field of transportation where the hydrofoil craft can show a definite advantage in terms of power economics. Moreover, certain military advantages are apparent outside this region where the hydrofoil craft presents the most economical means of transport.

The existence of such a regime, and the state of the art up to 1953 have been comprehensively recorded in a paper by Buerman, Leehey and Stilwell before the Society of Naval Architects and Marine Engineers in 1953, and the Institute of Marine Engineers in 1954. In general it may be said that subsequent advances have been a matter of detailed improvement and the collection of data rather than one of basic changes of principle.

Mr. Niedermair has referred to the "barrier" in speed which results from cavitation of the lifting hydrofoils. Particularly in surface-piercing hydrofoil configurations, however, a further "barrier" can present problems at a lower speed than that at which the hydrofoils begin to cavitate. Drawing an over-simplified picture, it is clear that cavitation will commence when the pressure on the upper surface of a hydrofoil has dropped to the vapor pressure of the surrounding water. However, in general a much lower speed will suffice to reduce the pressure on the upper surface to that of the atmosphere, and once the pressure falls below this value the phenomenon of ventilation can occur. This is the process by which air is drawn down the upper surface of the hydrofoil, possibly via a supporting strut, resulting in much the same effect on performance as a well developed state of cavitation.

The mechanics of ventilation has received little fundamental study in the past, and the process is not yet fully understood. While means of prevention and cure for particular hydrofoil craft have been successfully developed, the "barrier" cannot be said to have been truly overcome. The essential reason for this statement is that the scaling laws applicable to the phenomenon remain incompletely understood, and until they are fully appreciated the extrapolation of proven results and methods on small test hydrofoil craft to full scale prototypes will be subject to a degree of uncertainty.

It is in the hope of stimulating further interest in the study of ventilation that these remarks are made, and it is considered a significant addition to Mr. Niedermair's most interesting survey because it is probably the least understood of all hydrodynamic "barriers."

E. Moberg

Mr. Niedermair's excellent talk has brought to mind a hydrodynamic barrier which recently appeared and eventually was lifted, and is the type of practical problem which I believe will interest those who are not already familiar with it. A recently designed ship experienced a severe hull vibration during her builders trials, and also on subsequent trips. This vibration appeared at about $\frac{3}{4}$ speed, and at top speed increased to a severe level about 4 times the lower value. It was a constant frequency hull mode which was being excited. This mode was of lower frequency than either the shaft rotational or blade impulse frequency, which are the more common troublesome hull vibration frequencies. Considerable instrumentation and several sea trials later it was agreed by all involved that the rudders appeared to be transmitting the exciting forces to the hull. These ships are twin screw two-rudder ships, and the rudders were originally designed so that the rudder trailing edges were toed in forming an angle of 3° with the ship's centerline. As one attempt at a solution, adjustable rudder links were made so that the toe-in angle of the rudders could be varied. A series of runs, with toe-in angles varying from $4\frac{1}{2}^\circ$ toe-in to 3° toe-out was conducted. These produced a family of curves which showed very severe vibration at $4\frac{1}{2}^\circ$ toe-in and a quiet ship between $1\frac{1}{2}^\circ$ and 3° toe-out. Subsequently, the rudder angles were set at $1\frac{1}{2}^\circ$ toe-out, and no further vibration trouble has been experienced. Upon drydocking of this ship after only 1800 hours sea operation, about 20 of which were in the upper $\frac{1}{4}$ speed range, it was seen that nearly all of the paint had been removed from the suction faces of the rudders, almost as if sand-blasted, while on the pressure face only small areas in the maximum thickness section had been removed.

In closing, I will say that such problems of this nature, and their solutions, should, and do, stimulate research so that practical experience can be related to theoretical and empirical data, thus providing ship designers with more effective tools with which to work.

J. C. Niedermair

The primary objective of the paper was to indicate the many uncertainties which exist in the basic knowledge of hydrodynamics related to the design and performance of ships. This objective is amply supported by those who so kindly contributed to the discussion of the paper. Some exceptions were taken to a few of the statements made in the paper. These exceptions were particularly in regard to the cost of speed and the emphasis on the lack of basic theory. Fortunately, there is no disagreement in principle but only with the relative degree of importance which should be placed on the various "barriers".

I am grateful for the comments contributed by Dr. R. W. L. Gawn, chairman of the session at which this paper was presented, Captain H. E. Saunders, U.S.N. (ret'd), Dr. J. F. Allan, Dr. K. S. M. Davidson, Dr. F. H. Todd, Mr. H. deLuce, Mr. M. C. Eames, and Mr. E. Moberg. Their comments form an important addition to the paper. Rather than a personal reply to these comments I prefer to direct attention to Mr. Gawn's discussion wherein he sums up the situation regarding the current status of hydrodynamics in ship design and performance.

VII

ON THE DEVELOPMENT OF THE THEORY OF MARINE PROPULSION

H. W. Lerbs

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An attempt is made in this report to survey the essential results which have been obtained in the problem of hydrodynamics of marine propulsion. In this problem the predominant interest is directed towards the flow and the interaction which arise when a body is propelled in the usual way by means of a stern propeller. The Naval Architect is interested in these phenomena since rules for the most advantageous propulsion of the system may be expected if we are sufficiently able to analyse the flow. A great deal of work has been done to find such rules by means of model testing. It is not intended in this paper to enter into this experimental work but to describe the analytical attempts which have become known in this field.

It should be mentioned from the beginning that it has not yet been possible to deal with the problem in its full generality since the assumption of a given hull is made. That is, variations of the ship form are not taken into account when establishing the minimum condition for the propulsive power. We then arrive at a partial solution of the problem which essentially means a discussion of questions related to the theory of a wake adapted propeller. The complete problem, that is considering effects from variations of both the hull and the propeller on the power input, is still far from an analytical solution. Only a very first step in this direction is known in literature [17]. Additional restrictions as well as imperfections of the theory will be mentioned later on.

The paper is divided into 3 parts. In the first one the results of the general propeller theory are reviewed, in the second one the theory of interaction between hull and propeller is briefly discussed. As a synthesis of these two parts the basis of the theory of a wake adapted propeller is developed in the final chapter. New results are not given in the paper. However, it appeared useful to collect the scattered results in order to recognize the stand of the theory which has been achieved.

I. GENERAL PROPELLER THEORY

The central problem lies in the determination of the potential flow which is related to semi-infinite and symmetrically spaced helical vortex sheets. The sheets may be assumed of being built up by spiral vortex filaments of which both the diameter and the pitch remain constant when going aft. As follows from considerations of energy, this restriction on the shape of the vortex sheets is permissible with so-called "moderately loaded" propellers, that is, with conditions for which second and higher powers of the induced velocity may be neglected. Relative to the radial coordinate no restrictions on the shape of the sheets are imposed, which is an essential point for the following considerations.

The velocity field of the spiral filaments is determined first and that of the sheets is then obtained by an integration. Two ways are feasible to deduce the field of the filaments, viz., an integration of Laplace's equation or an application of the

integral by Biot-Savart. Both ways are found in literature, the first one in papers by Kawada [1] and Lerbs [6], the second one in papers by Maikapar [2], Schubert [3], Guilloton [4], Strscheletzky [5] and Lerbs [6]. Of main interest are the axial and tangential components which are induced on a propeller blade. If this blade is replaced by a lifting line for the present, the components of the velocity induced on the radius $x = r/R$ of this line are represented by the following expressions:

$$d \left(\frac{w_a}{v} \right) = \frac{1}{2} i_a \frac{dG}{dx_0} \cdot \frac{dx_0}{x - x_0} \quad (1)$$

and correspondingly for the tangential component if the subscript a is replaced by t . This element of velocity is generated on the point of reference x by a semi-infinite spiral vortex line of radius $x_0 = r_0/R$ and of circulation dG/dx_0 . By $G = \Gamma/\pi Dv$ the non-dimensional bound circulation on the radius x_0 is denoted. The "induction factors" i represent the ratio of the velocities induced from a spiral vortex line to that of a straight vortex line. They are introduced to make possible a finite representation of the element of the induced velocity if $x \rightarrow x_0$.

On the basis of the integral by Biot-Savart the induction factors may be ascertained in the following way [6]. As a first step, the expressions for dw_a and dw_t as established from Biot-Savart's law may be considerably simplified since both of these expressions may be reduced to a derivative of an integral over $1/T$, T being the distance between an element of the spiral vortex line and the point of reference. One finds that

$$d w_a = - \frac{1}{4\pi} \frac{d\Gamma}{dr_0} \sum_{n=1}^z r_0 \frac{\partial}{\partial r_0} \int_0^\infty \left(\frac{1}{T} \right) d\alpha \quad (2)$$

$$d w_t = \frac{1}{4\pi} \frac{d\Gamma}{dr_0} \frac{k_0}{r} \sum_{n=1}^z \left[\frac{1}{k_0} + r_0 \frac{\partial}{\partial r_0} \int_0^\infty \left(\frac{1}{T} \right) d\alpha \right] \quad (3)$$

where $k_0 = r_0 \tan \beta_i$, β_i being the pitch angle of the vortex line. The integral may be solved if $1/T$ is expressed first by an integral over the Bessel function J_0 applying the Lipschitz integral and if J_0 is then developed into a series of Bessel functions of the first kind by means of Neumann's addition theorem. In this way one is finally led to the Hankel integral which makes possible the following representation of the induction factors:

Exterior field ($x > x_0$)

$$i_a = 2 \left(\frac{x}{x_0} - 1 \right) \left(\frac{z}{\tan \beta_i} \right)^2 A \quad (4)$$

$$i_t = \left(1 - \frac{x_0}{x} \right) z \left(1 + 2 \frac{z}{\tan \beta_i} A \right) \quad (5)$$

$$A = \sum_{m=1}^{\infty} m I'_{mz} \left(\frac{mz}{\tan \beta_i} \right) K_{mz} \left(\frac{mz}{\tan \beta_i} \frac{x}{x_0} \right) \quad (6)$$

Interior field ($x < x_0$).

$$i_a = \left(1 - \frac{x}{x_0}\right) \frac{z}{\tan \beta_i} \left(1 - 2 \frac{z}{\tan \beta_i} B\right) \quad (7)$$

$$i_t = 2 \left(1 - \frac{x_0}{x}\right) \frac{z^2}{\tan \beta_i} B \quad (8)$$

$$B = \sum_{m=1}^{\infty} m I_{mz} \left(\frac{mz}{\tan \beta_i x_0} \right) K'_{mz} \left(\frac{mz}{\tan \beta_i} \right) \quad (9)$$

One recognizes that the induction factors depend on 3 variables, viz., on the ratio x_0/x , the number of blades z and the pitch angle β_i . For an application, it is important to note that these factors are independent of the circulation and are solely determined by the geometry of the vortex system. Numerical calculations of the factors have been carried out for $z = 3$ to 5 [6].

It is now possible to solve a fairly general propeller problem, viz., to determine the velocity components which are induced at any radius x of a bound vortex line if the bound circulation is an arbitrarily given function compatible with the end conditions at both the hub and the tip. This problem requires an integration of (1). This integration has two difficulties, firstly that the integral (1) is an improper integral and secondly that the induction factors depend on the unknown induced velocity components by way of the pitch angle β_i . The first difficulty is removed by expanding both the given circulation function and the induction factors into Fourier series. It is then possible to determine the principal value of the integral [3, 6]. The second difficulty requires successive approximations starting in the first step with both w_a and w_t equal zero to determine a first approximation for β_i . The convergence of this process is very rapid for functions $G(x)$ which are met in practical applications. Having determined the components of the induced velocity the force components which correspond to $G(x)$ follow immediately from the law of Kutta-Joukowski.

It should be mentioned that the analysis of a free-running propeller with minimum kinetic energy within the slipstream for a given thrust arises as a special case of the afore written general equations. In this case the vortex sheets are of a true helical shape corresponding to a rule by Betz [7]. From this rule the relation

$$(w_a/v) + (\lambda_i/x)(w_t/v) = w^*/v \quad (10)$$

follows, within which relation the quantities λ_i and w^* are independent of the radius. Introducing expressions (1), an integro-differential equation for the bound circulation $G(x)$ of the free-running optimum propeller is obtained:

$$\int_{x_n}^1 \frac{dG}{dx_0} \left(i_a + \frac{\lambda_i}{x} i_t \right) \frac{dx_0}{x - x_0} = 2 \frac{w^*}{v} \quad (11)$$

This equation permits a numerical calculation of the Goldstein factor which plays an important role in the application of propeller theory. From Stoke's law, this factor is defined by the following relation:

$$\kappa = (zG) \left/ \left(2x \frac{w_t}{v} \right) \right. \quad (12)$$

From both (11) and (1) the function κ is obtained in a way which is independent of Goldstein's former analysis carried out on a basis of Betz's displacement theorem [8].

Difficulties with slow convergence which arise with Goldstein's series expansions for great values of λ_i are avoided in numerical computations when applying the integral equation.

The relations mentioned so far hold both for an airplane and for a marine propeller. For the latter, however, the replacement of the blade by a line vortex does not suffice because its aspect ratio is usually small. This arises from the requirement to avoid the onset of cavitation from which condition an upper limit for the lift coefficient follows. Replacing a blade by a lifting surface instead of by a lifting line introduces a boundary value problem, viz., to determine the shape and the geometric angle of attack of an infinitely thin section such that a prescribed pressure distribution is realized within the propeller flow. There are two ways known in literature to treat this problem. In a direct way, the velocity components induced from the free and bound vortex sheets are calculated at several stations of the chord length and the boundary condition is satisfied at these stations [4, 5]. These calculations are fairly lengthy also when using precalculated functions. In a faster working approximate method one starts from a skeleton which generates the prescribed pressure distribution in 2-dimensional flow. The propeller flow requires corrections on both the shape and the angle of attack of this skeleton. These corrections follow from the curvature of the propeller flow at the half-way point, which is known from papers by Ludwig and Ginzler [9], and from the boundary condition at the $3/4$ -point. This latter means an application of Weissinger's approximate lifting surface theory to the propeller flow [10].

Marine propellers also differ from airplane propellers relative to the ratio of the hub diameter to the propeller diameter which is greater for marine propellers. The question arises whether or not the boundary condition on the hub requires corrections on the components of the induced velocity in the case of marine propellers since this boundary condition is not taken into account within equations (4) to (9) for the induction factors. Attempts to determine the order of magnitude of these corrections are made for the free vortex sheets in a paper by Lerbs [6] and for the bound vortices in a paper by Ackeret [11]. Numerical calculations have shown that this correction may still be neglected for a hub-diameter ratio of 0.2.

II. THEORY OF INTERACTION BETWEEN HULL AND PROPELLER

The older method to explore the interaction is based on the laws of conservation of energy and of momentum. The outstanding papers are those by Fresenius [12] and by Horn [13]. From these investigations it became apparent that there is a marked difference in the effects of a viscous and of a non-viscous fluid on the propulsive power. It is shown that a positive viscous wake has a favorable effect on the propulsive coefficient and that a positive displacement wake is unfavorable. The reason for the opposite behaviour of these two components of the wake lies in the energy of the absolute motion which is left far behind the propeller body. In the case of a viscous fluid the absolute motion consists of 2 parts of opposite direction, viz., the viscous wake of the body and the induced velocity of the propeller race, whereas in the case of potential flow only the propeller field is present. From these considerations it appears that a wake adapted propeller should be designed such that the remaining absolute motion becomes as small as possible. This requires a large axial component of the induced velocity and, therefore, a large thrust element to be generated on those radii on which a large viscous wake is found.

These general statements on the interaction have been considerably extended and refined by Dickmann [14]. In addition to the afore mentioned effects on the propulsive coefficient an influence of the free surface is also considered. From the interference of the wave systems of the hull and of the propeller it is shown that an improvement of the propulsive coefficient takes place if the propeller is placed below a hump of the hull's wave system.

For the purpose of this paper the relation between interaction force and wake is of predominant interest. It follows from the foregoing discussion that the wake and correspondingly the interaction force consist of three parts, viz., a viscous, displacement and wave part, the latter two being potential phenomena. These three parts are mutually dependent, that is, writing the total wake or the interaction force as a sum of the three components each one will be different from that which is obtained when the two other ones are neglected. For instance, calculating the displacement wake from potential flow, effects from viscosity will change the result since, so to speak, the hull is changed by the displacement thickness of the boundary layer.

It is to be expected that the relation between wake and interaction force is represented by different laws for the different components. Our knowledge of these laws is still fairly limited and is advanced best for the case of a deeply submerged body of revolution in non-viscous flow. For this case, Dickmann has introduced singularity methods. That is, the body as well as the propeller are replaced by proper singularities from which follow both the perturbation velocity created by the hull, which is the displacement wake, and the interaction force between hull and propeller, the latter by an application of Lagally's law. This concept has proven very fruitful and has made accessible the interaction problem to a detailed analytical treatment [14 to 20].

The singularities which replace the body with respect to its action on the flow are known to be either axially distributed sources and sinks or surface distributions. The absolute flow created by these singularities in the plane of the propeller is denoted the "nominal" displacement wake.

To replace the propeller by simple singularities infinitely many blades are assumed. This reduces the vortex system of the propeller to cylindrical sheets of ring vortices and straight line vortices. The latter are responsible for the tangential flow and may be neglected since this flow component is of minor interest in these considerations. To further simplify the problem the bound circulation is assumed independent of the radius. What then remains of the vortex system is a cylindrical sheet of ring vortices in the boundary of the slipstream. Relative to the inflow this sheet of ring vortices is equivalent to an uniform distribution of sinks on the propeller disc which may be replaced by a point sink in the propeller center if the action at large distances ahead is considered. It should be mentioned that the equivalence of the propeller inflow and the flow ahead of a sink disc, which has been obtained in the foregoing discussion employing ideas of vortex theory, may also be directly deduced as a first approximation from the fundamental equations of hydrodynamics as shown by Burgers [21].

On the basis of Lagally's law the following fundamental relation for the interaction force ΔR between the body singularities and the propeller point sink is obtained:

$$t_d = \Delta R/T = \rho E_P u_d / T \quad (13)$$

From the potential of the propeller singularity and from considerations of momentum it follows that

$$E_P / F_P = 2w_d \quad (14)$$

so that

$$t_d = 2w_d / (1 + \sqrt{1 + C_T}) \quad (15)$$

The notation is as follows:

t_d = displacement thrust deduction coefficient = $\Delta R/T$, T = thrust, E_P = input of the propeller sink which is related to the loading coefficient of the propeller, viz.,

$C_T = T / \frac{\rho}{2} v_s^2 F_p$, v_s = speed of the body, F_p = propeller disc area, u_d = velocity

induced from the body singularities at the propeller sink, $w_d = u_d / v_s$ = displacement wake fraction.

For a numerical application of (15) the difficulty arises that the body singularities should include the image of the propeller singularity into the body. The absolute flow generated by the complete system of body singularities in the propeller plane is called the "effective wake" and it is this quantity which should be introduced into (15) instead of the nominal wake. This complicates an application of (15) since the images are difficult to determine for a general body of revolution or, from the experimental point of view, since only the nominal displacement wake can be measured. It is therefore important to know the general structure of the relation between effective and nominal wake. For this purpose, it suffices to assume a simply shaped body. Considering the interaction of a sphere with a point sink, Martinek and Yeh have deduced the following relation [17]:

$$w_{d,e} = w_{d,n} \left\{ 1 + \frac{E_P}{36\pi v_s} \left(\frac{w'_{d,n}}{w_{d,n}} \right)^2 [w_{d,n}^{2/3} - 1]^{-2} \right\} \quad (7) \quad (16)$$

By the subscripts e and n the effective and nominal displacement wakes are denoted, respectively, and by $w'_{d,n}$ the derivative of the nominal wake in the axial direction. From this expression, it follows that the effective wake is greater than the nominal wake which is in accord with experimental experience.

A numerical evaluation of the sphere-sink interaction [18] shows that the interaction force decreases both when the radius of the sphere decreases, the distance of the sink from the sphere being constant, and when the distance increases, the radius held constant. However, the decrease of the force is shown to be much more rapid when the distance increases than when the radius decreases. From this it is concluded that it will be more favorable for the propulsion of a deeply submerged body to move the propeller away from the stern of the body than to refine the stern. This holds in a non-viscous fluid. In viscous flow the opposite effects of displacement and frictional wake on the propulsive power determine an optimum position of the propeller.

An obvious extension of the theory is possible when introducing a sink disc for the propeller instead of a point sink and when assuming that the input of the disc depends on the radial coordinate corresponding to the thrust distribution of a propeller. From Lagally's law the element of the interaction force $d(\Delta R)$ which is generated on an annular element of the disc is represented by the relation

$$d(\Delta R) = \rho u_{d,e} dE_P = \rho u_{d,e} e_p dF_P \quad (17)$$

where both the effective displacement wake and the surface sink distribution e_p depend on the radial coordinate x of the disc. The dependence of $d(\Delta R)$ on radius necessitates definition of a local thrust deduction coefficient, viz.,

$$t_d(x) = d(\Delta R)/dT \quad (18)$$

Expressing the surface density e_p by the thrust distribution dC_T/dx by means of the energy and momentum equations the local displacement thrust deduction coefficient becomes

$$t_d(x) = d(\Delta R)/dT = 2w_{d,e} \left/ \left(1 + \sqrt{1 + \frac{1}{2x} \frac{dC_T}{dx}} \right) \right. \quad (19)$$

We now face the problem of determining the image of a sink disc of variable density into a body of revolution in order to ascertain the effective wake. For this problem, solutions have become known recently. In a paper by Martinek and Yeh [18] certain surface singularities are studied in the presence of both a sphere and a prolate spheroid, among them the sink disc. For the spheroid the solution is given in terms of Legendre functions using spheroidal coordinates. In a subsequent paper by Hunziker [19] the restriction to bodies of revolution of special shape is abandoned and the problem of rotational symmetry is formulated generally. Introducing a distribution of

sources of strength e_H over the hull surface H , the boundary condition is expressed by the following integral equation:

$$e_H(x_1, x_2, x_3) + \frac{1}{2\pi} \iint_H \frac{\cos(r, n)}{r^2} e_H(\xi_1, \xi_2, \xi_3) dH = q(x_1, x_2, x_3) \quad (20)$$

$$q(x_1, x_2, x_3) = 2 \left[v_S \cos(n, x_3) + \left(\frac{\partial \phi_P}{\partial n} \right)_H \right] \quad (21)$$

$$\phi_P = -\frac{1}{4\pi} \iint_{F_P} \frac{e_P}{r_P} dF_P \quad (22)$$

$$r = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{\frac{1}{2}} \quad (23)$$

$$r_P = [(x_1 - x_{1P})^2 + (x_2 - x_{2P})^2 + (x_3 - x_{3P})^2]^{\frac{1}{2}} \quad (24)$$

By x the coordinates of the point of reference within the hull are designated, by ξ the running coordinates of the hull and by x_P the coordinates of the propeller disc.

The solution of this Fredholm equation of the second kind is known to be feasible by means of the method of iteration of kernels.

Having determined e_H , the effective wake in the plane of the propeller may be ascertained from the potential of the source distribution. The paper includes a numerical example for a body of revolution which has been tested formerly by Weitbrecht [22]. The thrust deduction coefficient obtained experimentally amounts to 0.128 and that by computation to 0.119. The difference lies in the right direction and the order of magnitude of the differences is in agreement with the generally held opinion, viz., that the frictional thrust deduction is a small quantity in the case of rotational symmetry.

In papers by Korvin-Kroukowsky [20] the interaction theory is developed on a basis of the singularity concept in the form of a computational procedure. The boundary condition is satisfied at a certain number of arbitrarily chosen control points on the surface of the body which leads to a set of linear algebraic equations for the strength of axially distributed sources and sinks. For the propeller, infinitely many blades are assumed and the radial distribution of its bound circulation is replaced by a rectangular one. Correspondingly, two free vortex sheets of different diameters each consisting of a semi-infinite row of ring vortices are introduced. Their velocity field is derived from Laplace's equation. Taking into account the velocities induced from the propeller vortex system when satisfying the boundary condition at the control points on the hull modifies the strength of the body singularities. The modification approximately corresponds to the image of the propeller system. From the modified singularities of the body the effective wake follows. The thrust deduction force is determined in two ways, viz., on the basis of Lagally's law from the perturbation velocities of the propeller at the body sinks and, alternatively, using surface pressure integration. Numerical results indicate that, for the example treated, the wake fraction is not sensibly affected by the action of the propeller. These results show further that the thrust deduction force decreases rapidly on the body when going forward from the stern. The first four sinks nearest to the stern situated within 12% of the body length already contribute 90% to the total force.

As far as a non-viscous fluid is concerned singularity methods have proven successful to understand the body-propeller interaction and to express this interaction analytically. The results which are known today are restricted to axi-symmetrical flow. The next step should be to explore the effect of a circumferentially non-uniform wake on the interaction force. Replacing the propeller by a vortex system, additional vor-

tex sheets built up from radial filaments become necessary to account for the change of bound circulation in a circumferentially non-uniform inflow. These additional sheets generate additional perturbation velocities on the body singularities which affect the interaction force. Studies are indicated by Martinek and Yeh on a general ellipsoid and by Korvin-Kroukowsky on 3-dimensional bodies. It is to hope that these studies will shed some light on the question whether the increase of the interaction force in a non-homogeneous wake may be explained from non-viscous flow or whether the frictional wake is responsible. The first opinion has been expressed by Dickmann [14] on the basis of first order estimates; the latter by van Manen [23] on the basis of tests on ship models conducted by van Lammeren. In these tests it has been found that the potential nominal wake is fairly evenly distributed over the propeller disc. From this is concluded that a circumferentially non-uniform wake arises essentially from frictional wake which would then be responsible for the increase of the thrust deduction coefficient.

We will now briefly consider a viscous fluid. Unfortunately, the methods to establish expressions for the frictional parts of both the wake and the interaction force cannot yet be regarded as sufficiently complete. What has been done essentially in the afore mentioned papers by Dickmann [14] and Martinek and Yeh [17] is to utilize the laws of the boundary layer of a flat plate. That is, the increment of the frictional force is determined from the increment of the free stream velocity which takes place from the action of the propeller race. The modifications of the boundary layer of a flat plate which arise from the pressure fields of both the body and the propeller and the modification of the displacement flow from the displacement thickness are left out of consideration. With these assumptions the frictional part of the interaction force is found to be a small quantity as compared to the displacement part. Somewhat more detailed are the calculations made by Korvin-Kroukowsky [20]. In this paper velocity profiles as measured experimentally on a body of revolution are introduced into the law of momentum. Further, the effect of the propeller flow on the boundary layer is estimated. Again, the result is obtained that the frictional thrust deduction is small and is essentially independent of the frictional wake fraction. However, these results refer to axi-symmetrical flow. Trying to estimate the influence of a non-homogeneous frictional wake on the interaction force on a basis of boundary layer theory great difficulties are encountered and it has not yet been possible to establish an order of magnitude of this effect. In this situation one is compelled for the time being when designing a wake-adapted propeller to use semi-empirical relations for the dependence of the local thrust deduction coefficient on the wake fraction, the latter being known as a function of the radius from experiment. Either, the assumption is made that the local thrust deduction coefficient is independent of the wake fraction, or the following relation is employed which has been given by van Manen [23]:

$$[1 - t(x)]/[1 - t_0] = \{1 - w(x)/[1 - w_0]\}^{0.25} \quad (25)$$

The quantities t_0 and w_0 are average values as determined by analysis of a model propulsion test. The relation is an interpolation formula obtained from numerical results for $t(x)$. Denoting the thrust deduction coefficients which are related to the elements of an annual ring of the propeller disc by $t(x, \varphi)$, $t(x)$ is considered the average value of $t(x, \varphi)$. To calculate $t(x, \varphi)$ two essential assumptions are made. Starting from the axi-symmetrical case, it is assumed that the local frictional thrust deduction is proportional to the local frictional wake fraction. Secondly, when the loading changes because of a circumferentially non-uniform wake, that the frictional thrust deduction is inversely proportional to the frictional wake. This means that

$$\begin{aligned} t_f(x, \varphi) &\approx t_f(x)/w_f(x, \varphi) \\ &\approx w_f(x)/w_f(x, \varphi) \end{aligned} \quad (26)$$

is written. Whether or not these assumptions are physically sound is unproven. It should be mentioned that it makes only little difference for a propeller design whether

the assumption $t = \text{const}$ is introduced or whether the function $t(x)$ as follows from this relation is utilized.

In spite of the imperfections of the existing theory of hull-propeller interaction it has proven of great value in explaining experimental findings on the thrust deduction coefficient which cannot be understood otherwise. In recent work on the scale effect of propulsion coefficients by van Lammeren, van Manen and Lap [24] the result has been obtained that the thrust deduction coefficient decreases both when the dimensions of the model are decreased and when the roughness of the model surface is increased for the same model. The explanation follows from the afore mentioned interplay of the displacement thickness of the boundary layer with the singularities of the body which may be axially distributed sources and sinks. To describe properly the displacement flow in a viscous fluid, an "effective" hull should be introduced which is the hull of the body augmented by the displacement thickness. The strength of the singularities depends essentially on the curvature of the hull such that the singularities become feebler if the curvature becomes smaller. This happens with the effective hull if the displacement thickness becomes greater either because of a smaller Reynolds number or because of increased roughness. Reducing the input of the singularities decreases the interaction force from Lagally's law which gives the explanation for the afore mentioned experimental results.

III. THEORY OF THE WAKE ADAPTED PROPELLER

The basis of the available theory is a steady relative flow, i.e., the wake is assumed to depend only on the radius. The problem is to ascertain the bound circulation such that the useful power of the system "ship and screw" becomes a maximum value for given quantities of power input, advance ratio and wake distribution. As a generalization of the optimum condition of a free-running propeller, the condition for a wake adapted propeller may be written in the form

$$\tan \beta_i = F(x)/k \quad (27)$$

where k is independent of the radius. The optimum function $F(x)$ is still to be determined. In the case of a free running propeller, $F(x) = \lambda/x$.

Combining the optimum condition with the geometrical relation

$$\tan \beta_s = \left[(1 - w) + \frac{w_a}{v_s} \right] / \left[\frac{x}{\lambda_s} - \frac{w_t}{v_s} \right], \quad \lambda_s = v_s/R\omega \quad (28)$$

which follows immediately from the diagram of the relative flow, and expressing the components of the induced velocity by their respective induction factors, an integro-differential equation for the bound circulation of the optimum wake propeller has been deduced by Lerbs [6]:

$$\int_{x_n}^1 \frac{dG_s}{dx_0} [ki_a + F(x)i_t] \frac{dx_0}{x - x_0} = 2 \left\{ F(x) \frac{x}{\lambda_s} - k[1 - w(x)] \right\} \quad (29)$$

$$G_s = \Gamma/\pi Dv_s$$

With $w = 0$ and with $F(x) = \lambda/x$ this equation passes over into the equation (10) for a free-running optimum propeller. An approximate solution is obtained if both G_s and the induction factors are developed into Fourier series. This leads to a system of linear algebraic equations for the coefficients of the bound circulation. For a practical application in propeller design work an assumption relative to the geometry of the relative flow is introduced which simplifies the calculations considerably without losing too much accuracy. This is explained in detail in a paper by Eckhard and Morgan [25].

It remains to determine the optimum function $F(x)$. This function follows from a rule which has been expressed first by Helmbold [26], viz., that the propulsive

coefficient of an element of the propeller which is related to an increment of the bound circulation at any radius be independent of the radius for optimum conditions. There are several attempts found in literature to establish $F(x)$ on a basis of this rule, see [27], [23] and [28], from which the form

$$F(x) = -\frac{\lambda_s}{x} \phi [w(x), t(x)] \quad (30)$$

follows. In the opinion of the author, however, the assumptions made in the afore mentioned papers include oversimplifications so that the results can not yet be considered final.

This is about the status to which the theory of marine propulsion is developed today. Improvements which are desirable have been mentioned in the paper. It is essentially the theory of interaction which needs to be extended whereas the propeller theory has been developed in a sufficiently general form, in the opinion of the author. In particular, the effects of non-homogeneties of the wake, including effects from the rudder need further considerations. This point is essential. In addition, the hull-propeller interaction effect of the unsteady relative propeller flow arising from a finite number of blades in a non-homogenous wake should be investigated.

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DISCUSSION

A. J. Tachmindji

Dr. Lerbs has given us a very good summary of the fundamental and basic work that has been done in the last few years in the development of propeller theory. His personal work has contributed materially in the investigation of vortex theory when the circulation distribution is considered arbitrary, and when the induced velocities can be expressed in terms of induction factors. By means of this method it has been possible to compare the validity of the circulation theory as derived by Betz's and Goldstein's work for lightly loaded propellers where the condition of normality of the induced velocity is assumed to hold true. The induction factors method, therefore, gives a means of evaluating the applicability of the lightly loaded propeller theory to

moderately loaded propellers. Furthermore, it also allows the evaluation of such effects as wake distribution and hull interaction on the loading distribution of the propeller.

The paper has given an excellent outline of the work conducted on propeller-hull interaction, a problem which is of primary concern to the propeller designer. It is by increasing development in such areas that a more fundamental understanding of propeller action will be accomplished.

The increasing use of propellers with relatively large hub diameters has emphasized the need of a solution for the circulation distribution for these cases. Such a circulation distribution may be considerably different than the one obtained by considerations of zero hub diameter. Through the use of corrective induction factors, Dr. Lerbs has investigated the effect of the presence of the hub. The problem has been pursued further at the David Taylor Model Basin and it was possible to extend Goldstein's work and derive the potential for the optimum circulation distribution¹ to include the effect of a finite hub. Similar work has also been conducted by McCormick who, however, solved the problem for simplified boundary conditions at the hub radius. Figure 1 shows a plot of the Goldstein factor K , which is defined as the ratio

Figure 1 - Effect of Hub Diameter for $\mu_0 = 4$, $p = 4$

Determined by Induction Factors for $\frac{r_1}{R} = 0.2$

Figure 2 shows the actual circulation distribution for the same propeller and the same cases. These calculations have been performed by using approximations to the solution but it is planned to calculate the exact solutions at DTMB.

of the circulation for a propeller having a finite number of blades to the circulation for one having an infinite number of blades, plotted against the propeller radius. This has been obtained for a four bladed propeller operating at a $\lambda_i = 0.25$ and shows the effect of hub diameters of 0.2, 0.3 and 0.4 as compared to a propeller having zero hub. Points calculated by means of the induction factors are also shown.

¹ DTMB Report 1051

Figure 2 - Effect of Hub Diameter on Distribution of Circulation for $\mu_0 = 4$, $p = 4$

Induction factors have also been used for the development of the theory for wake adapted contrarotating propellers. Although considerable effort has been expended, particularly by Lieber, Yen and Spiegel, in trying to determine the potential function for contrarotating systems, calculations cannot yet be performed by the theory. The use of induction factors is presently the only method available without going into arbitrary assumptions on the applicability of Goldstein's work for such propellers. It would be interesting to hear Dr. Lerbs' opinion as to the magnitude of the time dependent effects on contrarotating propellers and the need of theoretical work in this subject.

A. Silverleaf

I wish to add a few footnotes to Professor Lerbs' lecture rather than comment directly on the theory he summarised. One of the duties of the propulsion and cavitation group at Ship Division, N.P.L., is to design propellers which will be built and fitted to ships, and we are therefore concerned with both propeller theory and its practical consequences. The theoretical developments of the last ten years have undoubtedly been of very great value as a firm guide to practical design, and, as Mr. Tachmindji mentioned, Prof. Lerbs' own part in this has been a major one. However, it is necessary to check the consequences of using these theoretical design methods, and, whether we like it or not, at present the only reasonably precise way of doing so is by experiments with models. This produces some disconcerting features and also presents some difficulties. For instance, propellers designed by modern theoretical methods quite often have a knack of being slightly less efficient than those designed by what may be called rather 'slap-happy' methods. Further, these 'empirical' propellers are often less sensitive to variations from their designed operating conditions which, of course, occur owing to changes in loading, and this adds to the difficulties of using theoretical methods in designing propellers for ships which have to operate under widely varying conditions.

Many of the problems in the model experiment techniques can be grouped together under the general heading of scale effects. Thus we have found at N.P.L. that propeller models often suffer heavily from laminar boundary layer flow, affecting their performance under both non-cavitating and cavitating conditions, while differences in free stream turbulence levels between model and full-scale must also be taken into account. Another problem is that of boundary wall interference effects in propeller

water tunnels; this is susceptible to much more detailed mathematical study than it has yet been given. From this point of view it is possible that the effective length of a marine propeller is much greater than its physical dimensions would suggest, and is perhaps related to the axial dimension of the non-parallel induced flow and wake.

I am a little disappointed that Prof. Lerbs did not say a little more about the future of marine propulsion. Undoubtedly machinery powers are going to increase—as we have been reminded earlier today—thus the powers to be transmitted by orthodox propellers will also increase. Like Mr. Tachmindji, I believe that for this reason contra-rotating propellers may have quite a lively marine future, particularly as recent developments have been made in their theory. While they may not be applied to spectacular ocean liners, they may well be of value in future cross-channel type ships. It is also clear that we need, and shall increasingly need, further developments in the theory of fully cavitating propellers. Much work is at present going on in this field, of course, and opportunities for merchant ship applications will probably soon occur. However, there are many of us who believe that orthodox propellers will not be entirely adequate for future really advanced conditions. The overlapping problems of heavier loadings, the incidence and effects of cavitation, and propeller induced vibration may well force us to consider rather less orthodox propulsion methods. Although Prof. Lerbs uses the word 'propulsion' rather than 'propellers' in his title, he has in fact confined his comments to orthodox propellers; some remarks on other propulsion devices would have been a valuable addition.

The equipment needed for model experiments with these advanced designs will be considerably different from that used now. In this country and in Britain we are attempting to meet this need by building much larger water tunnels than we have at present, but I am perturbed that we are embarking on these expensive pieces of equipment with such an inadequate knowledge of how propellers perform in them, their limitations, and the relation between tunnel results and full-scale performance.

Finally, I should like to register a mild disagreement with the apologies made this morning for the naval architect's laggard approach to his problems compared with the aeronautical engineer. I believe that the ship designer has not a great deal of which to be ashamed, and I would suggest that Prof. Lerbs' exposition of the theory of broad bladed propellers is an excellent example of one field in which the marine approach has been at least as thorough as anything comparable in aircraft work.

T. Y. Wu

I wish to compliment Professor Lerbs on his interesting lecture which presents some general and instructive viewpoints. As was described in his lecture, the vortex theory for propeller problems has been developed to account for the flow conditions where both the inflow and the bound vortices are arbitrary functions of the radius. This contribution then narrows the gap between theory and application, for the marine propellers in general operate with a nonuniform inflow. Concerning the theory itself, I wonder whether a general expression has been obtained for the optimum configuration of the trailing vortex sheets when the propeller operates at the highest efficiency with a given inflow and loading distribution. I am also interested to know the feasibility of generalizing the theory to include two additional factors: 1. the effect of heavy loading due to the contraction of the slipstream and 2. the case of general nonuniform inflow, such as a shear flow with no axial symmetry, as is typical for stern propellers.

In the problem of the interaction between hull and propeller, some further improvements perhaps can be visualized. In practice, most conventional ships are likely too "fat" for the thin ship theory to remain a good approximation, especially for those ships with blunt stern section for which the interaction problem is concerned. The idea thus seems inviting that improvements could be obtained by using the slender body theory to represent the hull with the added displacement thickness of the boundary layer and by using a simplified vortex distribution for the propeller disc. In the case where the boundary layer separates from the hull in front of the propeller, it would

be of interest to study the interaction under that adverse condition and the effect on separation phenomena due to the propeller action.

S. F. Hoerner

It has been mentioned here this morning several times that there is a lack of understanding and contact between "mathematicians" and the naval architects. Now, as one who has been on the inside of aviation as well as naval architecture, I would like to tell you that that situation prevails in the field of aviation just the same.

There are highly skilled mathematicians working on theory, and there is the engineer who tries to apply the results. Usually, the engineer is lost. He does not fully understand the language of theory.

Since the advent of modern computers there is also the idea that those computers would solve all problems. This is not so at all. The computer is a "perfect" idiot; the scientific term is "idiot-savant." Now, the man who handles the computer must provide the brains to put the right information into it, and to give an adequate interpretation of what is coming out of it.

In the field of propellers, this is also true to a certain extent, and this has been realized in aviation for some time (for some twenty-five years). To help the engineer, methods have been evolved known as "polar methods," replacing the propeller blades by "equivalent" wings. The characteristics of the propeller are then reduced to those of wings.

Such an interpretation is much more understandable to the engineer.

I would suggest that for the benefit of the engineer, we would revive these methods a little bit. This does not mean, however, that I would criticize the efforts of men such as Dr. Lerbs. He is in his perfect right to develop an accurate theory.

R. R. Hunziker

I would like to comment on the subject of "thrust deduction" and interaction phenomena of propeller and hull in deeply submerged submarines, a subject which was a part of the extensive and sound lecture of Dr. Lerbs [1]. Dr. Lerbs started from the general propeller theory and considered later the hull-propeller system, pointing out the interaction character of the hydrodynamic phenomena. He mentioned the use of Lagally's theorem [2], as starting point of the theory, and the introduction of a sink-disk as a "first approximation" for the propeller. I would refer to my recent theoretical work investigating the interaction phenomena of the hull-propeller system.

Details may be found in my last paper [3] (30 August 1956) which supersedes my earlier paper cited by Dr. Lerbs.

First, I have dealt with the construction of the hull-propeller velocity field that satisfies the hydrodynamic equations in the form of Oseen [4], as an extension of Burger's [5] analysis for the open water propeller.

Departing from the Oseen equations, I have proved that in the steady flow hypothesis (used also by Burgers), the velocity field in the first approximation contains a system of helical vortices shed by a propeller of an infinite number of blades.

In the exterior of the slipstream and around the body, the field has zero tangential component, and is given by the superposition of the flow for the body alone, plus the axial and radial fields of a circular sink-disk over the propeller circle, plus an interaction harmonic field with which the hydrodynamic boundary condition remains satisfied over the body. This idea of the interaction field, characterizing the effective wake over the propeller circle, was introduced by Helmbold in 1938 [Ref. 6 p. 354 Eq. 5a]. In the interior of the slipstream the flow is rotational and at infinity has a net augmentation of the axial asymptotic velocity U , which is equal to the strength c of the simple layer over the disk. It is not too much to observe that the constructed velocity field which is singular over the propeller disk, implies a removal of the D'Alembert paradox for the body.

As the axial component of the pressure integral over the body is characterized by the velocity field and Bernoulli theorem, it is legitimate to apply Lagally's exterior theorem [2] to calculate this force of resistance in the potential flow that comprises the body. In this way, a general expression for the "thrust deduction fraction" is derived:

$$\Theta = \frac{\Delta T}{|T|} = \frac{\psi_1 + \left(\frac{c}{U}\right)\psi_2}{1 + \frac{1}{2}\left(\frac{c}{U}\right) - (\rho c UA)^{-1}\Delta P} \quad (31)$$

where T is the thrust force over the propeller support. $U\psi_1$ is the average of the normal velocity component over the propeller circle, for the body without propeller ($U\psi_1$, the "nominal wake"), and $c\psi_2$ is the average of the normal velocity component of the interaction potential. The density is denoted by ρ , the propeller circle area by A , and ΔP is the increment in the integral of the pressures over the intersection of the rotational slipstream with the plane at infinity.

ΔP is a consequence of the rotation of the water in the slipstream and introduces a dependence of Θ on the torque. Since in general the angular velocities impressed on the particles of the slipstream are small compared with the angular velocity of the propeller [7, 8, 9] it is assumed $\Delta P = 0$. In the limiting case of very light loadings ($c \rightarrow 0$), it is $\Theta \rightarrow \psi_1$, and the thrust deduction may be approximated by $\Theta = \psi_1$, which is a generalization of Fresenius's expression [10]. The expression for

Θ (generalization of Helmholtz formula [6]) for the moored condition, with $\left(\frac{c}{U}\right) \rightarrow \infty$,

must be taken with reservation due to the asymptotic character of the first approximation to the solution of Oseen equations.

Therefore, the consideration that $\lim_{U \rightarrow 0} \Theta = \Theta_0 = 2\psi_2$ is the thrust deduction coefficient for $U \rightarrow 0$ (moored condition) must be taken as a conjecture of empirical validity.

The hypothesis of a boundary layer with extreme backward separation corresponding to a favorable distribution of pressures due to the propeller inflow, seems to accord with bodies of reasonable shape such as torpedoes and streamlined submarines.

The solution of the corresponding harmonic problem for the case of the ovoid ellipsoidal hull was developed in Legendre functions and ψ_1 and ψ_2 have been tabulated for different configurations [3]. The results indicate that the thrust deduction fraction exhibits a rapid diminution with the increase of the distance of the propeller to the hull and with the increase of propeller diameter. (See Fig. 1.)

These theoretical results are slightly lower than the experimental values for the ovoid hulls tested by Weitbrecht [11]. This confirms the assertions of Dickmann [12] and van Manen [13] that for the case of radial symmetry the "frictional thrust deduction" is small. The theoretical values of the figure correspond to one of the cases investigated experimentally by Weitbrecht and show a typical variation of Θ with the

distance ratio $\frac{f}{a}$ of propeller to hull, and for different ratios $\frac{c}{U}$. (a, b , ellipsoid axes, R radius of propeller.)

Dr. Lerbs [1] has pointed out that "imperfections of the theory of interaction exist relative to viscous flow and to time-dependent flow on the propeller which arises with a finite number of blades in a non-homogeneous wake." A possible source of small discrepancies with experiments [11] is that the hull is not exactly our prolate ellipsoid.

Experiments with such prolate hulls may be performed and it is very reasonable to expect that the results would be also in good agreement.

Another more important source may be the separation of the flow, due to viscosity, in the hull after-body. Further, the simplifying assumption $\Delta P = 0$ gives lower values for the theoretical thrust deduction fraction.

Besides the proof of the hydrodynamic equivalence of the infinite blades propeller to steady flow, the approximate steadiness of the flow, depends essentially on the angular velocity Ω of the propeller, since the periodicity of the flow field is $\propto (2\pi\Omega z)^{-1}$, where z is the number of blades [8]. The thrust of the propeller and the thrust deduction should be quasi-non-periodic, if Ω is great enough.

Because our results are in substantial agreement with the observed values of time average thrust deduction [10], it is very plausible to expect that a more involved time-dependent theory, that is, for a finite number of blades, would not introduce differences significant for practical purposes.

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J. S. Florio

The purpose of my comment is to emphasize some critical aspects of the theory of the "thrust deduction" set forth by Dr. Lerbs [1] and commented on by Dr. Hunziker [2], as essentially a theory of hydrodynamical interaction, and particularly the generalized formula of the Helmbold type [3], which is a definitive conclusion, on some established hypotheses, following the line of thought mentioned by Dr. Lerbs.

The primitive Helmbold formula was derived directly from the principle of conservation of energy [4], and was the most important contribution, without the application of the Lagally theorem. In the other development of the theory (Dickmann) [5, 3] the starting point is the fundamental "exterior theorem" of Lagally [6], based on the Bernoulli equation which, as is well known, expresses the same Conservation Principle; this theorem, which goes beyond the earlier work by Cisotti on the "D'Alembert Paradox," taking account of continuous distributions of singularities of a general field (ideal fluid) expresses the corresponding resistance force and torque. Betz and Dickmann [5] applied

partially Lagally's theorem only to calculate the force exerted in some special cases of submerged bodies. Some authors have used the particular forms of Lagally's theorem, for the case of point singularities, as developed by Betz, but sometimes with improper interpretations that exceed its true physical meaning, or ignoring the true power of the theorem, which avoids some usual considerations—[Dickmann: "Kräfte auf seine Senke," "Widerstand der Senke," [5] etc., (pp. 468, 649, etc.). Also in references [7] and [8].] It is evident that forces are exerted only when there are bodies which experience pressure, and no more. There are not "forces acting between singularities." These considerations are not justified in any strict field theory of hydrodynamic phenomena.

The new formulas given [3] for the "thrust deduction" phenomenon, as essentially an interaction phenomenon, not only signify the correct application of Lagally's theorem in its original form, but also arise from the direct construction of the total velocity field satisfying Oseen's equations, as extension of Burgers' solution for the open-water propeller [9], containing an "interaction field."

Dr. Lerbs, in his interesting article entitled "Marine Propulsion" [10] has well observed that (for infinite bladed propeller) ". . . the tangential component of the induced velocity is of minor interest in these considerations." Indeed, in reference to the equivalence of the velocity field to a flow with infinite helical vortices over the slipstream, there is an interesting result. It has been proven [3] that even with the vortices over a non-cylindrical slipstream, the total velocity field has zero tangential component (in cylindrical coordinates) around bodies with a plane of symmetry. This result is valid in the "first approximation" in Oseen theory.

In this velocity field, the singularities are those for the flow grad ($U\phi_j^*$) around the body alone, plus a simple layer over the propeller circle with potential ϕ_1 , plus a simple layer over the hull. The latter corresponds to the "interaction field" grad ($c\phi_i^*$) with which the hydrodynamic boundary conditions remain satisfied over the body, subjected also to the propeller's inflow, grad ϕ_1 . This modification of the field around the hull is the cause of a resistance to the advancement of the hull. This determines with precision the meaning of the term "interaction."

The whole field, which has a region of singularities, satisfied the conditions for which the D'Alembert paradox is removed; and the hull is subjected to a hydrodynamic resistance (or a "thrust deduction," if a reduction of the thrust of the propeller is referred to by this resistance), which is expressed, as derived directly from Lagally's theorem, by [3]

$$\Delta T = -\rho c \iint_s \left(\frac{\partial \phi^*}{\partial x_3} \right)_s ds \quad (32)$$

where $\phi^* = U\phi_j^* + c\phi_i^*$. The total velocity potential is $\phi = Ux_3 + U\phi_j^* + c\phi_i^* + \phi_1$. The propeller thrust over the support is: $T = -\rho c(U + c/2)A + \Delta P$, where ΔP is an increment due to the rotation of water in the slipstream, T depending only on the values at infinity, U and $U + c$, of the velocity field in the exterior and interior to the slipstream respectively. The net thrust $T + \Delta T$ of the propeller, at a certain steady velocity of advancement, $-U$, is assumed to be equilibrated by the frictional resistance transferred to the hull by the boundary layer. The substantial agreement with observed values of the time average thrust deduction shows the plausibility of the theory (ref. [3] p. 40).

Without further comment, Dr. Lerbs has mentioned [1] (also ref. [10], p. 283), a type of approximation for the velocity field of the propeller. It consists in an unjustified conjecture replacing the helical free vortex sheets, that corresponds to a non-steady theory, by an infinite number of equidistant vortex disks along the propeller axis. Besides the latter field is computed as steady, we have to recognize that this arbitrary geometrical assumption cannot give a representation of the propeller field with its natural slipstream. Otherwise, in the steady case of the infinite number of blades, the mentioned "representation" fails completely since the actual flow around the body is

uniquely characterized only as the field of a sink disk plus a perturbation potential field $\text{grad } \phi^*$. This results as a steady solution of Oseen's equations in the first approximation, which contains a continuous (never discrete) ring vortex distribution over the slipstream. No other distribution is compatible with the motion equations, and a discrete distribution is not an approximation. For a steady flow theory it is not necessary to consider more than a general simple layer distribution over the propeller circle. For a non-steady theory it is at least necessary to compute a field compatible with the existence of a finite number of helical rotating free vortex sheets.

The great importance of the theory and formulae mentioned by Dr. Lerbs and here commented, lies in its broad applicability, especially the results and general formulae of ref. [3].

A solution for general ellipsoids is at present in development as extension of the above results [3, 12].

I think that the extension of the theory, formulae, and method of calculus for submerged hull with two propellers is immediate, with the appropriate "interaction potential" for the total field.

The interaction theory of "thrust deduction" for the surface ship is in development [11, 12], considering the free surface of water. The "interaction field" has a quite different form. The velocity field for the hull-propeller combination is characterized as boundary value problem with a linear boundary condition on the free surface. The perturbation potential is a modified Michell's potential that includes an interaction term due to the propellers inflow [12]. The essential result is that the "Potential thrust deduction" appears as an augmentation of the wave resistance.

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J. Martinek and G. C. K. Yeh

We wish to congratulate Dr. Lerbs for his fine and comprehensive presentation of a subject to which he has contributed so much in the past and in which he has

pioneered to such an extent that classical hydrodynamics has again been established as a powerful tool for solving so complicated a problem as marine propulsion. We are referring particularly to the theory of interaction between hull and propeller and the theory of wake-adapted propeller. Not all in the audience may have been aware of the fact that the researches in the last hundred years in hydro- and aerodynamics were essentially concerned with the motion of bodies where the disturbance on external forces due to the presence of the bodies was usually negligible. If we consider, however, the fact that in the case of a self-propelled vessel such disturbance is of first order, we need an analysis to study the interaction of flows and bodies. To Dr. Lerbs must go our deep appreciation, for it was he who introduced us to these subjects and whose wise counsel and encouragement were vital for whatever improvement and understanding we have been able to achieve on these problems.

We should like to mention now some of our recent results and their interrelation with Dr. Lerbs' work, by which they were inspired.

a. The application of the sphere theorem by the discussors to the problem of hull-propeller interaction is of course an approximation. Nevertheless, it was the simplest conceivable model of interaction available and it was applied with the objective of searching for a relationship which can utilize model testings. The reward was indeed surprising, as a relationship could be found which in order to obtain the interaction force requires only measurements of the flow field at the domain of the propeller location of a towed vessel and the free-running propeller itself, which complies with the conventional testing practice. This result encouraged our endeavor to study more complicated bodies such as oblate and prolate spheroids, and ellipsoids, as well as general bodies of revolution, with the objective of finding similar laws which may govern the problem of interaction. Indeed, in our Final Report II [4] a theorem has been developed for the prolate spheroid which was not directly stated, but was indirectly a result of the analysis *, as was found later by inspection. This theorem opens the way to finding a general method which will lend itself to the evaluation of the disturbance potentials and velocities for those cases where bodies of more complicated shapes are encountered. It indicates that the customary testing procedure is essentially sound and needs only a slight modification. Most important of all, it becomes apparent that more extended and refined measurements at the propeller disc of a self-propelled model, which to date have defied any such attempt, are not necessary. For the prolate spheroid we can demonstrate this in a few lines. The potential function of the totality of singularities which includes discrete ones as well as distributions in an infinite ideal fluid domain can always be expressed in the form

$$\phi_0 = \sum_{n=0}^{\infty} \sum_{m=0}^n [\cos(m\varphi) + A_{mn} \sin(m\varphi)] P_n^m(\eta) \begin{cases} B_{mn} Q_n^m(\xi) & \text{for } \xi > \xi_0 \\ C_{mn} P_n^m(\xi) & \text{for } \xi < \xi_0 \end{cases} \quad (33)$$

(ξ_0 is ξ of the singularity)

The disturbance potential ϕ_1 due to the presence of a prolate spheroid $\xi = \xi_1 < \xi_0$ is a solution of the Laplace equation and can be written as

$$\phi_1 = \sum_{n=0}^{\infty} \sum_{m=0}^n [\cos(m\varphi) + D_{mn}(m\varphi)] E_{mn} P_n^m(\eta) Q_n^m(\xi) \quad \text{for } \xi \geq \xi_1 \quad (34)$$

* of finding an exact solution for the interaction potential due to the action of an infinite-bladed propeller in the presence of a hull of prolate spheroidal form.

The general exterior theorem [6] of the prolate spheroid reads then

$$\left. \begin{aligned} D_{mn} &= A_{mn} \\ E_{mn} &= -\frac{P_n^{m1}(\xi_1)}{Q_n^{m1}(\xi_1)} C_{mn} \end{aligned} \right\} \quad (35)$$

which is not confined to the case of axial symmetry. Hence we see that the remarks above referring to the problem of testing are substantiated. Furthermore, numerical examples of such spheroids compared with Weitbrecht's tests show excellent agreement. Theorems for more general bodies have been found and are now in the process of publication. In conclusion, we can say that the vital part of the interaction force or, more generally, thrust deduction, namely the interaction potential, has been successfully clarified and established.

b. The topics and formulations discussed in a. are still confined to two restricted cases: the propeller is representable by a singularity distribution and, second, the hull form is supposed to be completely given. In practice, however, we like to know after the stage of preliminary design what minor changes in the hull form and the propeller characteristics could be effected in order to obtain an optimum combination of hull and propeller in the sense that for a given constant input power a maximum useful power can be obtained. This requirement was first stated by Dr. Lerbs [2] and expressed in terms of the local thrust deduction coefficients and the local induced velocity components. The formulation of this optimum problem whereby the average thrust deduction coefficients and the average induced axial velocity components were expressed in terms of the hull form parameters, and the disturbance velocity has been given in Final Report I [3]. At that time the general method of a. was not known and the sphere was used as a substitute. Nor has the thrust deduction formula been in its general form, as it was first derived by R. Hunziker [7]. This problem can now be analyzed where the mathematical work can be performed in a straightforward manner.

c. Finally, the general method applied recently by R. Hunziker (see Discussion by Dr. Hunziker) has confirmed the assumption that the essential information necessary for the thrust deduction coefficient is the disturbance potential of the body in the presence of the propeller in an ideal fluid flow, an assumption which was adopted *a priori*, without proof, by the discussors in their previous work.

In conclusion, we can say that some of the criticism expressed by other discussors on the fruitful use of the propeller theory will definitely be removed if the state of the art of wake-adapted propellers as already available is applied to its fullest extent to deeply submerged vessels.

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J. J. Eisenhuth

The paper by Dr. Lerbs was indeed very interesting and touched on some important aspects of the theory of marine propulsion. The comments which are about to be made are only intended to bring to the attention of the readers some other comparatively recent work that has been done in this general field, specifically in the field of torpedo propulsion. It is felt that certain phases of this work might be of interest to the ship propeller designer. The torpedo propeller work that will be mentioned here will reflect primarily the writer's direct familiarity with what was done at the Ordnance Research Laboratory of the Pennsylvania State University. The phases that will be touched upon will not be all-inclusive since some of the work has been of a classified nature.

Mention should be first made of work done on the theory of optimum propellers. In Reference [1], F. Lane describes a variational method applied to the problem of optimum wake-operating propellers. His method simply involves the variational problem of minimizing the integral of power input while holding constant the useful or thrust work integral. The solution of the problem establishes the condition of optimum distribution of bound circulation along the blade for minimum power loss at any required value of thrust or useful power.

On the problem of representing a blade of finite chord with a bound vortex or lifting line, Reference [2] presents an interesting way of handling an otherwise very complicated calculation. In this reference B. W. McCormick applies an electromagnetic analogy to the Ginzler-Ludwig theory for the correction of cambers of wide-bladed propellers. McCormick has also reported on the investigation of such subjects as the effects of a finite hub on the optimum propeller and the minimum pressure in a trailing vortex system (Reference [3 and 4]).

In Reference [3], Goldstein's analysis of a propeller satisfying the Betz condition is extended to the case where hub radius is appreciable relative to the propeller radius. The boundary condition which is satisfied is that of an infinitely long cylinder representing the hub. It is shown that for a trailing vortex sheet of given pitch, the insertion of a hub increases the value of the bound circulation of the line vortex generating the sheet. This change is more pronounced the smaller the number of blades.

In Reference [4], a report is made on a fairly extensive study made on the subject of tip vortex cavitation. In that study, the problem of predicting the minimum pressure in the trailing vortex system of a planar lifting surface was investigated both theoretically and experimentally. A semiempirical method of making these minimum pressure predictions was finally evolved, and later applied successfully to the design of torpedo propellers.

The items mentioned thus far are summarized in Reference [5]. This reference was more or less intended as a handbook for the design of torpedo propellers.

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R. W. L. Gawn

The development of theory to the stage that many propeller designers have been encouraged to apply the methods to practice has really only materialised during the last decade or so, although the basic circulation theory was propounded long ago. This is not at all surprising, since detailed corrections are involved for blade width, number of blades and other characteristics and it was necessary to establish these with the degree of accuracy essential to the propeller designer. In fact no more than a few years ago the induced velocities on the lifting line could only be reliably determined for an optimum distribution of circulation. Dr. Lerbs extended this to an arbitrary distribution and later evolved a method for determining the pitch corrections to take account of lifting surface effect. Such details generally are more important for propellers with wide blades as the theory is then less accurate and progressively so with increase of width. Dr. Lerbs' approach has been an invaluable guide in propeller design at the Admiralty Experiment Works, Haslar, and this opportunity of paying tribute to his admirable work is warmly welcomed.

There are, of course, many major aspects of design that the theory of propulsion does not cover. For example, full regard must be paid to strength, stiffness, manoeuvring qualities, propeller induced vibration and erosion, as well as propulsive efficiency over the operational speed range of the ship. The propeller must be suited to the hull and to the machinery. Many admirable researches have already been devoted to the various aspects mentioned but there seems ample scope for further development of the theory and it is suggested to Dr. Lerbs that among other important matters calling for further elucidation are cavitation, distribution of pressure, the interaction between hull and propeller and non axial flow to cover inclinations up to at least 15 degrees.

The designer has for long years been fortified by charts showing the variation of thrust, torque and efficiency of propellers over a wide range of slip obtained from tests of a series of model propellers in which one or more basic parameters of design have been systematically varied. One of the latest results of such tests published by myself three years ago covered the full range of pitch ratio and blade area ratio of practice. Each propeller was of constant pitch and segmental blade section which is a type that has long been recognized as efficient. One contribution of theory is to indicate the modifications that can be made to this comparatively simple shape with possible advantage albeit small. Dr. Lerbs' theory has been of great help in modifying the shape to suppress tip vortex or other type of cavitation to a higher speed and generally the indications of theory have been confirmed by tests of the model so modified in a Cavitation Tunnel. It frequently happens however that the propeller design so developed requires some further adjustment, to suit the particular hull and machinery and this is determined from the results of speed trials of a previous comparable type of ship. The final propeller design is thus a blend of theory with model experiment and ship trial results to which theory is making a contribution of increasing importance.

For completeness propeller charts should cover variations of characteristics other than the two mentioned above, namely pitch grading, blade section, camber, blade outline, rake and skew. Some information on these variables has been published from time to time but there is a need for a really systematic programme of tests. A com-

plete programme would be a formidable undertaking were it not for the fact that the development of theory points the way to a strict limitation of the range of useful variation of the characteristics to such a degree that the requirements for experiment can be reduced to a scope that do not involve an excessive demand on the limited test facilities afforded by ship tanks and water tunnels.

H. W. Lerbs

I should like to thank the gentlemen who contributed discussions, for enhancing the value of the paper and for indicating interesting related problems which fell beyond the intentionally limited scope of this treatise on the basic theory.

VIII

HYDRODYNAMICS OF HIGH-SPEED WATER-BASED AIRCRAFT

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I. ABSTRACT

Post-war trends in aeronautics have brought with them a significant increase in the scope of hydrodynamics research and development for water-based aircraft. Recent contributions include the high length-beam ratio hull and the hydro-ski landing gear, both of which are compatible with the aerodynamic requirements of high-speed flight. The hydro-ski provides a solution for the pantobase airplane, a means of reducing water loads and resistance at high water speeds, an effective ditching aid, and a land-water mode of operation that minimizes or eliminates the need for a prepared runway.

Fundamental problems on which recent progress has been made include that of the planing surface and the low aspect-ratio submerged lifting element near the water surface. The take-off resistance of hulls at high water speeds, the emergence

(unporting) resistance of ski-body combinations, resistance due to spray, and the transition resistance of land-water skis have been investigated.

Other continuing problems include that of separated flows with negative pressure regions ventilated to the atmosphere, the increase in wave impact effects with landing speed and their alleviation, and high aspect-ratio hydrofoil configurations for modern aircraft. The need for higher-speed hydrodynamic test facilities to keep pace with the increasing minimum flight speeds associated with transonic and supersonic airplanes has been recognized.

II. INTRODUCTION

The last fifteen years have seen many revolutionary changes in the power plants, performance, and geometry of aircraft, and in their missions and modes of operation. At the same time there has been an almost complete disappearance of the seaplane from the aeronautical scene. History has begun to repeat itself, however, in that some of the original reasons for water-basing are again becoming valid and some cogent new ones are emerging. Without dwelling on these reasons, it may be said that the present military interest in the use of water as a landing and take-off medium lies in its still unexploited potentialities for aerial warfare from advance bases and its unequalled invulnerability to conventional or high-energy bombing.

The term "high-speed" in aeronautics is typical rather than restrictive since the trend has always been toward higher powers in smaller packages and higher useful speeds. The jet engine has greatly accelerated the curve of attainable speed against time so that high speeds are now transonic and even supersonic. It has also brought with it for water-based aircraft a liberation from the propeller-clearance dimension and a true size parity with other aircraft. This fact leads for the first time to water-based airplanes having mission capabilities comparable to those of the best landplanes, and greatly encourages a new utilization of the water-basing concept.

This paper reviews briefly the basic considerations involved in the design of water-based aircraft as influenced by the present era of high-speed flight, the new concepts for water operation that have arisen, and some of the associated hydrodynamic problems that have been investigated or formulated. The contents for the most part represent the activities and views of the author and his associates in the National Advisory Committee for Aeronautics, which, in cooperation with the military services and the aircraft industry, has continued to devote a substantial research effort to this area.

III. SYMBOLS AND DEFINITIONS

Planing and Submerged Surfaces

C_L	planing lift coefficient, $\frac{L}{qS}$
$C_{m_{t.e.}}$	planing trimming-moment coefficient about trailing edge, $\frac{M}{qSl_m}$
C_{L_L}	lifting-line component of C_L
C_{L_C}	cross-flow component of C_L
A	planing aspect ratio, $\frac{b}{l_m}$
τ	trim or angle of attack with respect to still-water surface.
β	angle of dead rise from keel to chine

l_p	distance of center of pressure of total lift forward of trailing edge
l_m	mean wetted length from trailing edge to stagnation line
$\frac{l_p}{l_m}$	center-of-pressure ratio for total lift
a_1	center-of-pressure ratio for lifting line component of lift
a_2	center-of-pressure ratio for cross-flow component of lift
b	maximum wetted width of planing surface
$\frac{l_m}{b}$	wetted length-beam ratio
R	longitudinal radius of curvature of planing surface
Z	vertical clearance from trailing edge of planing surface to water bottom
L	planing lift
D	planing drag
M	planing trimming moment about trailing edge
q	dynamic pressure $\frac{\rho}{2}V^2$
S	wetted area
ρ	water mass density
V	water speed
α	angle of attack of submerged surface with respect to still water level
Submersion	distance of highest point of submerged surface below still-water level

Aircraft Configurations

$(R + D)$	total resistance (water resistance plus aerodynamic drag during take-off)
V_g	take-off or "getaway" speed
C_Δ	Froude load coefficient, $\frac{\Delta}{wb^3}$
C_{Δ_o}	Froude load coefficient at rest, $\frac{\Delta_o}{wb^3}$
C_R	Froude resistance coefficient, $\frac{R}{wb^3}$
C_V	Froude speed coefficient $\frac{V}{\sqrt{gb}}$
Δ_o	take-off gross weight
Δ	water-borne load during take-off
R	water resistance plus hull air drag
b	maximum wetted width over forebody chines
w	specific weight of water
g	acceleration of gravity

IV. BASIC CONSIDERATIONS

The obvious and important factors to be considered in the design of a high-speed water-based aircraft in the sense defined are listed in Figure 1. What is not so obvious and depends greatly on current trends is their relative importance with regard to the basic compromises that must be made in procurement specifications and major design decisions.

Flight

In the first place, there can be little or no concession to established aerodynamic principles that provide favorable drag levels near and above the speed of sound in relation to currently available thrust levels. Once the desired speed is achieved, range and payload remain critical so that various factors in the range equation must be carefully balanced to attain useful mission radii. The characteristically low lift-drag ratios at supersonic speeds and the large adverse effects of small weight increases on range demand a great deal of design ingenuity to obtain useful performance, and more than ever before place flight considerations first.

Take-Off and Landing

The wing geometries and areas associated with high-speed flight have led to very high minimum flight speeds and the maximum water speeds with which the hydrodynamicist must deal. Contemporary landplanes are operated at runway speeds approaching 250 knots, and unless STOL or VTOL techniques can be applied their water-based counterparts will be required to take-off and land at twice or three times the water speeds of post-war propeller-driven seaplanes. The questions of whether such speeds

- A. FLIGHT
 - SPEED
 - RANGE AND PAYLOAD

- B. TAKE-OFF AND LANDING
 - SPRAY CLEARANCES
 - STABILITY AND CONTROL
 - TAKE-OFF PERFORMANCE
 - ROUGH-WATER CAPABILITIES

- C. MANEUVERING AND HANDLING

Figure 1. Basic considerations.

are practical and less or more so than for land-based operations appear of fundamental importance, and worthy of intensive investigation.

Spray clearances.—High-speed jet-propelled configurations are characterized by high fineness ratios, small frontal areas, high hydrodynamic loadings, and little or no spray clearance for vital components such as the engines themselves, the flaps when extended, and the horizontal tail. This problem in some respects dominates the basic design, although not necessarily adversely. Spray can be tolerated to some extent, and spray damage can be minimized by auxiliary devices and operating techniques. It must, however, be recognized and lived with as far as aerodynamic considerations permit, and in the treatment of mechanical or electrical gear subject to wetting.

Stability and control.—Adequate hydrodynamic stability and control during water operation are relatively difficult to achieve because high-speed aircraft are never too well off aerodynamically in these respects and their marginal characteristics reflect on their water behavior more than is generally appreciated. In the case of new and untried hydrodynamic design solutions, which abound in this era, the configuration must be made operable before other characteristics can be evaluated or optimized. "Runability" must therefore receive a great deal of attention in towing tank investigations, and is the basis for much of the work there with dynamically similar models.

Take-off performance.—Take-off performance is defined in terms of the total time and water distance required and of the ease of acceleration through critical spray or stability regimes. The latter is probably of greatest importance since it determines the adequacy of severely compromised hydrodynamic solutions found desirable from flight considerations. High-speed aircraft are blessed with high thrust-weight ratios for take-off; on the other hand, their high take-off speeds still prolong this phase of the flight. With both thrust and wing loadings determined primarily from mission requirements, the actual take-off performance remains properly a dependent characteristic except as it may be optimized in the detailed hydrodynamic design.

Rough-water capabilities.—The surface of the water in nature is seldom perfectly smooth or, on the other hand, too rough for seaplane operation except in large operating areas with long fetches. It is a moot question what should be demanded of a high-performance water-based airplane in this respect. Even relatively small waves and swells can inflict local damage on light aircraft structures, and at the high water speeds for take-off and landing can impart excessive motions and accelerations to the airplane as a whole. Adequate provision for these contingencies in severe sea states can be made only at the expense of the structural weight-gross weight ratio which is already critical. The alleviation of rough-water effects is recognized as a major goal in hydrodynamic research and some progress had been made, but it is believed strongly that the rough-water capabilities of a transonic or supersonic airplane is a derived rather than a primary specification, subordinate to rigid flight requirements.

Maneuvering and Handling

The maneuvering and taxiing problems of high-speed water-based aircraft are at this point probably more severe than those of propeller-driven seaplanes because of the relative inflexibility of jet engines, their lack of reversibility, and generally close-coupled transverse locations. Wing-tip floats may still be required for static transverse stability and safe cross-wind taxiing, but may have to be retracted or jettisoned in flight because of their drag penalty at sonic speeds.

While outside the scope of this paper, the subject of basing requirements must be included in any listing of considerations for advanced water-based aircraft. The point of such aircraft is, of course, their ability to operate in unprepared areas and independently of elaborate facilities; nevertheless, they will require bases of some sort where they can be fueled, loaded, and repaired. The airplane's part in these operations must be determined early and provisions made for it, subject, however, to the same flight restrictions that apply to all the factors discussed.

V. NEW CONFIGURATION AND CONCEPTS

High-Speed Hulls

The classic hydrodynamic solution for a large water-based airplane remains the flying-boat hull which combines the functions of the fuselage, floatation, and landing gear. Such a hull for a high-speed jet-propelled configuration reflecting several features resulting from post-war research and development is shown in Figure 2.

Figure 2. Typical high-speed hull.

Overall proportions.—One of the basic contributions has been an emancipation from arbitrary beam formulas, provided the hydrodynamic lengths are selected to suit. When the beam is reduced to reduce frontal area, the required lengths, particularly the forebody length, increase at a slower rate, resulting in a smaller bottom area without sacrifice of essential hydrodynamic qualities (ref. [1, 2, and 3]). The higher beam loading results in lower wetted aspect ratios and more intense side spray at the hump and planing speeds (ref. [4]); on the other hand, it provides a significant alleviation in water impact loads and motions (refs. [2 and 5]). This beneficial process can be carried to a practical limit from the construction point of view and embraces hydrodynamic length-beam ratios entirely compatible with the high body fineness ratios and low frontal areas mandatory for low drag at supersonic speeds.

Forebody.—With a hull of the type shown, the long streamlined forebody is hydrodynamically necessary for buoyance, cleanness of running, and low water resistance at pre-hump speeds. The relatively high angles of dead rise favorable for absorption of impacts at high water speeds are also advantageous with regard to low-speed spray (ref. [6]). The heavy main spray can be controlled to some extent by horizontal chine flare, or even vertical chine strips or “spray dams,” but the close coupling of high-speed configurations still makes high wing and tail positions desirable for seaworthiness.

Step.—The trend in step design as shown is the use of a highly tapered plan form with some form of step fairing. These features minimize the aerodynamic and structural discontinuities and appear hydrodynamically acceptable if not carried too far. The tapered plan form tends to reduce impact loads (ref. [7]) and to improve the landing stability with shallow step depths.

Afterbody.—The long clean afterbody and low angle of afterbody keel are almost mandatory in a high-speed hull for buoyancy and fairing purposes. This type

of afterbody improves rough-water loads and motions (ref. [8]), but is detrimental with regard to stability and resistance at high water speeds, as will be discussed later.

Area rule application.—In addition to the features shown, the form may include body indentations and bulges to minimize aerodynamic interference effects at transonic speeds. Such distortions of the form will be located above the chines, and, except near the after end, should not influence the hydrodynamic characteristics. In arriving at the proper distribution of hull volume from area-rule considerations, the graphical line-fairing methods of the naval architect have been employed to advantage to blend the various aerodynamic and hydrodynamic features desired. Trial solutions in the research laboratory for transonic and low supersonic configurations have indicated the continued usefulness of the flying-boat hull for aircraft in this category.

Hydro-Skis

Probably the most significant of the post-war hydrodynamic developments has been the so-called hydro-ski landing gear. This concept in its original form, stimulated by the trend to jet power and the need to approach more closely an optimum aerodynamic configuration, is illustrated in Figure 3.

Figure 3. Typical high-speed fuselage with hydro-ski.

General.—Basically the hydro-ski is the hydrodynamic equivalent of the landing wheel rather than the conventional seaplane float. As such, the floatation function is dispensed with and it is deliberately made small enough for complete retraction in flight. It then becomes comparable in weight and complexity with a wheel gear, and localizes the landing loads near the airplane center of gravity.

Stability.—Early investigations with free dynamic models indicated that simple configurations like that shown may be expected to emerge (unport) stably, and to operate under control of the pilot at high water speeds very much like a wheel gear. They likewise may be made to land stably, and as the speed is reduced to the point where the ski gear submerges, they settle gracefully into the water without uncomfortable motions or changes in course (refs. [9 to 11]).

Resistance.—It was accepted at the start that the emergence water resistance would be high, since the fuselage would preferably have a minimum of hydrodynamic form and the necessary ski lift would be generated at a high angle of attack to make

up for its small area compared to a hull forebody. At high planing speeds where the ski wake clears the fuselage, the water resistance can approach that of the planing surface alone, depending primarily on wetted aspect ratio and angle of attack.

Spray.—Large amounts of spray are generated by a small ski gear, particularly during emergence when the angle of attack is high and the ski nose is breaking the surface. The spray remains severe at planing speeds because of the high hydrodynamic loading. Most of this spray is relatively harmless, but makes a high degree of water-proofing and corrosion resistance important in the engineering design.

Rough-water qualities.—Highly loaded skis increase the effective stroke during water impacts to a degree dependent on their size and shape, and thus offer a further fundamental improvement in rough-water capabilities. This quality has been observed generally in model experiments as well as full-scale operation, and is substantiated by impact theory. Since the loads are localized in the ski struts, additional alleviation for the airplane can be provided by mechanical shock absorbers (ref. [7]), although this elaboration may be fundamentally redundant if the ski size can be made small enough.

Ski arrangements.—Many ski arrangements are possible for applications of the principle; they may be single or multiple, twin or tandem, retractable into different airplane components, etc., In any event, variations in the system must obviously, as for landplane gears, suit an established operating sequence requiring the minimum of attention and special control from the pilot.

Additional Hydro-Ski Applications

The Convair F2Y-1 Sea Dart represents the only literal application so far of the bold hydro-ski principle as originally conceived. Several additional concepts have, however, appeared from military attempts to evaluate the usefulness of the idea, which are of potential importance. These concepts are summarized in Figure 4.

Figure 4. Additional hydro-ski applications.

Pantobase gear.—The similarity of water and snow skis leads immediately to arrangements suitable for operation on water, snow, and ice, and even other “wet” surfaces such as mud and sod. Such arrangements, combined with a conventional landing gear for normal runway use, provides the so-called “pantobase” airplane for universal operation in prepared and unprepared areas. These concepts have been investigated by hydro-ski conversions of a JRF-5 amphibian for Arctic service and of a C-123 assault transport for pantobase service (ref. [12]). The conversions of course increase utility rather than performance, but the principles explored are applicable to high-speed aircraft in other categories without the flight penalties so far incurred.

The logical ski-wheel configurations for a pantobase gear are indicated in the figure. For water operation, the skis are locked at a positive trim angle for hydrodynamic lift and stability, while the landing wheels are retracted above the skis to minimize their adverse effect on resistance. For soft surfaces and ice, the skis are free-to-trim as in the case of snow skis to follow surface contours and climb over obstructions. In this case the wheels may or may not be used depending on the nature of the medium. For hard-surface runways, the wheels are extended and the skis retracted above them so that the landing gear can function in a normal fashion.

Hull auxiliaries.—The employment of existing hull-type seaplanes as test beds for hydro-ski evaluations has confirmed their inherent superiority in rough water and the advantages of the large increase in afterbody clearances provided near take-off. This experience and continued model experiments have suggested the possibility of adding auxiliary skis to high-speed hulls to improve their hydrodynamic performance. If the existing aerodynamic stability and control permit, the ski can be made very small and extended to completely penetrate the surface during wave impacts and thus provide a larger amount of energy absorption. Because even small skis increase water resistance in the early stages of take-off, they preferably remain retracted in these stages and be jacked out at speeds near getaway where water loads and afterbody interferences are most critical.

Ditching aids.—In emergency ditchings of land-based aircraft in rough seas, hazardous motions and extensive structural damage result from wave impacts at high water speeds leading to serious casualties and rapid flooding of the fuselage after the airplane comes to rest. The concept of the hydro-ski landing gear offers a positive means of eliminating these hazards from the ditching operation, and greatly increasing the chances of survival and rescue (ref. [13]). In this case, the gear would be used only once and a manual arrangement for extending it would be sufficient, hence its weight should be small as compared to that of a normal installation. It is conceivable that such a device would permit the fuselage to stay intact during a ditching and thus able to float indefinitely while awaiting rescue of the occupants.

Land-water operation.—Small airplanes have been operated successfully on and off beaches and ramps using snow skis, and several landplanes have been converted to water operation by the addition of small hydro-skis to the wheel gear. This experience leads to an entirely new concept to free high-performance aircraft from present runway limitations, and at the same time eliminate the handling, floatation, and low-speed hydrodynamic problems associated with complete water-based systems.

Essentially the airplane is land-based adjacent to one of the sheltered bodies of water which abound in most parts of the world. To take-off it accelerates down a beach or short prepared ramp to a suitable low planing speed. At this point, it enters the water and completes the high-speed portion of its take-off run on the hydro-skis. Coming back, it reverses the process by landing on the water, decelerating to near the minimum planing speed, and going up the ramp to a convenient point for service and maintenance.

The essential features of a gear for land-water operation are similar to those for a pantobase gear in that the skis must be free-to-trim on land and fixed in trim on the water and the wheels are preferably retracted during the water runs. Both the

ski and wheels can be smaller and lighter than for a true water-based version or a land-plane, and the airplane fuselage requires little or no hydrodynamic modification. The minimum planing speeds are low compared to take-off and landing speeds, hence accidents resulting from power failure would be less severe than for the equivalent maneuvers on restricted land runways.

The employment of this mode of operation could be of great military or commercial significance since it embodies the primary advantages of land basing and water basing without the serious disadvantages of either. There are, of course, many design and operational problems which must be solved, but which can best be evaluated by economical conversions of existing high-speed land-based equipment and operation of these conversions under actual service conditions.

VI. HYDRODYNAMIC RESEARCH RESULTS AND PROBLEMS

Planing Surfaces

With trends toward higher landing speeds and the developments described, experimental research on planing by the NACA has been largely directed toward extensions of the ranges of available data to higher speeds, trims, and length-beam ratios. At the same time, the scope of the work has included practical variations in geometric parameters introduced by the new concepts and configurations (refs. [14 to 20]).

Most of the previously determined and extended ranges of experimental pure planing data for the classic flat plate have been satisfactorily correlated by Shuford of the Langley Aeronautical Laboratory (ref. [21]) using the methods of modern low-aspect-ratio airfoil theory. This investigator has recently extended his contribution to include cross-section and plan-form effects as indicated in Figure 5.

Lift.—The hydrodynamic lift coefficient based on wetted area is a simple function of a lifting-line airfoil term and a crossflow lift term. In this equation, A is the hydrodynamic aspect ratio $\frac{b}{l_m}$, where b is the maximum width and l_m is the mean

LIFTING LINE TERM, C_{L_L}

CROSS-FLOW TERM, C_{L_C}

$$C_L = \frac{0.5 \pi A \tau}{1 + A} \cos^2 \tau (1 - \sin \beta) + C_{D_C} \sin^2 \tau \cos^3 \tau \cos \beta$$

$$\frac{l_p}{l_m} = \frac{a_1 C_{L_L} + a_2 C_{L_C}}{C_L}$$

Figure 5. Pure Planing lift and center of pressure (Shuford).

wetted length. τ is the trim angle, and β is the effective angle of dead rise from keel to chine. The introduction of a constant crossflow drag coefficient in the second term correlates the experimental data from different sources, and for different cross sections and plan forms over a very wide range of the independent variables; for example, speeds up to 80 feet per second, trims up to 30° , and length-beam ratios up to 8.0.

Center of pressure.—The center-of-pressure ratio derives from corresponding ratios for the lifting-line and crossflow lift terms, a_1 and a_2 . These ratios remain constant for either rectangular or triangular plan forms throughout the range of available data. The two equations are therefore generally representative of the recent research on pure planing, and numerical solutions of them provide a convenient summary of the significant results.

Chine rounding.—The variation of lift coefficient and center-of-pressure ratio of a sharp-edged flat plate with trim for a typical wetted length-beam ratio of 3.0 is shown in Figure 6. The curves fit recent experimental data from metal models best

Figure 6. Effects of chine rounding, deadrise 0° , $l_m/b = 3.0$.

using a crossflow drag coefficient of $4/3$. A small amount of chine rounding ($1/64$ inch in a 4-inch beam) such as might be found on a sanded wooden model appreciably reduced the lift coefficient and the apparent value of C_{D_c} which might account for some of the observed discrepancies among data from various sources. With a given load, this effect would be expected to increase the wetted area, frictional resistance, and total resistance at low trim angles. At high trims where the frictional resistance is small, the effect on planing efficiency is negligible.

For rectangular-plan-form flat plates, the center of pressure of the lifting-line component of lift apparently remains at $7/8$ of the mean wetted length, and of the crossflow component at $1/2$ the mean wetted length. The total center-of-pressure ratio moves from approximately 0.80 at low trims to $2/3$ at the high trim of 30° . Chine rounding has only a small effect on the position of the center of pressure.

Figure 7. Effects of deadrise, $l_m/b = 3.0$.

Dead rise.—The results for various angles of dead rise are compared in Figure 7. With sharp-edged models, the three coefficients established for the flat plate appear to remain constant for angles of dead rise up to 50° . Variations with dead-rise angle are uniform through the trim range and exhibit the usual nonlinear characteristic associated with theoretical dead-rise functions.

Cross section.—Practical hulls and hydro-skis with dead rise usually require some form of chine flare to control spray, and many model tests have indicated additional benefits from the use of flare on resistance and stability. This experience is substantiated in Figure 8, which shows the progressive increase in lift coefficient obtained with horizontal flare and vertical strips on a rectangular-plan-form surface with a basic angle of dead rise of 20° . The experimental data correlate with the theory entirely on the basis of an increase in the crossflow-drag coefficient as noted. The center-of-pressure coefficients remain constant as before, hence the center-of-pressure ratio is reduced only slightly by the chine modifications.

The corresponding effects for a basic dead rise of 40° are shown in Figure 9. The same trends are obtained to an even greater degree with larger apparent values of the cross-flow-drag coefficient and the same constant center-of-pressure coefficients. These results indicate that the Shuford formulation provides a means of obtaining the planing characteristics of an arbitrary cross section over a wide range of operating parameters with only a brief experiment to determine the constant value of the cross-flow-drag coefficient.

Triangular plan form.—The application of the theory to a flat surface with a pointed plan form such as preferred for hydro-ski trailing edges is shown in Figure 10. Good correlation with unpublished experimental data is obtained with the same sharp-chine crossflow-drag coefficient as for rectangular plan forms. In this case, however, the lifting-line component of lift apparently effectively acts at the leading edge,

Figure 8. Effects of cross section, basic deadrise 20° , $l_m/b = 3.0$.

Figure 9. Effects of cross section, basic deadrise 40° , $l_m/b = 3.0$.

Figure 10. Effects of planform, deadrise 0° , $l_m/b = 3.0$.

and the crossflow component at two-thirds the wetted length. Going to a triangular plan form does not change the lift coefficient appreciably, and shifts the center-of-pressure forward by an approximately constant amount.

Resistance.—The drag associated with planing surfaces is partly skin friction and depends on boundary-layer conditions which are influenced by pressure gradients and Reynolds number. In the case of prismatic surfaces, the induced drag is assumed to be simply the load times the tangent of the trim angle, and the boundary layer to be fully turbulent. Analyses of model measurements have largely substantiated the validity of these assumptions for engineering estimates. It is then preferable to calculate the resistance of full-size elements, using the wetted area-lift coefficient relationship derived from tank model tests rather than extrapolating model resistance values.

The relative planing efficiency for various cross sections (refs. [14 to -8]) is shown in Figure 11. This is the hydrodynamic lift-drag ratio from model data plotted against trim for a mean wetted length-beam ratio of 3.0. The peaks of the curves for various cross sections occur at trims of from 5° to 9° depending on the dead rise. At high trims, the curves tend to approach the $\cot \tau$ boundary curve representing the induced lift-drag ratio. It has been noted that at high trims and low length-beam ratios, measured lift-drag ratios for a flat plate actually lie a little above this boundary because of forward flow in the spray root ahead of the stagnation line.

The L/D is reduced as the dead rise is increased throughout the range of trim shown as would be expected. As shown previously the addition of horizontal chine flare or vertical chine strips increases the lift of the dead rise surfaces. These additions also increase the lift-drag ratios by a useful amount in the best trim range in the same manner as a decrease in the effective angle of dead rise. There is little to choose between the two forms of chine with regard to their beneficial effect on the efficiency of either the 20° or 40° surfaces.

Figure 11. Planing efficiency for various cross sections, $l_m/b = 3.0$.

Longitudinal curvature.—In the case of a planing surface with longitudinal curvature, the effective trim and the induced drag are not simply defined. A correlation of model resistance data to indicate curvature effects on planing efficiency is shown in Figure 12. Here the maximum lift-drag ratio obtained at the optimum trim is plotted against length-beam ratio for a rectangular flat plate (ref. [18]) and plates with curvature. The maximum obtainable ratio for the flat plate approaches a value of 8 at low length-beam ratios (high aspect ratios) and falls off to less than 6 at length-beam ratios of practical interest for heavily-loaded skis. Corresponding unpublished data for a longitudinally convex surface, which is commonly found in practice, indicates a large penalty on planing efficiency that increases with length-beam ratio. The brief data of Sottorf (ref. [22]), on the other hand, has indicated an improvement in maximum lift-drag ratio attainable with increasing degrees of longitudinal concavity. Practically, concave surfaces have had adverse stability characteristics which have discouraged their use on seaplane hulls. The available evidence presented, however, indicates not only the marked inferiority of convex planing elements, but the possibility of exceeding the efficiency of the flat plate by introducing camber as for airfoils, particularly for the very high length-beam ratios of current interest.

Shallow-water.—It has been observed many times by service pilots that shallow water seems to improve seaplane take-off performance. As yet, a theoretical solution of this effect has not been made, although it should be possible by available methods. Its magnitude has been evaluated, however, by model tests in a tank with a variable-depth bottom (ref. [23]). This brief investigation indicated that lift, drag, and trimming-moment coefficients about the trailing edge increase with decreasing bottom clearance at clearances less than one beam, and significant changes in the wake occur.

A condition of practical interest is the case of a hydro-ski airplane approaching a constant-slope ramp as shown in Figure 13. Here it may be assumed that the water-

Figure 12. Effects of longitudinal curvature on planing efficiency.

Figure 13. Effects of shoaling water, constant load, speed, and beam, $\frac{l_m}{b} \times C_L = K$.

borne load, speed, and trim do not change appreciably for the short interval influenced by the bottom clearance, or that the product of the length-beam ratio times the lift coefficient is constant. The plots against the clearance parameter z/b for various trim angles indicate a large reduction in wetted-length-beam ratio and increase in moment coefficient as the clearance approaches zero. The hydrodynamic center of pressure actually moves aft with the decrease in wetted length. The lift coefficient increases because the hydrodynamic lift is constant. At high angles, the L/D remains the same, but at low trims it increases sharply because of the decreased wetted area and frictional resistance. These effects are relatively unimportant for transition to and from the ramp, as has been demonstrated by operation with light ski airplanes and dynamic model tests of jet airplanes with high ramp speeds, because of the very shallow depths at which they appear.

Self-excited vibration.—A tendency for flat planing surfaces to vibrate in a vertical plane at high speeds and short wetted lengths has been encountered as shown in Figure 14. The oscillations are self-starting in smooth water, have high fundamental frequencies, and can be very violent. The speed for inception is a function of the trim angle and load (wetted length-beam ratio) as indicated, and the inception conditions are the same for both the systems diagrammed. The phenomenon is thus not explainable on the basis of coupling between trim and vertical motions since it occurs without freedom in trim (ref. [24]).

It can be seen that the inception moves to higher speeds with decrease in trim or increase in load, both of which are in the direction to increase wetted length. The vibration can be reduced or eliminated by the use of V plan forms or cross sections, which limit the wetted aspect ratio to low values. It can also be minimized or avoided by stiffening the planing surface itself, or restraining the vertical motion by reduction in overhang or the use of auxiliary struts.

The evidence so far shows that the problem is not inherent and where it exists, may be reproduced in the towing tank with a Froude model. The basic mechanism involved however, is not well understood, and theoretical means to predict its occurrence in practical cases are not yet at hand.

Figure 14. Self-excited vibration of planing surface.

Underwater Lifting Surfaces

The problem of low-aspect-ratio flat plates moving under and close to the water surface at high speeds has been investigated experimentally by the NACA, and a theoretical method has been developed to predict the lift for unseparated flow at large depths (ref. [23]). The problem remains of taking into account in the theory the influence of the surface, and this phase of the research is continuing.

In addition to the familiar phenomenon of cavitation at large depths, a dynamic lifting element at shallow depths experiences a type of separated flow brought about by ventilation from the atmosphere as shown in Figure 15. At high angles of attack, air is drawn in through the strong trailing vortices as the speed increases until it reaches the plate and the flow separates from the upper surface. At first the flow reattaches, but which further increase in speed, it completely clears the plate forming the so-called planing bubble. At this point sharp discontinuities in the forces occur as upper surface lift is completely lost (ref. [25]).

At high speeds, the fully ventilated flow occurs suddenly at low angles of attack without the progressive development described for high angles. The form of the angle of attack speed boundaries for inception of the two types of fully-ventilated flow are shown in the figure for two depths of submersion (unpublished data). The inception speed for the high-angle type does not vary appreciably with angle of attack but increases with depth of submersion; the angle for the low-angle type decreases with speed and varies only slightly with depth of submersion. Between the two branches, the boundaries are essentially independent of speed and submersion as indicated.

Since the inception of ventilated flow is generally a function of speed, the question arises as to the effects of scale in model tests of submerged elements. Experiments with two sizes of plate have indicated that the high-angle portion of the boundaries at corresponding depths scale according to Froude's Law, a useful result for configuration tests in a towing tank. The low angle portions of the boundaries

Figure 15. Ventilation of underwater lifting surfaces.

however are relatively independent of scale as for cavitating flows. The whole subject is therefore amenable to a better understanding and theoretical treatment before extrapolations of model data involving ventilated flows can be made with confidence.

Take-Off Resistance

Typical take-off resistance curves for three types of water-based aircraft are shown in Figure 16. In order to make them comparable, the ordinate is the water

Figure 16. Take-off resistance of water-based aircraft.

resistance plus air drag of the configuration divided by the gross weight, and the abscissa is the ratio of water speed to getaway speed in each case. The curves are for a high length-beam ratio, long-afterbody, hull-type seaplane with a take-off speed corresponding to wing parameters of current interest (unpublished data); the original application of twin hydro-skis to a transonic research airplane (ref. [10]); and the pantobase C-123 assault transport (ref. [12]). None of these curves are necessarily optimum, but they represent operable arrangements for widely different purposes.

The hull configuration has low resistance at low speeds with a hump resistance of 22 percent of the gross weight or an average L/D up to $0.8 V_G$ of 4.5. The pronounced second hump near take-off rises to over 30 percent of the gross weight. A part of this rise is attributable to the increasing air drag with speed, but it is largely an increase in water resistance due to wetting of the long afterbody by the forebody wake.

The hydro-ski D-558 has a very high resistance, 45 percent of the gross weight, at speeds near ski emergence. As mentioned previously, this low-speed peak is characteristic since it is related to a small ski size and a deliberate disregard of hydrodynamic concessions to take-off resistance that would increase ski size and weight or the drag in flight. Lower peaks have been obtained, but this type of ski application is properly associated with airplane take-off thrust-weight ratios of 0.4 or greater. The high-speed resistance is generally higher than for the hull because of high ski trims and wetting of airframe components; however, there is a significant improvement near take-off where the afterbody clearance is greater than can possibly be achieved with a hull.

The pantobase ski example has the benefits of a large existing fuselage with a flat bottom, large skis with side wings to add planing area, and a relatively low take-off speed. These characteristics add up to favorable resistance levels as compared to the ski D-558 at low speeds and to the hull seaplane near take-off.

Hulls.—The nature of the afterbody interference problem when hulls are applied to high-landing-speed aircraft is illustrated in figure 17. This is a plot of the optimum (best trim) load-resistance ratio of a high length-beam ratio hull (ref. [32]) plotted against the parameter $\sqrt{C_{\Delta}/C_V}$ which is the square root of the conventional planing lift coefficient based on the square of the beam. When towing tank resistance data at a given trim or at the best trim are reduced to this form, the points for various loads and speeds fall along a single curve which extrapolates to zero at infinite speed or zero load (ref. [33]). Corresponding data for a prismatic planing surface having the same 20° dead-rise cross section with horizontal chine flare as the forebody of the hull (ref. [14]), and obtained behind a wind screen to eliminate all air drag, show a constant $(\Delta/R)_{max}$ of about 6.0. Adding the air drag of the hull as determined from separate wind tunnel tests (ref. [34]) to the drag of the planing surface gives the intermediate curve, which also goes to zero at zero planing coefficient. Since the hull data include the air drag of the model, the difference represents approximately the decrement due to afterbody wetting. It is because of these two factors that hull efficiency becomes progressively poorer at higher speeds and lighter loads.

The results of a simple calculation of the water resistance plus air drag of the hull alone for a high-speed water-based bomber, based on the tank data curve of Figure 17, are shown in Figure 18. A design beam loading C_{Δ_0} of 6.0 was assumed, giving a beam of 7.48 feet for a gross weight of 160,000 pounds. With parabolic unloading, the ratio of the water-borne load to the gross weight, $\frac{C_{\Delta}}{C_{\Delta_0}}$, is shown by the dashed line.

The corresponding ratios of hull resistance to gross load $\frac{C_R}{C_{\Delta_0}}$ for various assumed values of take-off speed are shown by the solid curves. It is seen that for conventional seaplane take-off speeds, the hull resistance near take-off is relatively low, in fact it has not been a serious problem. At take-off speeds of current interest, it is three times as high, while at take-off speeds which may be associated with supersonic aircraft it could become as much as six times as high. It must be remembered that these curves are extrapolations of the data shown, and are of value chiefly as the formulation of a problem requiring further investigation.

Tank data similar to that presented indicate that the resistance due to afterbody interference increases with afterbody length-beam ratio, so that the problem is more severe for high-speed hull applications than for low length-beam ratio hulls with short afterbodies. The situation actually is similar to that of seaplane floats for small personal-type aircraft, which also operate at high water speeds in proportion to their size, that is at high Froude numbers, and whose water resistance is critical near take-off (ref. [33]). The resistance in practice is not as high as indicated by tank data which is not corrected for Reynolds number effects on skin friction, but experience with small underpowered seaplanes has substantiated qualitatively the existence of the effects described.

Tank data for hulls of the planing-tail type having large afterbody clearances (ref. [33]) lie very close to the forebody plus air drag-curve in Figure 17, indicating one solution to the problem if such a hull can be integrated into the overall design in other respects. A second method to increase clearance is the use of auxiliary skis at high-water speeds, extended far enough below the hull for their wakes to clear the afterbody.

Figure 17. Variation of hull efficiency with planing coefficient.

Figure 18. Effect of take-off speed on hull resistance.

Ski-body combinations.—The nature of the relatively high hump resistance of an aerodynamically-refined hydro-ski water-based airplane is shown in Figure 19. The hydrodynamic resistance of the aerodynamically-shaped fuselage alone rises rapidly with speed to a prohibitive value (refs. [35 to 38]). The planing resistance of

Figure 19. Take-off resistance of hydro-ski airplane.

the skis alone intersects that of the fuselage at a speed depending on ski size and angle of attack, and this speed is selected to transfer the water-borne load to the skis before the resistance becomes too high for the available thrust. The resistance of the combination is somewhat higher than the intersection because of spray interferences, the resistance of the ski gear before emergence, and the energy required for the dynamic emergence maneuver itself. Above emergence, the resistance decreases rapidly with speed until it approaches that of the skis.

The hump resistance can obviously be decreased by designing the skis for earlier emergence, but they then approach the proportions of a hull forebody and become more difficult to retract in flight. It can also be decreased by hydrodynamic treatment of the fuselage, which can be accomplished to some extent without a large departure from an aerodynamically clean body. Even with a complete forebody and afterbody on the body, however, the hump resistance is considerably higher than that of a hull alone.

Because of these considerations, the application of skis to airplanes with low thrust-weight ratios (below 0.3) is not strictly in accord with the concept. The emergence resistance of practical ski-body combinations remains a research problem of some complexity, and means for reducing it to the level of the hump resistance of a hull have not yet been found.

The extensive involvement with spray of the aerodynamic components of a hydro-ski airplane raises the question of how the forces due to spray vary with scale. A preliminary experimental investigation (ref. [39]) has indicated that the lift force generated on a flat plate by impinging spray from a planing surface can be scaled by the conventional Froude methods. The spray drag forces, however, are relatively

smaller for a larger model, evidencing a Reynolds number effect similar to that for turbulent skin friction in solid flow.

Land-water gears.—In the case of a land-water airplane, the hump resistance of the hull or emergence resistance of the ski-body combination are by-passed entirely by the use of the ramp and taxiing wheels as shown in Figure 20. Here the governing

Figure 20. Take-off resistance of land-water airplane.

factor is the minimum planing speed below which "water stall" occurs and the resistance is very high. Above this speed, the resistance decreases rapidly as shown, and the transition speed must be enough higher to keep the resistance to an acceptable value after leaving the ramp. In general the planing resistance will be higher than for the larger skis of a completely water-based combination because of the associated heavier ski-loadings and lower hydrodynamic aspect ratios. Thus although the problem is alleviated by the mode of operation, it cannot be neglected entirely, particularly in the choice of design transition speeds and in the need for a hydrodynamically clean configuration in the water-borne part of the operation.

Hydrofoils and Struts

Hydrofoils.—The use of submerged hydrofoils as lifting elements for water-based aircraft has always been an interesting idea because of their potentially high lift-drag ratios and favorable rough-water qualities as compared with flying-boat hulls. The hydro-ski is essentially a low-aspect ratio hydrofoil that can be integrated into airplane design and operation, and avoids the flow problems associated with negative pressures by planing on the surface at high water speeds. Apart from early Italian applications, however, competitive high aspect-ratio hydrofoil systems for aircraft have not yet appeared, although fundamental research on the subject has been continued (ref. [28 to 31]) and applied in fields other than aeronautics.

The incentives for the development of practical hydrofoil configurations remain high, since even with moderate aspect-ratios and separated flows they continue to

offer efficiencies somewhat superior to those of hydro-skis, and their rough-water advantages over planing hulls have been further demonstrated by recent surface-craft applications. There is therefore a need for further basic research on high-speed cavitated and ventilated flows, and new design and operational concepts leading to their effective integration into high-speed aircraft configurations.

Struts.—Underwater lifting elements for aircraft must be supported by strut systems extending below the hull or fuselage. When completely submerged, the struts are a source of drag predictable from airfoil data, except when the speeds are high enough for the onset of cavitation. Rake reduces the section drag coefficient and increases the incipient cavitation speed as would be expected because of the reduction in effective thickness ratio (ref. [40]).

When piercing the surface, struts produce wave drag also predictable from theory (ref. [41]), and are subject to ventilation effects. As for the underwater plates, the mechanism of the ventilated flow and the laws of similitude that apply are not completely understood and merit further investigation.

Impact Loads and Motions

The loads and motions resulting from wave encounter increase with forward speed and must continue to be dealt with in the design of high-speed water-based aircraft. Theoretically, the acceleration resulting from an isolated wave impact varies as the square of the vertical velocity, and, with the same flight-path angle as the square of the forward speed. Actually, other things do not remain equal and maximum accelerations increase at a slower rate (unpublished experimental data), but even the first power is prohibitive when landing speeds are being doubled or tripled.

The results cited have shown that rough-water loads and motions of hulls are greatly reduced by increased beam loadings, afterbody lengths and angles of dead rise (ref. [5]). They have also been ameliorated in practice by the use of reversible thrust during landing and thrust augmentation during take-off to reduce the number of critical impacts. Perhaps the greatest improvements are offered by the use of heavily-loaded elements, to achieve high orders of penetration and localization of the loads. The development of auxiliary systems for this purpose constitutes a fruitful field for further investigation in order to hold high-speed water-based aircraft competitive with their land-based equivalents with regard to structural weight while retaining a reasonable degree of seaworthiness.

High-Speed Test Facilities

Along with increases in airplane landing speeds, there is an obvious need for a corresponding increase in the water speeds available for experimental investigations of hydrodynamic elements and configurations. The maximum towing speeds of the NACA seaplane tanks (50 knots) are now regularly employed in fundamental experiments and Froude model evaluations. Full-scale experience has approached 160 knots for seaplanes and 200 knots for racing craft. The latter speeds are still below the runway speeds of current high-speed landplanes.

Planing data has been obtained at the NACA's Langley Aeronautical Laboratory up to 120 knots by the use of a free-water jet propelled by compressed air (ref. [42]) as shown in figure 21. This apparatus has proved useful and capable of further development; it is subject, however, to boundary corrections and free-surface effects not present in a still-water towing tank.

This laboratory has recently placed in operation a high-speed hydrodynamics facility capable of test speeds up to 130 knots. A view of the facility is shown in Figure 22. It consists of a concrete tank 2,000 feet long, 8 feet wide, and 5 feet deep adjacent to an existing track and towing carriage provided for research on wheel landing gears. The hydrodynamic model and its instrumentation are suspended from a temporary boom on the carriage overhanging the water tank, and data are obtained during decelerated runs after the carriage is brought up to speed by a water-jet catapult.

Figure 21. NACA water-jet apparatus.

Figure 22. NACA high-speed hydrodynamics facility.

A second lighter carriage is being provided for hydrodynamic testing that will raise the available speeds up to the order of 150 knots, or three times the still-water range hitherto available.

VII. CONCLUDING REMARKS

In conclusion, it may be repeated that the present era of high-speed flight has a profound influence on the relative importance of the various considerations and conflicts involved in the design and procurement of water-based aircraft. Aircraft of this era are transonic and supersonic with fantastic speed penalties associated with departures from optimum aerodynamic design. Their water-based counterparts at present may have minimum flight speeds beyond the range of past experience; at the same time, they have take-off thrust-weight ratios that encourage the employment of radically different hydrodynamic configurations and concepts.

The changing picture brings with it new hydrodynamic problems, and new justification for the continued attention of qualified authorities in the field. The opportunity to review the state of the art in this Symposium is therefore appreciated by the author and the agency represented.

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DISCUSSION

W. A. Schoech

I would like to congratulate Mr. Parkinson for a timely and important contribution to the science of the hydrodynamics of high speed water based aircraft. This field is of great interest to the Bureau of Aeronautics and we have attempted to keep abreast of the art with research vehicles and prototype aircraft. The Navy has now flown and obtained experience with three different flying boats having hull length/beam ratios greater than 12. The excellent hydrodynamic characteristics predicted by the model tests has been generally borne out by the flight tests. As might have been expected, the low speed maneuvering characteristics of these hulls were unsatisfactory and sparked off additional research.

An extensive flight test program has been conducted with a variety of aircraft equipped with hydro-skis. One of these aircraft, the XF2Y-1, was not only the first aircraft designed with hydro-skies as part of the original concept, it was also the first flying boat to exceed the speed of sound. The flight tests of hydro-skis have uncovered new and novel hydrodynamic problems. Mr. Parkinson has been instrumental in solving these problems and providing us with a clear physical understanding of the phenomena so that they can be avoided in the future. The full scale experience with flying boats equipped with hydro-skis has also clearly demonstrated improved ability to operate in rough water. This is of considerable interest to the Navy.

Some years ago All American Engineering Co., in an unsolicited proposal, suggested the application of wheeled hydro-skis to landplanes. This concept has been found to increase greatly the operational flexibility of Marine Corps observation airplanes. Mr. Parkinson's work has done much to put the design of these hydro-skis on a rational and straight forward basis.

Hydrofoils have offered promise of greatly improved rough water characteristics and the Bureau of Aeronautics has actively supported research in this field for at least twenty years. The very great difficulty of providing both stability and control has seriously hampered efforts to incorporate hydrofoils on a man-carrying aircraft. Recent work by Mr. Parkinson has indicated that solutions to these problems may be in sight. If this turns out to be the case, hydrofoils will open the door to much improved performance.

Mr. Parkinson's paper offers much fresh data which will be invaluable to designers of high speed flying boats. Again, I would like to offer my congratulations

to Mr. Parkinson for his most interesting and informative contribution to the hydrodynamics of high speed water-based aircraft.

C. L. Fenn

Mr. Parkinson's complete coverage of the high speed hydrodynamic situation has left me with practically nothing to say. I might, however, make one or two points. They are very small.

It might be possible in future development to arrange a hydroski or hydrofoil, but for the moment let us consider a hydroski, of such a nature that it would lift the aircraft out of the water at a speed at which the aerodynamic, but not the hydrodynamic, loads would design the fuselage. This would put the seaplane on an even footing with the land plane. It is something to think about.

The next thing that occurs to me is where does the hydrofoil sit in relation to the hydroski, development-wise? It seems to be somewhat further back. At the moment, we have no correlation between the model tests and full scale operation.

It also appears—and I will not say the dangerous thing—the dangerous thing to say is that there has been no successful hydrofoil aircraft. When you say that, someone says "Oh, they had one in Hammondsport in 1908." But, let's put it this way—there has been no hydrofoil aircraft so good that it spawned a family of hydrofoil airplanes.

Such is not true with the hydroski. There have been complete families of hydroski airplanes. I know of one, four in number, which started at 8,000 pounds and which has grown to 55,000 pounds, and they have lent themselves to high-speed operation.

R. M. Hopkins

Mr. Parkinson is to be congratulated on a very interesting and comprehensive paper. While listening to it, however, several thoughts arose which might be of some interest to you. On looking back—particularly over the past year—I can imagine that Jack may well feel that we never meet without complaints on my part over things he hasn't done. It may therefore be appropriate at this point to mention a few of the things he has done.

First, Jack and his group at Langley did the pioneering work on hydro-skis, which in conjunction with the development of the turbojet engine was the initial step toward high speed equality between land-based and water-based aircraft. More recently Jack has combined forces with the aerodynamics people at N.A.C.A. to show that there are no fundamental reasons why the famed "area rule" concept may not be applied equally as well to water-based as to land-based aircraft. This type of leadership has caused many to re-evaluate the future role the seaplane may play.

I should now like to bring to your attention one or two high speed hydrodynamic problems in somewhat more detail. In the slides just presented, you will recall the one showing the very large facility designed to examine the problems involved in a land-plane landing gear when it makes contact with a nice smooth runway. It is suggested that the high speed towing tank auxiliary to that facility which Jack discussed might be used to examine the much more severe problems of high speed seaplane landing gears in contact with a runway that is far from smooth.

These problems are closely allied to the self-excited or hydroelastic vibrations which Jack discussed, and for which some insight may be gained through the brief theoretical treatment of the flow involved contained in Professor Milne-Thomson's book on "Theoretical Hydrodynamics." It is believed, however, that the crux of the matter is best stated as that case for which the unsteady input forces (whether self-excited in character or due to wave impacts) excite the natural or resonant frequency (or frequencies) of the aircraft structure. The serious nature of this problem may be appreciated by a reference to an early incident in Sea Dart development. This seaplane

originally incorporated twin skis with rectangular trailing edges for low resistance reasons. We soon became aware of our mistake, however; for soon after the first taxi run, and while taxiing over what may be very roughly described as four to six inch wind waves, about 12 to 15 feet apart, we had a failure of the nose-mounted airspeed boom. The dynamic characteristics of the Sea Dart structure were such that these seeming insignificant waves produced a failure in an airspeed boom designed for in excess of 50 g's. It is believed that the structural and hydrodynamic implications of this incident might well be sufficient justification for test runs of full scale hardware in this new N.A.C.A. facility, particularly in view of the trend toward greater airframe slenderness and flexibility.

It is also believed possible that hull bottom plating fatigue failures may be partially attributed to the higher stresses resulting from resonance effects. It is known that the pressure pulses imposed upon seaplane bottoms during high speed planing over waves may exhibit a pressure-time history so steep as to approach a so-called "step function" in character. Since a step-function theoretically contains all frequencies, it would seem plausible that these very steep pressure pulses might, through resonance effects, produce a plating stress in excess of that which would be produced by the same peak pressure acting statically. It therefore seems that the new facility would lend itself very nicely to an investigation of this type through runs of a segment of actual seaplane bottom structure at high speed over waves and with pressure pickups and strain gauges installed.

One other area might be mentioned for which the new facility could supply answers of interest to the high speed seaplane designer. Most of the impact theories presently available are built around the concept of the single, isolated, design landing. They are based on a mental image of a relatively large wave, and the detailed calculations therefore treat an impact on an unlimited expanse of water. The preceding discussion makes it clear that waves which are relatively small compared with the airplane and having relatively steep slopes may be of considerable importance. It is therefore suggested that a theory is needed to account for the finite expanse of the wave—an allowance for breaking through the back side of the impacted wave.

R. W. L. Gawn

We should all be grateful to Mr. Parkinson for an excellent paper with a clear account of many important developments in flying boats. It serves as a reminder that there are some common hydrodynamic problems between the flying boat and the craft known as fast patrol boats or in your country P.T. boats and it is instructive to consider these and ascertain the extent to which experience matches up.

The "Princess" flying boat is much the same all-up weight as the fast patrol boats known as the BOLD Class. There is a notable difference in the hull in that the flying boat is longer and narrower. In fact the length is 8.9 times the beam whereas the ratio is only 5.0 in the fast patrol boat.

An interesting point in common is that the running trim is controlled in both craft. The elevators serve this purpose on the flying boat and operate in air. Transom flaps are the counterpart on fast patrol boats and are noticeably smaller than the aircraft's elevators, one reason being that they operate in water. The purpose is to reduce the running trim primarily to make the fast patrol boat more seaworthy at the hump speed by reducing slamming. It is very satisfactory in this respect and also leads to less hull resistance.

There has been controversy as to the relative merits of hard chine and round bilge hull shapes for fast patrol boats. A boat has been built of each type in recent years and trials carried out at sea and in addition there has been a series of comparative tests on models. The latter indicated that the ship motions including accelerations are very much the same, the hull resistance of the round bilge form is less up to a speed length coefficient approaching 4 but is greater at higher speeds and the only other notable difference is that the hard chine form is drier in a seaway. The comprehensive sea

trials are not yet complete but so far the comparative results do not seem to differ significantly from those obtained in model tests.

A matter of great importance is the pressure and acceleration which the hulls of fast patrol boats have to sustain at sea. Pressures have been recorded up to 25 lbs. per square inch with maximum accelerations ranging from about 2g to 3g. The latter are of course well below the figure of 50g mentioned by the last speaker in another connection. The seas to which the fast patrol boats were subject are probably more severe than those for flying boats and in the particular trial for which results are quoted the waves were 3 to 4 feet high and about 70 to 80 feet long. The large pressures build quickly in the fast patrol boats and generally in about a thirtieth of a second. My impression is that impact stresses develop at a slower rate in aircraft.

It would be most useful if Mr. Parkinson could add to the value of his most instructive paper by giving some complementary information from his wide experience with flying boats regarding the various points mentioned.

J. D. Pierson

Mr. Parkinson has presented an excellent, comprehensive review of the trends in seaplane design particularly with regard to high speed aircraft. I have only a few comments to add with regard to the water loads imposed in take-off and landing.

It appears that the trend towards longer and thinner hulls required for high speed flight is consistent with the improvement in seaworthiness in rough water ship designs. Unfortunately, the aircraft trend includes an increasing speed for take-off and landing.

In a more descriptive sense the aircraft speeds on the water are approaching the conditions at the bottom of a great waterfall such as Yosemite Falls. From a height of over 1500 feet the free fall velocity would be over 300 feet per second. It is in such a relative stream that the thin-skinned, lightly built seaplane must be successfully landed. And not only must this be accomplished many times in calm conditions but also in waves.

Many ways of doing this have been noted. On wheeled aircraft the tire and oleo strut provide the energy absorbing stroke. For the hull the deadrise of the bottom effectively provides this stroke, and devices such as skis and hydrofoils may be extended below the keel to extend further the effective stroke.

The actual application of these configurations to seaplane designs requires the detailed calculation of loads for structural design. Because of the trend towards the longer, slimmer hull forms these loads calculations have been made more complex by the interaction of the body flexibility and the dynamic water forces. It has only been in the past few years that the development of the rapid automatic computation systems has made such combined solutions possible.

The water loads problem is treated by dividing the hull into a number of transverse slices which are acted upon by the water forces and react according to the overall motion of the aircraft and the major bending modes. The water forces on each slice are determined on the basis of the local hull shape and water conditions (including waves) and the dynamical motion of the slice. The summation of all slices gives the net effect on the body.

Even for a very limited number of divisions of the hull, manual calculation of this loads problem is well-nigh impossible. However, with the large computers such as the IBM 701 and then the 704 a large number of sections may be handled simultaneously for the required solution.

So far in the computational work, uniform waves of trochoidal form, have been used to represent the sea as a matter of convenience. Other wave shapes and combinations of water conditions are within the capability of the computational machinery, and we look forward to a greater combination of the results of oceanographic studies and practical seaplane design.

Mr. Parkinson has presented a fine summary of the "status of the art" of high speed seaplane hydrodynamics—both as regards the special hydrodynamic problems which have been accentuated by the high speed water based aircraft and as regards some of the new hydrodynamic developments which are available and necessary to the designer of high speed seaplanes.

I would like to amplify further Mr. Parkinson's reference to the beneficial potentials which may develop with the application of hydrofoil alighting gear to the aerodynamically refined high length beam ratio hulls. In particular, it appears that a hydrofoil system shows a good deal of promise in improving and extending both the high and low speed rough water capabilities of the seaplane. As pointed out by Mr. Parkinson, high speed hulls, while having a satisfactory lift drag ratio, are subjected to heavy impact loads and motions in a seaway. Small, highly loaded, penetrating hydro-skis alleviate the severe rough water loads and motions but have excessive water resistance during ski emergence. The high aspect ratio hydrofoil, on the other hand, has a lift drag ratio comparable to the hull and can be designed to alleviate the impact loads and motions to a degree comparable to the hydroski. A significant design feature which is inherent in a hydrofoil system is that the hydrodynamic support area of the main foil will probably be distributed vertically rather than longitudinally as in the case of either a hull or hydroski. This would keep the center of pressure essentially in a fixed longitudinal position and hence eliminate generation of excessive pitching moments. Further, the vertical arrangement of support area can be designed to contact the water surface with small areas during the impact process and hence lead to alleviation of the impact forces.

Another advantage of the hydrofoil system which has not been generally discussed, is the effective damping in pitch that a tandem foil system provides at prehump speeds. In these speed ranges, aerodynamic damping is negligible and the high density, low frontal area hulls are subjected to severe pitching and excessive bow spray generation when operating in a seaway at low speeds. In some cases the development of severe wetting of engines and structure may preclude take-off. The tandem hydrofoil system has the capability of providing significant pitch damping which in turn would reduce the pitch motions and the corresponding bow spray development.

To achieve these hydrofoil potentials we must continue our research on the basic hydrodynamic problems attendant with hydrofoils, e.g. cavitation, ventilation, longitudinal and lateral stability of foil systems suitable for seaplane application.

The high landing speeds associated with projected high speed water based aircraft has put a strain on the existing facilities of the Stevens Experimental Towing Tank. It appears that the top speed of our No. 3 Tank will have to be increased to accommodate the increasing landing speed of seaplanes and to handle some of the cavitation problems which exist with the hydrofoil systems. Also, an irregular wave maker has been installed in our No. 3 Tank to provide for more realistic evaluation of the rough water characteristics of seaplane forms. Our present No. 2 rectangular tank facility which is being altered to examine ship behavior in oblique seas, will also be able to accommodate tests of seaplanes at oblique headings to a sea. Combining these new test facilities with the so-called ETT "free-to-surge" towing apparatus, which permits a model to be towed with complete longitudinal freedom, will make for a more realistic evaluation of the new seaplane designs.

M. C. Eames

The writer was most gratified to learn of the progress being made in the study of the mechanics of ventilated flow on submerged lifting surfaces, but was disappointed to discover the relative states of development of the hydro-ski and the hydrofoil for application to water based aircraft.

It is to be hoped that recent work by Tulin and Wu on the theory of hydrofoils operating in a fully cavitated state will provide the additional incentive required to initiate an active research program in this field.

Sufficient evidence must now exist to convince the aircraft designer of the hydrodynamic superiority of the hydrofoil. One's thoughts, therefore, naturally turn to the structural problems involved, and it seems inevitable that these will be very much more severe in the case of the hydrofoil. One thought in this direction appears appropriate to a symposium on hydrodynamics. The writer is treading on dangerous ground, but it is believed that several of the problems met in high speed flight have a counterpart in the design of hydrofoils. (One elementary example that comes to mind is the simple relation that exists between the critical Mach number and the critical cavitation number of a foil section.) It might not, therefore, prove to be beyond feasibility to design a hydrofoil system which, rather than being an awkward encumbrance in flight, could perform a useful aerodynamic function—thus obviating the need for retraction. If this were possible the weight normally associated with retraction gear would presumably be the hydrofoil structure, and it is even feasible that an increased payload might result.

J. B. Parkinson

The discussers of this paper have enhanced its value by supporting and amplifying two major points which were made but not developed to any extent. The first is that water loads and their effects on airplane structures are of increasing concern because of current design trends and operating requirements. The second is that there remain valid reasons for further research and development in the area of hydrofoil-supported aircraft. The two are interrelated since the hydrofoil provides an additional promising approach to alleviation of prohibitive loads and motions due to wave impacts. With this clarification of objective, it now should be possible to arrive at practical hydrofoil applications for water-based aircraft that are competitive with or perhaps superior to presently available solutions.

An extended discussion of water loads is beyond the scope of the paper. It may be mentioned that authorities on this subject are reluctant to prescribe upper limits on loads, pressures, and rates of build-up resulting from rough-water operation. Transient pressures up to 50 psi have been measured on a seaplane and up to 100 psi on a large hull model in the Langley impact basin. The effects of local resonances and flexibilities, finite expanse of the waves, etc., are not too well understood, but in general are believed to be small compared to the total loads, statistical variations of the dominating parameters, and the orders of alleviation required for significant improvements in rough-water capabilities.

The author wishes to express his thanks and those of his colleagues in the NACA for the favorable comments on the paper and the stimulating thoughts contributed in its discussion.

IX

RECENT CONTRIBUTIONS TO BASIC HYDROBALLISTICS

R. N. Cox and J. W. Maccoll

Armament Research and Development Establishment

I. HYDROBALLISTICS AND THE BALLISTIC SCIENCES

As the word implies, hydroballistics is primarily concerned with the motion of projectiles through water or any other liquid medium. The subject is thus the hydrodynamic counterpart of aeroballistics which concerns the corresponding behavior in air. As it is usual for missiles to have an air flight path before entering the water, water entry is also included as an important section of hydroballistics.

Hydroballistics has however connections with other branches of the ballistic sciences. Thus certain problems of terminal ballistics are closely related to similar problems in hydroballistics: the latter subject can in fact be regarded as the terminal ballistic study of the ideal liquid-like medium. Terminal Ballistics is a field in which the basic scientific background is still largely undeveloped and it may be useful to consider certain terminal ballistic problems from an hydroballistic standpoint. This topic will not be carried further here as the present symposium is concerned with hydroballistics in its relation to naval problems.

In the present paper we deal mainly with some of the basic researches carried out during the last decade: most emphasis will be on work that has appeared during the last five years.

Our subject is nowhere dealt with extensively in the open literature, but Professor Garrett Birkhoff's volume on Hydrodynamics (1950) makes frequent reference to selected topics of a hydroballistic character, and we shall assume some degree of familiarity with the matters touched upon there. Much of the literature on the subject, in both the United States and in the United Kingdom, is in the form of Research Establishment reports. Such reports are not distributed very widely and are not always readily obtainable. Because of this we trust we have not omitted from our review any major contribution to our subject.

II. THE SEVERAL SECTIONS OF HYDROBALLISTICS

The hydroballistic field divides conveniently into a number of phases which can be associated with the typical trajectory. Thus, referring to Fig. 1, we have

- a. *Shock phase* which covers an initial, and extremely short, period of shock when the water surface is hit (I).
- b. *Water Entry phase* which covers the subsequent cavity forming period (II).
- c. *Closure phase* which follows, and which includes either or both surface seal and deep closure (III).
- d. *Quasi-Steady Cavity phase* in which the missile moves with an attached cavity (IV).
- e. *The Fully-Wetted condition* which will exist at sufficiently great hydrostatic pressures, when a cavity cannot be maintained. The conditions of motion are here similar to those existing in subsonic motion in air at the corresponding Reynolds number: due to the density ratio, the forces in water are about 800 times greater than those in air.

Figure 1. The phases of water entry.

Although these divisions of the field are rather artificial in character, and do not embrace conditions such as those which hold for ricochet, they do serve the useful purpose of helping towards a specification of the several regimes which are amenable to mathematical analysis. During the last few years most basic research has been concerned with phase (d), with a more limited amount of activity in (b), and the following review is mainly concerned with these two phases.

III. SHOCK AND WATER ENTRY PHASES

When a missile makes contact with a water surface, the water is compressed near the point of contact and a shock wave is formed which radiates at a velocity a little higher than the speed of sound in water. This initial shock phase, in which the compressibility of the water is important, lasts for a few micro-seconds. During this period, the nose of the body experiences a localised peak pressure, the magnitude of which is approximately ρcv where ρ is the density of water, c the speed of sound in water and v is the vertical component of the velocity of entry. Apart from the possibility of damage to the body, the shock phase is relatively unimportant in the study of underwater ballistics because of its very short duration.

The first stage of importance is the inertial, or flow forming, phase during which the water, which is originally at rest, is accelerated around the nose of the missile. The rapidly changing flow can involve large accelerations and the production of high transient pressures. The determination of the high transient pressure distributions over the surface of the body is of practical importance since the forces produced determine the

initial conditions for the subsequent underwater motion of the missile. The most important of these conditions is the change in angular velocity, or "whip," experienced by a missile on water entry; design engineers have already at their disposal a variety of head-shape devices which provide pitching moments of varying magnitude and sign. These devices, most of which have been arrived at by empirical methods, give the designer considerable control over the subsequent trajectory.

a. *Normal Entry.* A theoretical study of the flow forming stage is difficult, even with normal entry, as it involves three independent variables: two space coordinates and time. An additional difficulty is that the exact boundary conditions on the free surface are non-linear. For these reasons no exact mathematical solution for a water entry problem, even with the simplest headshape, can be expected. The best we can hope for are either approximate solutions or solutions of the exact equations obtained by numerical methods.

The problem is simplified for wedges and cones entering at a constant speed since the flow pattern produced by such bodies may be pseudo-stationary; under these conditions the pattern of the disturbed flow increases in size uniformly with the depth of penetration. The vertical impact of a wedge on a water surface was investigated many years ago by Wagner (1933); this problem was further developed by Pierson (1950) who calculated in detail the conditions on wedges of several angles. Likewise, Schiffman and Spencer (1951) considered the similar problem of the vertical entry of a cone and dealt in detail with the case of a cone of 60 degs. semi-angle. Solutions of problems of this type can be obtained in a number of ways; for example Schiffman

Figure 2. Pressure and velocity distributions on 60° semi-angle wedge and cone during water entry.

and Spencer applied Green's theorem in an iterative process which alternated between conditions on the free surface and those of the solid body. Fig. 2 shows the calculated distributions of velocity and pressure over the surfaces of these wedges and cones. It is interesting to note that the peak pressure on the body is attained close to the origin of the spray jet.

An alternative approach to the normal entry problem is to construct a model flow which reproduces the main conditions of the problem. Von Karman (1929), for instance, used a technique in which he approximated to the conditions of impact of seaplane floats striking the water by the flow produced by the impulsive vertical motion of a plate expanding in the plane of the water surface. For similar horizontal water surface areas, the pressure distribution produced by this model flow approximates to that over the actual submersed body. In his original paper von Karman made no allowance for the rise of the free surface but methods can usually be developed to introduce this refinement, if required. Wagner's (1933) paper did this for the von Karman problem.

Most of the theoretical investigations of water entry carried out during the last decade have been concerned with extensions of the methods originally developed by von Karman and Wagner. Thus problems have been formulated to take account of the actual penetration of the body into the water. This is usually done by considering the flow about an expanding model consisting of the submerged portion of the actual body together with its image in the plane of the undisturbed free surface. An example of this type of investigation is the war-time work of Schiffman and Spencer, who considered the entry of a sphere by means of an expanding lens-shaped model.

Figure 3. Expanding hollow body models for water entry flow patterns.

Likewise Monaghan (1949) extended the work of Wagner by approximating to the entry of a wedge by the flow past an expanding prism.

In addition to the above, a number of model flows have been investigated by comparatively easy analytical methods and are usually found to be in fairly good agreement with experimental data. In this connection mention may be made of some war-time work at the California Institute of Technology on the potential flow about a spherical bowl as the basic pattern for the vertical entry of a sphere (Fig. 3(a)). In this model the $p = 0$ contour is taken to be the free surface. Although the condition of continuity is not satisfied in this model, most of the physical features of the actual problem appear to be reproduced. A similar treatment has been that carried out recently by Coombs at Fort Halstead for an expanding hollow wedge (Fig. 3(b)). Although infinite velocities are present at the rims of both these models, these singularities are outside the field of practical interest.

b. *Oblique Entry.* With oblique entry the problem is further complicated by the presence of an additional independent coordinate and for this reason little work has been carried out in this field. Trilling (1950) has devised a technique for calculating approximately the forces on a body of arbitrary shape during the early stages of submersion. He estimates the hydrodynamic forces on the body at a number of depths of penetration by considering the impulsive motion of half a general ellipsoid having the same depth of penetration, the same length and the same submerged volumes as the actual body. Trilling applied his method to the case of the water entry of a sphere at an angle of 45 degs. to the water surface and obtained for the vertical and horizontal components of the drag the results shown in Fig. 4. It will be seen

Figure 4. Variation of drag coefficient with depth for 45° oblique entry of sphere.

that the vertical force component is in good agreement with experimental results; it is also of the same type as that obtained by Schiffman and Spencer.

A new method of investigating the oblique entry of a cone has recently been developed by A. Coombs (1956) with promising results. In this method the pressure distribution along a generator is assumed to be the same as that over the corresponding generator of a cone of the same apex angle when entering vertically into a conical water surface. The basic assumptions of this technique are illustrated in Fig. 5; the

Figure 5. Strip theory for oblique water entry of cone.

pressure distribution along the generator OB of the top diagram is assumed to be the same as that over the cone BOB' shown in the lower left hand diagram. Similarly correspondence is assumed for conditions on generator OA and cone AOA'; also for generator OC and cone COC'.

IV. THE STEADY CAVITY PHASE: AXI-SYMMETRIC PROBLEMS

Hydroballistics is mostly concerned with the motion of bodies of revolution and it is unfortunate that much of the extensive exact theory of two dimensional cavity flow cannot be applied directly to hydroballistic problems. During the past few years, research on the axi-symmetric problem has concentrated mainly on a. the development of analytical techniques capable of solving the exact differential equations and b. methods involving similarity between plane and axi-symmetric flows.

In the present section we propose to go into various aspects of axi-symmetric problems in greater detail.

a. *Exact Theory.* The calculation of the conditions of flow in axi-symmetric problems is much more difficult than the corresponding problem in two dimensions. This is largely because in the axi-symmetric field we have no technique corresponding to the theory of conformal representation of a function of a complex variable. Attempts have been made to overcome this difficulty, with some success in specific problems.

The Rankine method of using sources and sinks in a uniform stream was a first attempt to solve this problem. Although this model gives a superficial representation of the flow, and may be adequate for many engineering problems for certain contours, the source-sink technique often fails to give the finer detail, which is desirable to obtain an accurate representation of conditions near flow singularities, such as at pointed noses and at separation.

Several developments of the last decade have had the object of improving on the Rankine approach. Some of these researches have made use of mathematical singularities of a higher order: thus the simple sources and sinks of Rankine, which were distributed arbitrarily along the axis of symmetry, are replaced by vortex sheets situated along the surface of the body and along the free streamline surfaces. This type of model was proposed by J. W. Fisher in a war time report of 1944 and was later developed in mathematical terms by Vandrey (1951) when working at the A.R.L. in England. If, for example, this method is used to investigate the Riabouchinsky model for the cavitating flow around a head contour (Fig. 6), then the method involves the

Figure 6. Riabouchinsky model for axi-symmetric cavitating flow.

solution of integral equations representing the flow conditions over the head and tail contours H and T and over the free stream surface M. The vortex sheet is, of course, of varying strength over H and T, and of constant strength over M.

Vandrey did not apply his technique to any case of cavity flow but he showed that the method gave satisfactory agreement with experiment in three cases of fully wetted axi-symmetric flow; it is clear from these examples that his iterative method of solving the second order integral equations will, after a few iterations, usually lead to a satisfactorily convergent solution although the computational labour involved may be considerable. This work of Vandrey was followed up shortly afterwards by the investigations of Armstrong and Dunham (1953) who formulated a similar method for obtaining an exact iterative solution of the Riabouchinsky model for axi-symmetric cavitating flow around a flat disc held normal to the main stream. In order to simplify the numerical work associated with their method they made use of certain similarity principles between the two-dimensional and three-dimensional flows. The results of these calculations give cavity dimensions which agree well with the available experimental data; Fig. 7 shows satisfactory agreement between the calculated cavity length

Figure 7. Variation of length coefficient with cavitation number for flow past disc.

and the observations of Rouse and McNown (1948) in the Iowa cavitation tunnel. The modified drag coefficient $C_D' = C_D/(1 + Q)$ calculated by Armstrong and Dunham was in good agreement with the Fisher theory; the numerical values, were, however, some 3% greater than those calculated by Plesset and Schaffer using their similitude technique which will be described later (Fig. 8).

During the last year Garabedian (1955) has described further investigations of

Figure 8. Variation of modified drag coefficient with cavitation number for flow past disc.

axi-symmetric problems. His basic technique is to relate the stream functions in plane and axi-symmetrical flow by the partial differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\epsilon}{y} \frac{\partial \psi}{\partial y} = 0 \quad (1)$$

where $\epsilon = 0$ corresponds to the plane flow and

$\epsilon = 1$ corresponds to the flow with axial symmetry.

He considers the flow in the later case to be a perturbation solution of the case of plane flow, i.e. $\epsilon = 0$. Relationships connecting the boundary conditions on the free streamlines are similarly derived. Garabedian's method offers attractive possibilities: however, like all other techniques in this field, it entails considerable computation in its actual application. It is interesting to note that Garabedian's computation of the modified drag coefficient of a disc is in reasonably good agreement with the value obtained by Armstrong and Dunham.

b. *Approximate methods using Principles of Similitude.* The difficulties involved in applying exact theory have led to attempts to solve certain axi-symmetric problems by the use of similitude considerations linking the axi-symmetric flow with the corresponding problems in two dimensional flow. It can be shown (Armstrong and Tadman, 1954) that for the fully wetted flow past elliptical profiles, an exact similarity exists between plane and axi-symmetric flow for different but specific values of the

flow velocity. By assuming that the same similarity law holds for cavitating flows (for which the cavity is approximately ellipsoidal) it is possible to calculate the flow about cavitating ellipsoidal head shapes. Fig. 9 shows the similarity relationship for this case.

Figure 9. Similarity relationship between plane and axis-symmetric flow for ellipsoids.

To obtain the plane flow past curved surfaces it is, however, first necessary to determine the separation position. This type of problem was solved by Brodetsky (1922) and Schmieden (1929) using the Levi-Civita method. The separation positions obtained are in poor agreement with experiment, and it seems possible that the boundary layer plays an important part. The Brodetsky solution for a circular cylinder gives, for instance, separation at 55 degs. whereas experimental observations give values 10-20 degs. greater. A possible explanation of this is afforded by Fig. 10, which shows the calculated distribution of wall shear stress in laminar flow using the Brodetsky velocity distribution: the wall shear stress is high at the calculated separation position, and may inhibit cavitation inception, causing the separation point to move downstream.

There are also some analytical doubts as to the validity of the Brodetsky and Schmieden solutions near the separation point since the calculated position turns out to be a singular point of the head profile for derivatives of the curvature. Complete removal of the infinity is only possible if the free streamline coincides with the body.

Such considerations have to be borne in mind when any attempt is made to obtain solutions of any steady cavity problem by means of high speed digital computers. The application of relaxation or similar techniques is liable to conceal the

Figure 10. Velocity and laminar wall shear stress distribution for Brodetsky flow past circular cylinder.

true conditions of flow near the singularities even using a fine network near these points.

Professor Garrett Birkhoff and his colleagues (1953) have used computer techniques extensively on plane flow problems with curved profiles: they have in fact extended the Brodetsky approach to cavity flow to a much higher degree of refinement. Their investigations have given valuable experience of the formulation of plane flow problems and should prove of considerable value when the physical conditions of flow in the neighbourhood of the separation point are better understood.

These difficulties are no longer present if separation is defined by a corner, as for instance in the flow past wedges and discs. Although the singularity of the flow still exists at the corner in these cases, its behaviour is well understood and can be allowed for. The velocity distribution for abrupt separation at a corner can be seen in Fig. 11 and may be compared with that for smooth separation with a curved profile as shown in Fig. 10.

M. S. Plesset and P. A. Schaffer (1948) calculated the drag coefficients for a series of cones and discs over a range of cavitation numbers using the assumption that the pressure distribution along the generator of the cone or disc was the same as that on a wedge having the same semi-angle at the same cavitation number. The implications of this assumption have been investigated more recently by Armstrong (1953(a)) who, as a result, proposed improved similitude rules for wedges and cones. Armstrong showed that, if the velocity distributions near the apex of a wedge (semi-

Figure 11. Comparison of velocity distributions over 5° wedge and 14° cone.

angle β_2) and a cone (semi-angle β_3) be expanded as power series then, the velocity near the tip of the wedge was

$$\left| \frac{dw}{dz} \right| \propto Z^{n-1} \quad \text{where} \quad n = \pi/(\pi - \beta_2) \quad (2)$$

whereas for a cone the velocity was of the form

$$r^{n-1} P'_n(\cos \theta) \cdot \sin \theta \quad (3)$$

where n must here satisfy the condition $P'_n(-\cos \beta_3) = 0$. The exponents of the first terms of the two series for the velocity distributions can be made the same by means of (2) and (3), which define an "angular transformation" and the coefficients of the leading terms become the same if the uniform stream ratio $q_{\infty 2}/q_{\infty 3}$ is defined by

$$\frac{q_{\infty 2}}{q_{\infty 3}} = |P_n(-\cos \beta_3)| \quad (4)$$

By means (2), (3) and (4) conditions near the apex can be made similar in plane and axi-symmetric flow; Armstrong further shows that the conditions near the rim are the same for the two and three dimensional cases. Fig. 11 illustrates the similarity obtained between a 5 deg. wedge and a 14 deg. cone.

Fig. 12 gives a comparison between drag coefficients of cones calculated by

Figure 12. Variation of drag coefficient with cavitation number for cones.

Plesset and Schaffer with the corresponding calculations by Armstrong and with experimental data. It would appear from this diagram that the Plesset and Schaffer results are satisfactory for cones of large apex angle but that Armstrong's results give a marked improvement when the cone is slender. It is interesting to note that this diagram confirms that the drag coefficient can be represented as two components, in the form $C_D = C_{D(Q=0)} + \sigma$, the head drag $C_{D(Q=0)}$ being practically unaffected by the cavitation number for small values of Q .

Fig. 13 gives a general comparison of the two theories with experimental measurements for the limiting case of zero cavitation number. This diagram clearly shows the difference between the Plesset-Schaffer and Armstrong methods near the origin. The former curve approaches the origin with a finite gradient whereas the Armstrong curve approaches the origin with zero slope and infinite curvature.

c. Bodies with non-zero Incidence. During the steady running section of an underwater trajectory, most missiles with sufficiently high velocities will have a certain amount of yaw and the tail of the missile will be in contact with the wall of the cavity produced by the head. In these circumstances the moment of forces acting around the centre of gravity will be zero and the total lift force acting on the missile will determine the radius of curvature of its trajectory. The possibility of being able to calculate the forces and moments acting on both the head and tail of a yawing missile in steady cavity flow is therefore of considerable practical importance.

The amount of information concerning actual flow conditions near bodies held at incidence to the flow is still very limited. In this connexion A. D. Cox and W. A.

Figure 13. Variation of drag coefficient with cone angle. ($Q = 0$)

Clayden (1956) have recently investigated the two dimensional flow near a wedge of 30 degs. semi-angle at small angles of incidence at zero cavitation number. In this particular example, incidence produced a movement of the stagnation point on to the face of the wedge which was most inclined to the main flow; a subsidiary cavity was also formed on the leeward face at the apex of the wedge. Fig. 14 is a photograph of this wedge at 10 degs. incidence obtained in the A.R.D.E. cavitation tunnel. Fig. 15 is a diagrammatic representation of the mathematical model which was used in a theoretical investigation of the problem when the flow in the neighbourhood of the subsidiary cavity was treated according to the Gilbarg re-entrant jet technique. Analytical expressions were obtained for the lift-slope and moment-slope coefficients for small angles of yaw. Fig. 16 gives a comparison of the theoretical value of $\left[\frac{C_L}{\alpha} \right]_{\alpha \rightarrow 0}$ for different wedges with recent measurements made in the A.R.D.E. cavitation tunnel.

The above example shows clearly some of the additional complexities which arise when an attempt is made to investigate by exact theory the conditions near a yawed cone, or similar body. In spite of these difficulties it is of interest to explore this problem by means of much simplified models and techniques, such as the application of a. principles of similitude or b. a modified Munk-Jones cross-flow theory. Both these techniques have been under investigation at Fort Halstead.

In applying the similitude technique we use a simplified "Strip Theory" with a two dimensional wedge model having the stagnation point at the nose rather than the model with the subsidiary cavity described above. This is probably justifiable because of the extra degree of freedom in the flow past the cone. Thus we imagine a wedge AOB, of semi-angle β , at a small angle of yaw δ to the stream, we assume that

Figure 14. Photograph of flow past yawed wedge.

the pressure distribution on the upper face OA is the same as if this face were half a wedge AOA' of semi-angle $\beta - \delta$, symmetrically placed in the stream (Fig. 17). Similarly the pressure distribution on the face OB is assumed the same as on a wedge BOB', of semi-angle $\beta + \delta$. If we expand these two distributions by Taylor's theorem we then have, by resolving or taking moments, integrating and then letting tend to zero, expressions for the lift-slope and moment-slope coefficients which are readily calculated by numerical differentiation from the known pressure distributions. Experiments have shown that this approach to the problem, when applied to the cone, at least, gives good agreement with experiment for large cone angles (Fig. 18). This result is in line with the corresponding results obtained by Plesset and Schaffer described above.

An alternative theory, which might be expected to yield good results for small to moderate cone angles, is based on a modification of the well-known Munk-Jones theory for slender bodies. For very slender bodies this theory assumes that the cross-force may be obtained by considering the rate of change of momentum in the two-dimensional flow on a plane perpendicular to the flow direction, and moving with the free-stream velocity V . This yields the value $2/\text{rad.}$ for C_L/α . Fig. 18 shows that as $\beta \rightarrow 0$, $C_L/\alpha \rightarrow 2$ experimentally.

Since for the cavitating flow past moderately slender cones, the surface velocity V_s is constant over much of the surface (V_s is given by $\frac{V_s^2}{V^2} = 1 - C_{Do}$) the condition that the flow must be tangential to the surface of the cone may, over most of the surface, be met by letting the cross flow plane move with a velocity $V_s \cos \beta$ past the body.

Figure 15. Mathematical model for flow past yawed wedge.

Figure 16. Variation of lift curve slope with wedge angle (β_2).

This yields

$$\begin{aligned}
 C_L &= C_N - C_{D0} \\
 &= 2 \frac{V_\infty^2}{V^2} \cos^2 \beta - C_{D0}/\text{radian} \\
 &= 2(1 - C_{D0}) \cos^2 \beta - C_{D0}
 \end{aligned} \tag{5}$$

and substituting values of C_{D0} , good agreement with experiment is obtained—not only for moderate cone angles, but even over the whole range. (The good agreement for large angles may be due to the $\cos^2 \beta$ term which $\rightarrow 0$ as $\beta \rightarrow 90^\circ$.)

d. *Breakdown of the Axi-symmetric Cavity.* Conditions at the rear end of a cavity are important as they determine the rate at which the air contained in a detached cavity is dispersed, and hence the rate of collapse of the cavity. For fairly long cavities it was observed by Swanson and O'Neil (1951) that a pair of hollow core vortices was generated at the rear of the cavity (Fig. 19). A theoretical study of this problem by R. N. Cox and W. A. Clayden (1955) showed that the circulation round the vortices could be related to the buoyancy lift on the bubble. This flow model gives a more satisfying physical explanation for a resolution of the 'Brillouin paradox' than does the artificial—though extremely valuable—conceptions of the Riabouchinsky image body, or the re-entrant jet of Gilbarg-Rock.

Figure 17. Strip theory for flow past yawed cones.

Figure 18. Variation of lift curve slope with cone angle (β_3).

V. CONCLUDING REMARKS

Having now completed our review of some of the recent advances in theoretical hydroballistics it may be of value to comment briefly on the present position of research in the several sections of the subject and to remark generally on some of the outstanding problems which need attention if the science is to continue its advance.

In this connexion it is clear that research on the steady cavity phase, especially when the body has straight line contours so that the point of separation is known, is much more highly developed than research on other aspects of the subject. Axisymmetric problems in the steady cavity phase have, at least with abrupt separation, become mainly a question of the labour involved; this should resolve itself once adequate programming techniques have been developed on high speed digital computers. Before we can attain this happy state with curved profiles it will be necessary to understand better the physical conditions at the point of separation. There is also a need for exact solutions with yawed bodies.

The entry phase remains a most difficult topic for basic research and calls for further elementary solutions, especially for oblique entry, which may help us to understand the flow mechanism during this phase. A direct theoretical attack on this essential practical problem is still a distant objective. The need for carefully controlled experimental data concerning this phase has been evident for several years and it is to be hoped that such information will be available before long from the experimental facilities in the U.S.A. and U.K.

Figure 19. Twin hollow core vortex-tubes at rear of steady cavity.

Similar remarks can be made concerning conditions governing cavity closure. The principles involved in both surface closure and deep seal are now fairly well understood from war-time researches; there is, however, still scope for the development of improved theoretical techniques which would enable these features to be followed in greater detail.

As regards cavity breakdown, and the subsequent entrainment of the air within the cavity, the recent development of a theory which explains the effect of buoyancy on the generation of twin, hollow core, vortices at the rear end of the cavity is probably capable of theoretical extension.

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DISCUSSION

G. Birkhoff

My reactions to the preceding admirable review of recent contributions to hydroballistics tend to be those of the cautious critic.

A critical review of impact and water entry theory was initiated several years ago by E. Cooper, H. Wayland, and myself. A careful study revealed wider discrepancies between theory and recorded observations than might be inferred from the

literature. Some of these discrepancies may be due to faulty instrumentation, but real further progress would seem to require that they be explained. Some of our conclusions are contained in Ref. [1], Ch. XI, §2, but N.O.T.S. lost interest in our project before it was completed. Also it was not clear how much experimental data should be declassified, especially as regards the influence of air viscosity.

Again, and in spite of laborious calculations by Marchet, Miss Vaisey, Brunauer, and Young (see Ref. [1], Ch. X, §9), the axi-symmetric potential-theoretic cavity flow past a disc has never been accurately computed, for any cavitation number Q . Though $C_D(Q)$ is now accurately known, the variation in cavity length l as a function $l(Q)$ of Q is not. In these calculations, the free streamline location seems very sensitive to small variations in pressure. Also, the singularity at the known separation point is extremely troublesome, and I wonder if a more careful analysis of the computational implications of this singularity should not take precedence over the still harder problem of locating the unknown separation point on streamlined missiles.

In the potential-theoretic approximation, this separation point should presumably be assumed to be determined by the axi-symmetric analog of the Brillouin-Villat Separation Condition (Ref. [1], Ch. VI, §6). But again, the numerical application of this method promises to be extremely difficult.—The situation is much more favorable in the plane case, and I confess to feeling great confidence in the calculations by Zarantonello and myself, referred to by Drs. Cox and Maccoll.

The theoretical analysis of cavitating flows past yawed solid bodies strikes me as very much harder, except perhaps in the case of long, nearly cylindrical missiles. Moreover it should be remembered that, in hydroballistic applications, the problem is to predict the angle of yaw α and cavitation number Q , whereas these are assumed known in most theoretical analyses. In practice also, l and hence $Q(l)$ depend on the complex mechanism of air entrainment at the rear end of the cavity, described by Drs. Cox and Maccoll. The amount of this air entrainment, and the degree of delay of separation due to boundary layer turbulence and air viscosity, all of which are neglected in the potential-theoretic model, impress me as being extremely difficult to predict.

In spite of the above difficulties, and the probability of other complications as yet unsuspected, I hope that a determined effort to solve impact and water entry problems will continue. My critical remarks are not intended to discourage further research, but to emphasize the need for distinguishing plausible ideas from reliable methods of prediction. In hydrodynamical theory, this distinction is always essential.

Ref. 1: G. Birkhoff and E. H. Zarantonello, "Jets, wakes and cavities," Academic Press, approx. Dec. 1956.

F. S. Burt

Dr. Maccoll has taken hydroballistics to cover not the whole field of the hydrodynamics of underwater weapons but rather the various aspects connected with the entry of an uncontrolled air launched weapon up to and including its fully wetted underwater run. I think this is a perfectly proper view to take and it certainly leaves a sufficiently wide and difficult field to be considered under the heading of hydroballistics.

In the water entry phase, even geometrically simple shapes like the sphere are extremely difficult to deal with theoretically. Unfortunately many of the applications we are interested in for naval use are concerned with the oblique entry of head shapes which are spherical, ogival, or a combination of both. With such head shapes the similarity principle that applies to the entry phase of a cone is no longer applicable. The cone is a rather special mathematical case in that during the initial entry phase the situation at a time t is very similar to that at a time $t + \delta t$. Nevertheless, I am very interested to see the data which Dr. Maccoll has presented on the oblique entry of cones.

At A.R.L. we had at one time extremely high hopes of Trilling's theory which Dr. Maccoll has mentioned but we now suspect that no theory which fails to take account of the deformation of the free surface can satisfactorily predict the horizontal component of force in oblique entry. We have recently done a recalculation for the oblique entry of a sphere using an approximation to an oblate spheroid rather than a prolate ellipsoid. The general trend of the drag coefficient for horizontal and vertical components seem to agree generally with that obtained by the Trilling method. That is to say that the vertical component is much the same as that obtained at vertical entry rising to a maximum of 0.8 to 0.9 at some time after an immersion of about $\frac{1}{10}$ th of the radius below the undisturbed free surface, whilst the horizontal component of the force rises much more slowly.

Recently Hobbs, Backdon and Woodley at the National Bureau of Standards have produced some measurements of impact force on spheres at oblique entry. One of the results they obtained was for a 4 in. diameter sphere with an entry angle of 51° . Comparing the vertical and horizontal coefficients of drag they obtained with those predicted by Trilling's method, it can be seen that although the vertical components are of the right order, the horizontal components are way off in the time scale. At A.R.L. we are of the opinion that in calculating the horizontal force in oblique entry the free surface deformation has to be taken into account, whereas Trilling's method is only possible because the deformation of the free surface is ignored. Admittedly, satisfactory experimental data on impact forces during oblique entry are not numerous. In this connection, we at A.R.L. are in the early stages of a programme of whip measurements at model scale. A related programme of axial deceleration measurements, for which we have no spare effort at the moment, would ideally complement the whip measurements: between them they would provide much useful data on the impulse imparted to a weapon during the entry phase.

The recent work by A.R.D.E. on the lift on the nose of an inclined body on a fully developed cavity is extremely interesting. They are to be congratulated on their "strip theory" and their application of the Munk-Jones slender body theory to cavity-running missiles. A recent A.R.L. report No. ARL/R1/G/HY/18/1 entitled "Water Tunnel Boundary Effects on Axially Symmetric Fully Developed Cavities" by I. J. Campbell and G. E. Thomas, dealing with the special case of such a cavity near a solid boundary such as a water tunnel wall, may be of some interest. We are also hoping to devote some effort to such forces in connection with the problems of stability in the cavity stage and, in view of the difficulties of making such measurements in water tunnels of our type, we are planning an experimental programme for our large rotating arm which seems to offer a number of advantages for this type of work.

I can confirm that the application of Vandrey's method to the calculation of the pressure distribution on fully wetted bodies of revolution does indeed involve much complicated labour and at A.R.L. we are in the process of programming the method for a high speed computing machine. The evolution at A.R.D.E. of a similar method for application to cavities is very interesting.

J. D. Nicolaides

I had really only two remarks to make. One, I think that the timeliness and importance of this paper certainly should be recognizable to all who read the newspapers these days, and it is no breach of security to point out that the underwater weapons are beginning to gain on the pure air-flight weapons, and this problem of water entry and underwater motion is extremely important.

You will recognize with supersonic aircraft drops and rocket shoots, we are into areas in which we are now hardly thinking about in terms of water entry velocity.

Now I certainly acknowledge the many outstanding items which you included in your paper, particularly the lift force at angles of inclination, but these are indeed steady state phenomena, for the most part, and when one compares the accuracy of performance computations for missiles under water with missiles in the air, you will

note we are in a very serious situation with regard to mils dispersion. The argument that you travel a short distance under water so that the errors, though large in mils, are not important, is not sound. When we talk about submarines at very deep depths, and the errors in mils begin to imply large errors in distance, the dynamic forces and moments acting may well be equal in importance to the static ones which we now consider, and, thus, will have to be included for calculations of satisfactory accuracy.

Recent tests in fully wetted flow indicate that the Magnus forces which are ordinarily completely neglected by the hydrodynamicist, are indeed quite important, and create quite large transverse displacements. It might be expected that in the cavity flow these forces will be important also.

In other words, what I am saying is we have come a long way, but the knowledge that we now have compared to what we need in weapons development is poor, and thus we are very far behind indeed.

Now I might say in that regard, and this my second point, that in reference to the high-speed computers, they may in a sense be our main hope. For example, in connection with the double disk cavity problem in recent work by the Naval Proving Ground (NPG Report No. 1413 by Young, et al.) this problem has been completely coded, and the free boundaries have been determined.

Now this problem was somewhat unchallenging to some of the people at the Naval Proving Ground, and they felt that it might be interesting to consider the case of a disk travelling at constant velocity in water, and then suddenly stopped, and then attempt, using that Naval Ordnance Research Computer to compute the transient development of the cavity. It is too early to indicate any results. More than a year has been spent just bringing up to date the coding required to attack this formidable problem.

I might also mention that once you determine your forces and moments on a hydroballistic missile, be they linear or non-linear, you have the problem of determining where the missile will go. Assuming we have these forces, let me say to you, the coding is now available for this same machine to predict the underwater trajectories.

In summary I would like to suggest two things about which there is an awful lot we don't know: first, transient or dynamic forces and moments, including those of the Magnus type; and secondly, flow and performance problems for which the high speed computing machine, in the final analysis, may be our only hope.

A. G. Fabula

This paper gives a valuable summary of recent results and new directions for hydroballistics research. This summary is of especial value because as the authors have indicated, the open literature on the subject is limited and the research establishment reports have limited distribution. In an attempt to thank the authors for their excellent work, I would mention certain additional reports—in the order as the subject material of the paper.

SHOCK PHASE

In [1], Dergarabedian has summarized the work of Trilling, Owens, and Korkegi on effects of water compressibility in normal impact (two and three dimensions). Basic assumptions of these linearized theories are that the body is rigid and that the ratio of body speed to the speed of sound in the water is much less than one. The more general problem of an elastic body impacting on the surface of a compressible liquid has received some preliminary consideration in [2].

NORMAL ENTRY

In connection with the authors' mention of Monaghan's extension of Wagner's approximate treatment of normal symmetric wedge impact, it is interesting that related work was done by Bisplinghoff and Doherty [3]. Besides the effect of deadrise on the added mass coefficient, they included the effect on the rise of the free surface.

This was calculated as by Wagner, but computed with the velocity distribution along the line of zero potential of the prism instead of with that of the flat plate. Unfortunately an incorrect velocity distribution was used in [3]. Computations based on the correct velocity distributions have been made by the writer as part of a study of extensions of Wagner's method [4].

A new approach to the exact solution for normal symmetric wedge impact has been given very recently by Borg [5].

The authors tell how Coombs has utilized an expanding hollow wedge model for the wedge impact flow, and how this is an outgrowth of work (by Chou) with an expanding spherical bowl model of normal sphere impact. The direct two-dimensional analogue of Chou's work has also been considered, that is the expanding circular-arc model for the normal broadside impact of a circular cylinder [6].

OBlique ENTRY

In some recent work at the Naval Ordnance Test Station in Pasadena, miniature accelerometers inside two-inch diameter models have given data on axial and cross components of impact force during oblique impact. For a hemisphere-cylinder configuration, the results have compared well with the predictions of Trilling's zero surface-rise, ellipsoid-fitting approximation, after some necessary rederivation of the expressions for the added mass of half-submerged spheroids. Reports are in preparation.

WATER-ENTRY MODELING

While this facet of hydroballistics was not discussed by the authors, it seems worthy of note that recent work at NOTS under Dr. Waugh has demonstrated accuracy of the modeling technique wherein model cavitation index and Froude number are made the same as for the prototype. For a family of head shapes, the pitch versus penetration in diameters has been shown to be modeled accurately for at least several diameters of travel. Investigation is underway of the importance of gas-density scaling on transient cavities. This work is answering many of the questions raised by wartime research in modeling, e.g. [7].

In closing, this writer would like to mention two little known reports which may be of interest. The first of these, [8] by Birkhoff and Isaacs, expands in considerable detail the references to transient water entry cavities in Birkhoff's book "Hydrodynamics." The second [9] is a translation of Weible's summary of German wartime work on water impact loads, including Schmieden's extension to three dimensions of Wagner's approximation.

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H. M. Berger

We have heard a very comprehensive review of the state of the art in the theory of water entry and it was interesting to hear in particular of the recent work at ARDE on steady state cavitation. I would like to point out that one of the problems in hydroballistics that was not mentioned and which has received practically no theoretical attention in the past is that of water exit. It seems to me that this is a fertile problem for a theoretical treatment. Many of the difficult problems that plague us when we attempt to analyze water entry are not present in water exit, and therefore, I believe that a theoretical approach could contribute a great deal to an analysis of the forces that arise during water exit.

I have just a few brief remarks to make about the state of the experimental work in this field. Concurrent with the effort to develop an adequate theory for water entry that has been outlined by Dr. Maccoll has been an extensive experimental program. However, early in the history of water entry studies, those working in the field realized that the modeling techniques they were using did not model important full scale effects such as the sudden accelerations experienced when the missile strikes the water. Consequently the experimental program divided into two parts—one, a full scale investigation of water entry characteristics of particular missiles—the other, an investigation of the scaling laws. Also, much attention was given to the development of adequate instrumentation for both full scale and model tests.

Recently, studies have been carried out by Levy at the California Institute of Technology [1] and Waugh at the Naval Ordnance Test Station [2] that have shown that with both Froude and cavitation number modeling, good agreement can be obtained with full scale water entry tests.

Levy concludes that for the range of Froude numbers and cavitation numbers that are within the prototype range:

1. "The behavior of the model with regard to trajectory, inclination, time to tail slap, and drag during cavity motion depends much more on the cavitation number than on the Froude number."

2. "Model tests in which the cavitation number is considerably greater than that of the prototype may yield a model behavior that is completely different from that of the prototype."

3. "Within the range of parameters that are encountered in prototype operation, the behavior of the missile is nearly independent of Froude number. This means that good results may be expected in most cases by using cavitation scaling alone. This is not meant to imply that the cavitation criterion is the only one that need be considered. It is known that under some conditions water entry phenomena are affected by gravity, viscosity and surface tension. However, cavitation scaling does not specify the operating conditions of the model except by establishing a functional relation between speed and pressure.

Therefore, it should be possible to avoid these other complications by proper choice of variables."

Dr. Waugh has told me that preliminary tests that he has made show that with gas density modeling, in addition to Froude and cavitation modeling, the cavity shape and closure can be accurately reproduced by model tests. The above results should give

added impetus to experimental studies of water entry, especially in view of the difficulties Dr. Maccoll has pointed out that exist in the theory.

There are several questions that I would like to ask Dr. Maccoll. First, what test facilities does he consider best for studies of the various phases of hydroballistics that he has outlined? My second question arises from his mention of the gravity effects on the quasi-steady cavity phase—that is, the effects causing the rolling up of the rear of the cavity into two hollow-core vortices, I would like to know whether he knows of any tests that show a Froude number effect on the drag of a cavitating body, and if so, is the drag reduced or increased by such effects? My last question concerns the role of water entry theory in the development of hydroballistic missiles. I would like to hear Dr. Maccoll's opinion of the role water entry theory plays in such a development.

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These contributions represent a valuable extension to the topics discussed in the paper, and draw attention to some subjects, such as water exit, unsteady flow, and dynamics of an underwater body which it was not possible to deal with in the time available.

The remarks by Professor Birkhoff and Mr. Nicolaidis serve to emphasise the computational difficulties of even the simplest potential-theoretic flows, and underline the need for a closer mathematical and physical understanding of the separation phenomenon.

The remarks by Dr. Berger and Mr. Fabula on the experimental work on water entry modelling at the California Institute of Technology and the Naval Ordnance Test Station are of great interest. An extension of this work to the high speeds associated with rocket entry would be of great practical use. Dr. Berger's question on test facilities is rather difficult to answer briefly, except perhaps to mention that in the experimental field there is a need for detailed measurements of pressure distributions during the water entry phase, and, in general, for work on high speed water entry, in which the main problem is that of controlling the entry conditions. For steady flow problems there is already an impressive and varied array of cavitation tunnels and rotating arm facilities in existence.

The authors know of no tests which show a marked Froude number effect on drag at the low cavitation numbers at which vortex rolling-up takes place; it is unlikely that the flow over the head of a body will be very much affected by conditions at the rear of a cavity. Security requirements unfortunately preclude a full answer to Dr. Berger's question on the role of water entry theory in the development of hydroballistic missiles; the remarks by Mr. Burt and Mr. Nicolaidis point out the need for a great deal more work in this field before theory can be applied with confidence to the problem of water entry of missiles.

In conclusion the authors wish to thank Mr. Fabula, Dr. Berger and others taking part in the discussion for drawing attention to additional work which had escaped their attention.

X

HYDRODYNAMIC SOURCES OF SOUND

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ABSTRACT

The paper describes the various flow phenomena that result in underwater sound in flowing water, and reviews the recent experimental and theoretical investigations of the sound sources. Four distinct sources of hydrodynamic noise are discussed: entrained air bubbles; vaporous cavitation; surface disturbances; and unsteady flows. For each of these, brief descriptions are given of 1. the physical mechanisms involved in the noise generation; 2. the available experimental data; 3. the theoretical methods of analysis; and 4. the relations between the flow parameters and the amplitude and spectral distribution of the sound.

I. INTRODUCTION

Sounds associated with flowing air and flowing water are everyday experiences. Examples are the splash of water and the proverbial babble of the brook, the howl of the wind, the hiss of escaping air, and indeed all the sibilant and aspirate sounds of speech itself.

Despite their familiarity, sounds from flowing fluids have received comparatively little scientific attention. Although most text books on acoustics discuss at length the sounds produced by vibrating bodies, the equally-common sounds associated with flow are usually ignored.

Within the past decade, however, interest in hydrodynamic and aerodynamic sources of sound has increased. At the same time, a theoretical foundation for dealing with these acoustical problems has been provided by recent developments in fluid dynamics, particularly in the study of turbulence. As a result, important advances in our understanding of hydrodynamic noise have been made since Folsom, Howe and O'Brien [1] reviewed the subject in 1942.

The study of hydrodynamic noise involves a combination of hydrodynamics and acoustics. These two subjects were closely allied in the classical physics of the nineteenth century; Stokes, Helmholtz, Rayleigh, and Lamb, for example, contributed to both. But in recent years they have grown apart; acoustics becoming associated with electronics, while hydrodynamics concentrated on problems of steady motion. As a result, many workers in each of the fields are scarcely aware of developments in the other. The study of hydrodynamic noise requires a reunion of the two disciplines.

The object of this paper is to describe the various flow phenomena that can result in sound, and to review the recent theoretical and experimental investigations that have led to a partial understanding of the relations between the flow parameters and the intensity and spectral distribution of the resulting sound. Because this symposium is concerned with naval problems, it is appropriate to limit the discussion to underwater sounds associated with the flow of water. The discussion will also be limited to those sounds which can be considered to be "noise" in the sense that the sound is either undesirable or else an incidental concomitant of the flow.*

* The generation of sound by underwater sirens and other devices specifically designed to act as sound sources is discussed by Bouyoucos.(2)

Underwater sounds are ordinarily detected by devices sensitive to pressure fluctuations. These devices are called "hydrophones," and they are simply waterproofed microphones. Accordingly, the physical quantity of interest in hydrodynamic noise is generally the fluctuating pressure associated with the hydrodynamic flow. The magnitudes of the fluctuations of interest are very small compared with the pressures ordinarily met in steady hydrodynamics: a pressure fluctuation of only 100 dyne/cm² (a head of about 0.04 inch of water) is an intense sound in water. The power associated with these fluctuations is also very small: the entire underwater sound output of a noisy ship is only a few watts. It is thus apparent that hydrodynamic noise may stem from second-order effects which have no influence at all on the more obvious characteristics of the flow.

It is convenient to classify the various forms of hydrodynamic noise according to the grosser flow phenomena with which they are associated. Four categories will be considered here: 1. Air bubbles entrained in water, 2. Vaporous cavitation, 3. Surface disturbances such as splashes, and 4. Unsteady flows such as vortex shedding and turbulence. Each of these phenomena requires somewhat different methods of analysis; accordingly they will be discussed in separate sections, the order being chosen merely for convenience of exposition.

II. ENTRAINED GAS BUBBLES

Perhaps the most commonplace of all the sounds in water are those associated with bubbles. Sound is generated when bubbles form, when they divide or unite, and when they stream past an obstacle in a flow or through a constriction in a pipe.

The sound of bubbles forming at a nozzle was investigated by Minnaert [3] in 1933, and later by Meyer and Tamm [4], who showed that the sound was associated with volume pulsations of the bubble, and that in these pulsations the bubble behaved like a simple oscillating system with damping. The frequency of the pulsation was calculated by Minnaert and the damping by Pfriem [5] among others.

When bubbles are observed as they rise through a liquid, large oscillations in their shape are apparent. It is natural to ask whether the sound associated with these shape changes is significant relative to the sound from the volume pulsation. This question can be answered by representing the vibration of the bubble wall by a sum of spherical surface harmonics (see Lamb [6] §294)

$$R(\theta, \phi, t) = R_0 + \sum_n A_n S_n(\theta, \phi) e^{2\pi i f_n t}, \quad (1)$$

where R is the instantaneous radial coordinate of a point on the bubble wall, as a function of the spherical angles θ and ϕ , and time t ; R_0 is the mean radius of the bubble; $S_n(\theta, \phi)$ is the surface harmonic of order n giving the variation with the angles; $i = \sqrt{-1}$; and A_n and f_n are the amplitude and frequency of oscillation for the n -th order. The zeroth order, with $S_0 = 1$, corresponds to volume pulsation, the first order to translational oscillation, and the higher orders to oscillation in shape with n nodal lines.

If the amplitude of oscillation is small, i.e., if $A_n \ll R_0$, the oscillations of each mode contribute independently to the sound pressure. Far from the bubble, the sound pressure amplitude p_n associated with oscillation in the n -th mode at frequency f is given by

$$p_n = \frac{\rho A_n R_0^2 (2\pi f)^2 e^{-ikr} (ikR_0)^n S_n}{r(n+1) [1 \cdot 3 \cdot 5 \cdots (2n-1)]}, \quad (2)$$

where r is distance from the bubble, and $k = 2\pi f/c$. The derivation of this equation assumes that the bubble is small, so that $kR_0 \ll 1$, and that the distance is large so that $kr \gg 1$.

Each mode has a natural frequency of free oscillation. For volume pulsation, the natural frequency is given by

$$f_0 = (3\gamma P_0/\rho)^{1/2}/2\pi R_0, \quad (3a)$$

where ρ is the density of the liquid, and P_0 and γ are the mean pressure and specific-heat ratio for the gas in the bubble. The natural frequencies of shape oscillation depend on the surface tension T in accordance with the relation (see Lamb [6] §275)

$$f_n = [(n^2 - 1)(n + 2)T/\rho R_0]^{1/2}/2\pi R_0. \quad (3b)$$

By substitution of the natural frequencies given by Eqs. (3) into Eq. (2), it can be shown that the sound pressure associated with free oscillations is negligible except for the zeroth mode. For example, at the relatively large amplitude of $A_n = \frac{1}{4} R_0$ (with $R_0 = \frac{1}{3}$ cm, $P_0 = 1$ atmos) the sound pressures are about 3×10^4 and only 9×10^{-4} dyne/cm² in the zeroth and second mode respectively, at $r = 100$ cm. Accordingly, only volume pulsations are of importance in calculations of the sound radiated by bubbles.

For volume pulsation, the instantaneous sound pressure $p_s(t)$ and the instantaneous bubble volume $V(t)$ are related simply by

$$p_s(t) = \rho \ddot{V}(t')/4\pi r, \quad (4)$$

where the two dots indicate the second time derivative, and $t' = t - r/c$.

Sound of bubble formation.—When a bubble forms on a nozzle in water, a pulse of sound is emitted just as the bubble leaves the nozzle. An oscillogram of a typical sound pulse is shown in Fig. 1, together with synchronized frames from a

Figure 1. Oscillogram of the sound pulse from an air bubble formed at a nozzle, with synchronized photographs of the bubble. The time each photograph was taken corresponds to the point below the bubble on the oscillogram.

motion picture of the bubble. Although the oscillations of the bubble shape are apparent in the motion picture, the volume pulsations generating the sound are not visible because of their small amplitude.

If the pulsations are of small amplitude, the instantaneous volume of the bubble pulsates in accordance with the linear differential equation

$$(\rho/4\pi R_0)\ddot{V} + D\dot{V} + (\gamma P_0/V_0)(V - V_0) = 0, \quad (5)$$

where V_0 is the mean volume of the bubble; D is a constant which accounts for the

damping; and the dots indicate time derivatives.* The volume oscillates sinusoidally with decaying amplitude about its mean value. Accordingly, the instantaneous sound pressure is also a decaying sinusoid represented by

$$p_s(t) = p_0 e^{-\pi\delta f_0 t} \cos(2\pi f_0 t - \vartheta), \quad (6)$$

where f_0 is the natural frequency of pulsation as given by Eq. (3a), and $\pi\delta$ is the natural-logarithmic decrement of the oscillation. The amplitude p_0 and phase constant ϑ depend on the initial conditions exciting the bubble. (For an air bubble in water at atmospheric pressure, $f_0 R_0 = 330 \text{ cm/sec}$; and $\pi\delta = 0.0045 + 0.0014(f_0 \text{ sec})^{1/2}$, the two terms resulting from radiation and thermal dissipation, respectively.)

For a bubble leaving a nozzle, the constants which depend on the initial conditions can be evaluated from the radial velocity attained by the growing bubble just before it separates from the nozzle. The sound pressure was calculated in this way by one of us [7] using values of the radial velocity measured on motion pictures of the bubble. The calculated sound pressures agree quite well with the measured values. The frequency and decay rate also agree with the theoretical values. Accordingly, it can be concluded that the generation of sound at bubble formation is well understood.

When bubbles split or unite, a pulse of sound is generated just as at bubble formation. In this case the excitation is caused by a change in the pressure inside the bubble: the single larger bubble has less surface tension pressure. If a bubble splits into two smaller ones of equal size, the peak sound pressure of the pulse at distance r can be shown to be of the order of $0.5T/r$, independently of the bubble size. This value is considerably less than the pressure generated at bubble formation.

Sound from entrained bubbles.—Probably the most important sounds result from the flow of entrained bubbles past a body in a stream. The bubbles experience a transient pressure as they move through the hydrodynamic pressure field around the body. The transient pressure causes the bubbles to pulsate and radiate sound.

The pulsations are described by a differential equation like Eq.(5) but with a forcing term on the right side equal to $P_0 - p_e(t)$, where $p_e(t)$ is the instantaneous environmental pressure, i.e., the pressure that would exist in the liquid at the bubble location if the bubble were absent. The solution of this equation is conveniently expressed in terms of Fourier transforms.

If a Fourier transform $h(f)$ of the environmental pressure is defined by

$$h(f) = \int_{-\infty}^{\infty} [p_e(t) - P_0] e^{-2\pi i f t} dt, \quad (7)$$

then the transform $s(f)$ of the sound pressure radiated by the bubble can be calculated from

$$s(f) = \frac{(R_0/r) e^{-ikr}}{(f_0/f)^2 - 1 + id} h(f); \quad (8)$$

where the entire fraction is the response characteristic of the bubble. The spectral density $E(f)$ of the sound energy is related to these transforms by

$$E(f) = (8\pi r^2 / \rho c) |s(f)|^2, \quad (9)$$

$E(f)$ being defined so that the energy radiated by the bubble in a narrow frequency band of width Δf is $\Delta f E(f)$, and the total sound energy radiated over the entire frequency range in all directions is $\int_0^\infty E(f) df$. If a large number of bubbles radiate similar transients at random intervals with average repetition frequency N , the spectral

* When written in terms of the volume, Eq.(5) gives an approximate description of the pulsations even for a non-spherical bubble, if R_0 is taken as the radius of a spherical bubble of equal volume. Eq.(4), however, is exact regardless of the bubble shape.

density of the sound power is $N E(f)$, and the *rms* sound pressure in a band of width Δf is $(2N\Delta f)^{1/2} s(f)$.

The sound energy tends to be concentrated at the natural frequency f_0 of the bubble, because the denominator in Eq. (8) is a minimum when $f = f_0$. However, in many practical situations the fluctuating environmental pressure is such that $h(f)$ differs from zero only at low frequencies, much below the natural frequency f_0 of the bubble. In this case, the instantaneous sound pressure $p_s(t)$ can be expressed directly in terms of environmental pressure $p_e(t')$ at earlier time $t' = t - r/c$ by

$$p_s(t) = - (R_0/r) (2\pi f_0)^{-2} \ddot{p}_e(t'). \quad (10)$$

The calculation of the sound radiated by motion about a particular body requires a knowledge of the environmental pressure at the moving bubble as a function of time. This can be obtained from the known pressure and velocity field around the body. If, for simplicity, it is assumed that the bubble follows a streamline, its position as a function of time is determined by integrating the velocity along a streamline, and the pressure is then determined from the position. This procedure, although straightforward in principle, usually results in expressions which cannot be expressed in closed form.

These calculations have been carried out by one of us [7] for the flow past a circular cylinder, with the bubble moving around the surface of the cylinder. The relatively complicated result for the sound pressure is

$$p_s(t) = \frac{32\rho^2 U_0^4 R_0^3}{3\gamma P_0 R_c^2 r} \left[\operatorname{sech}^2 \left(\frac{2U_0}{R_c} t' \right) - \frac{3}{2} \operatorname{sech}^4 \left(\frac{2U_0}{R_c} t' \right) \right] \quad (11)$$

where U_0 is the free-stream velocity; R_c is the radius of the cylinder; t is measured relative to the instant the bubble passes the median plane; and it is assumed that $(2U_0/R_c) \ll f_0$. The waveform of the sound pressure is shown in Fig. 2. The sound pressure has a negative peak at $t' = 0$ and two smaller positive peaks at $t' = \pm 0.6R_c/U_0$. The peak pressure can be quite high: for the conditions $U_0 = 5$ meters/sec, $R_0 = 1/3$ cm, $R_c = 10$ cm, the negative peak from an individual bubble is about 16 dyne/cm² at 1 meter. No experimental work has yet been performed to verify these calculations.

For bubbles streaming past a cylinder, the sound pressure should increase with the fourth power of the velocity, as indicated by Eq.(11). In fact, the sound pressure should increase as U_0^4 for all similar forms of bubble motion, as is apparent from a dimensional consideration of Eq.(10): the environmental pressure fluctuations increase as U_0^2 and the second time derivative introduces another factor $(U_0/L)^2$, L being a characteristic length.

Large-amplitude pulsations.—The rapid increase of sound pressure with increasing velocity leads to a consideration of the limitations of the linear theory on which the above results are based.

The large-amplitude free pulsations of gas bubbles have been studied extensively in connection with underwater explosions [8]. It is known that the radiated pressure loses its sinusoidal form with increasing amplitudes, the negative excursion becoming flatter while the positive excursion develops a sharp peak. The same effect occurs in forced large-amplitude pulsations in response to a widely fluctuating environmental pressure.

As a first step toward a more-exact description of large-amplitude pulsations, the radial pulsation is described by a non-linear equation

$$\rho(R\ddot{R} + \frac{3}{2}\dot{R}^2) = P - p_e(t), \quad (12)$$

where R is the instantaneous radius of the bubble, and $p_e(t)$ is again the environmental pressure that would exist at the bubble location in the bubble's absence. Also, P is the actual pressure in the water at the bubble wall; for a bubble filled with gas obeying the

Figure 2... The sound pressure radiated by a bubble in response to an environmental pressure given by $p_e(t) = P_0 - P_a \operatorname{sech}^2 at$, which is similar to the pressure on a bubble moving around the surface of a cylinder in a flow. The curves indicate the development of non-linearity for successively smaller values of P_0 relative to P_a . The solid curve represents the linear range.

adiabatic gas law, $P = P_g(R_0/R)^{3\gamma} - (2T/R)$, where R_0 is the equilibrium radius of the bubble at the mean gas pressure $P_g = P_0 + 2T/R_0$; and T is the surface tension.*

The non-linear effects can be investigated most easily if the environmental pressure is assumed to vary in some definite way. For comparison with the linear theory, consider the fluctuation given by $p_e(t) = P_0 - P_a \operatorname{sech}^2 at$, which results in sound with a waveform like that of Eq. (11) if $P_a \ll P_0$. The changes in the waveform associated with larger values of P_a are illustrated qualitatively in Fig. 2. For $P_a \ll P_0$, Eq. (12) becomes linear and the waveform of the radiated pressure (solid curve) is just that given by Eq. (11). As P_a is made larger, non-linearity occurs first in the term $(R_0/R)^3$ in Eq. (12), and the negative peak in the sound pressure grows rapidly (dashed curve).

* All pressures are measured relative to the vapor pressure of the water, in order to avoid a repetitive constant.

If $P_a > P_0$, so that $p_e(t)$ is actually negative at $t = 0$, the bubble becomes statically unstable and the non-linear time-derivative terms in Eq.(12) control the radial motion of the bubble.** As $p_e(t)$ goes negative, the bubble grows to many times its equilibrium size and then, when $p_e(t)$ returns to positive values, the bubble suddenly collapses and radiates a very large positive pulse (dotted curve). Noltingk and Neppiras [9] have calculated the size to which the bubble grows and the magnitude of the pressure pulse as a function of the ratio P_a/P_0 .

During its growth, the bubble becomes filled primarily with vapor. The large growth and subsequent rapid collapse of the bubble are characteristic of the behavior of vaporous cavitation bubbles. Indeed, the transition from small to large amplitude pulsations is just the transition to cavitation, with the bubble acting as a cavitation nucleus. The sound associated with these large-amplitude pulsations will be discussed in greater detail in the following section.

III. CAVITATION

The origin of cavitation noise.—The earliest investigators of cavitation were aware of the noise it makes. It was probably the loud hissing sound which first attracted the attention of Osborne Reynolds [10] to the occurrence of cavitation in water flowing through a constricted tube. He recognized the cause of the sound to be the “boiling” of the water. During the First World War, it was known that cavitation of ships’ propellers radiated sound which could be heard underwater for great distances [11].

The sound is generated as a result of the growth and collapse of vapor cavities. A cavity, beginning as a microscopic nucleus, grows when its environmental pressure becomes sufficiently negative and collapses when the pressure is restored. Such behavior, with accompanying noise, is to be expected wherever nuclei in the water are subjected to sufficiently extreme transient reductions in their environmental pressure.

The sketch, Figure 3, depicts one common occurrence, the growth and collapse of individual cavities in the pressure field produced by flow past a curved boundary such as that of a propeller, a strut or similar appendage, or past a contraction in a conduit. The growth begins after a nucleus of some sort enters the region of low pressure and the collapse takes place when the cavity is carried downstream into a region of higher pressure. Various experimenters have employed high-speed photography to demonstrate the existence of individual cavities, roughly spherical, which grow and collapse in the manner indicated [12, 13, 14, 15].

Cavitation noise may be produced also in turbulent shear flows in pipes, jets, or boundary layers; by water hammer and by acoustic radiation; and even as a concomitant of “steady” cavities. In each instance, the sound is related directly to the kinematics of the individual transient cavities. The latter may appear as elongated “cores” at the centers of vortices; in shear flows, they may be rather amorphous and may undergo rapid distortion. Where the velocity gradients are mild, the cavities tend to a spherical shape. Nearly all analytical treatments of the hydrodynamical problems assume a spherical cavity. This is not completely unrealistic: the validity of many of the relations derived is not seriously affected by departures from spherical symmetry.

Interest in the inception, growth, collapse, and rebound of vapor cavities stems from various fields of interest: boiling of liquids, detergent and chemical effects of ultrasonics, absorption of sound in water, efficiency of hydraulic machinery, and pitting and corrosion of structural materials. A great amount of research has been

** If the equilibrium size of the bubble is very small, the condition for instability is not simply $P_a > P_0$, but rather $P_a > P_0 + (4\sqrt{3} T/9R_0) [1 + (P_0 R_0/2T)]^{-1/2}$. If P_a exceeds this value by an appreciable amount, the size to which the bubble grows before collapsing is relatively independent of its original size R_0 .

Figure 3. Sketch indicating growth and collapse of a vapor cavity.

directed toward each of the aspects mentioned. For our purpose, here, it will be sufficient to select a few examples of theory, experiment, and computation best suited to illustrate the way in which cavitation acts as a source of sound. Those interested in more general aspects of the various phenomena touched upon will find extensive discussions in bibliographical material relating to cavitation [16].

Calculation of the sound pressure: acoustic theory.—In the previous section, the sound pressure generated by a gas bubble was related to the fluctuating environmental pressure encountered by the bubble. It is of interest to ask whether, in the case of a transient vapor cavity, a corresponding relation can be given.

One of the characteristics of the behavior of transient vapor cavities is the violence of the collapse. In that part of the motion, the inward radial velocity of the fluid in the immediate vicinity of the cavity may exceed the velocity of sound in the liquid and pressures comparable to the modulus of compressibility of the liquid may be developed. The description of the motion in the final stages of collapse therefore requires consideration of the compression of the liquid. But during the period of initial growth and most of the period of collapse, the incompressible theory yields a correct description of the motion of the cavity, and the sound pressure (at sufficiently great distance $r \gg R$) is given accurately by Eq. (4) in Section 2.

If the temporal variation of the environmental pressure encountered by the cavity is known, the resulting motion, or at least the incompressible stages thereof, may be computed from Eq. (12). The roles of surface tension and of any minute quantity of gas which may be contained in the cavity are important only when the cavity is very small. For the purpose of the present discussion it is sufficient to consider that their combined effect is simply to forestall the inception of growth of the

nucleus until the environmental pressure has reached a certain value (smaller than the vapor pressure and, in fact, typically negative) characteristic of the individual nucleus (See Section 2). Once the growth has begun, the pressure P at the cavity becomes essentially zero*. The example of Figure 4 rather exaggerates the ordinary extent of the delay of the inception of growth.

Of special interest is the computation of the motion for situations like that depicted in Figure 3. If the temporal variation of the environmental pressure

Figure 4. Computed radial motion and sound pressure for a growing and collapsing vapor cavity.

encountered by a passing nucleus can be deduced from a knowledge of the velocity and pressure fields in the vicinity of the curved boundary, Eq. (12) can be solved by numerical integration to obtain the radius of the cavity as a function of the time. Plessett [17] has done this and, despite the obvious idealizations introduced, found the computed radial motion to resemble the observed behavior reasonably well. Figure 4 shows the results of a typical, though hypothetical, example of such a computation,

* Note that, to avoid needless repetition throughout the discussion, all pressures are referred to the vapor pressure.

illustrating not only the radial motion of the cavity but also the resulting sound pressure. The abscissa represents time. The curve labelled "External Pressure" indicates a temporal variation in environmental pressure typical of that encountered by a cavity under conditions like those being considered. The radius of the cavity is shown by the heavy solid curve. The constant P_0 is the value of the pressure difference, $(p_e - P)$, at the final instant of collapse, t_0 . A characteristic radius, R_1 , may be defined so that, in the final stages of collapse, the total energy, kinetic and potential, is $\frac{4}{3}\pi P_0 R_1^3$.

The sound pressure, shown by the light solid line, oscillates once and, at the end of the collapse, rises in an extremely high narrow "spike." The details of this high narrow pulse (which determine the high-frequency portion of the sound spectrum) are not given correctly by the acoustic theory. While most of the energy radiated as sound is associated with the pulse at the end of the collapse, the sound associated with the earlier part of the motion is by no means negligible.

The broken lines indicate the solution for the special case, considered by

Figure 5. Radial motion and sound pressure for a vapor cavity which rebounds after growing and collapsing.

Rayleigh [18], in which $(p_e - P)$ is constant and equal to P_0 . He found, by expressing the constancy of the sum of the kinetic energy, $2\pi\rho R^3\dot{R}^2$, and the potential energy, $\frac{4}{3}\pi R^3P_0$, that the radius and the wall velocity are related by the equation,

$$(R_1/R)^3 = 1 + (3\rho\dot{R}^2/2P_0). \quad (13)$$

For this case, R_1 is the maximum radius of the cavity.

Some cavities rebound after collapse. The mechanism by which the rebound comes about is not known. Apparently a small quantity of gas contained in the cavity plays an essential role, but details are uncertain. What is known is that the flow velocity and pressures attain such values that the acoustic theory is not applicable to the part of the motion for which the cavity is very small. Figure 5 continues the illustration, however, showing the behavior typical of those cases in which rebound occurs. The assumption of an empty cavity and an incompressible liquid have been retained. The cavity is assumed to collapse to an indefinitely small radius and to rebound with an arbitrarily postulated loss of energy. The whole sequence is clearly only an illustration; actual cavities show a wide range of behavior. Knapp and Hollander [13] photographed cavities which rebounded as many as five times. Benjamin [19] has also observed repeated rebounds under different conditions.

Figure 6. Spectral distribution of the sound, for motions shown in Figure 5, as computed for an incompressible liquid.

The spectrum of the sound: acoustic theory.—From the sound pressure, the spectral distribution of the radiated energy may be computed. Figure 6 shows the spectra corresponding to the growth and single collapse and to the growth and multiple collapse postulated in the example. The spectra exhibit maxima at frequencies of the order of the reciprocal of the time required for growth and collapse. At lower fre-

Figure 7. A measured spectrum of cavitation noise. The solid curve represents the spectrum of the steady noise observed at a distance of 1 meter from a metal rod, 2 inches long and 1/16 inch in diameter, rotated at 4300 rpm about a transverse axis through its center. The reference level for the decibel scale corresponds to sound pressure of 1 dyne/cm² in a 1 cy/sec band of frequency. The cavities observed were about 1 mm in diameter. The pressure was approximately 1 atmos.

quencies, the spectral density necessarily varies as the fourth power of the frequency.* The oscillations in the spectrum of the sound generated by the multiple collapse do not disappear from the computation at the higher frequencies. However, the oscillations will not appear in the spectrum of the sound generated by a succession of cavities which do not all grow, collapse, and rebound in exactly the same way. Accordingly, at the higher frequencies the "smoothed out" spectrum is shown. The asymptotes

* This follows from the fact that the volume of the cavity is necessarily positive during its lifetime and essentially zero before and after. The magnitude of the Fourier transform of the volume (the latter considered as a function of time) is then, by well-known properties of the Fourier transformation, to the first order, independent of the frequency at values of the latter which are small in comparison with the reciprocal of the total lifetime of the cavity. Since the sound pressure is proportional to the second time derivative of the volume of the cavity (Eq. (4), Sect. 2), the exponent relating the spectral density and the frequency at small values of the latter is thereby determined. The conclusion applies to the spectrum of the sound generated by a single cavity or by a random succession of cavities, but must be modified if referred to a sequence of cavities in which the behavior of successive cavities is in any way correlated. The presence of a reflecting boundary, especially a free surface will, of course, alter the frequency dependence also.

indicating that the spectral density at high frequencies varies as the reciprocal of the two-fifths power of the frequency cannot, of course, correctly describe the spectra of real cavitation noise, if only because of the physical requirement of finite energy. The departure is simply a manifestation of the fact that the high-frequency portion of the spectrum is determined by details of the "spike" of sound pressure which are not given correctly by the incompressible and acoustic theories.

The spectrum of the sound: experimental.—Figures 7 and 8 show examples of spectra of cavitation noise obtained experimentally by Mellen [20] and Jorgensen [21] respectively. Three features characterize the spectrum: the location of the peak, and the two exponents corresponding to the asymptotic behavior at low and at high frequency. At high frequencies, the spectral density is observed to vary roughly as the reciprocal of the square of the frequency (−6db/octave). This feature of the spectrum suggests that the sound pressure undergoes a sharp rise of the nature of a shock. Direct evidence of the sudden rise, such as an oscillograph record showing it, is difficult to obtain because of the extremely short time interval which must be resolved (perhaps 10^{-7} second or smaller). However, such records showing indications of a shock have been obtained through the use of tiny barium titanate transducers [22]. Shock waves emitted by collapsing cavities have also been shown by Schlieren photography [23, 24].

Compressive flow in the collapse of a cavity.—The explanation of the outgoing

Figure 8. The measured spectrum of noise produced by a cavitating jet. The symbol $\tilde{p}$ represents the rms pressure in a half-octave band of frequency, at a point four diameters off the axis and four diameters downstream from the orifice. The different spots represent different combinations of jet diameter, D ($\frac{3}{8}$, $\frac{3}{4}$, and $1\frac{1}{2}$ inch), and ambient pressure, P_0 ($\frac{1}{2}$, 1, and 2 atmos). U_0 is the efflux velocity; σ is the cavitation index, $2P_0/\rho U_0^2$. (From Jorgensen [21])

shock developed at the collapse of a vapor cavity must begin with a consideration of the effect of the compressibility of the liquid upon the motion. Various extensions of acoustic theory in which the wave equation in the velocity potential has been retained have provided relations which can be said to give "first order" corrections for the effect of compressibility of the liquid [25]. These results are useful in the consideration of the collapse of the gas globe formed by an underwater explosion, where the flow velocities do not exceed one-tenth of the velocity of sound and the pressures do not exceed about six percent of the compressibility modulus of the liquid. In the case of a collapsing vapor cavity, however, the significant effects of compressibility are manifest while the cavity contains essentially only condensing vapor: the small amount of gas inside is not sufficient to arrest the collapse before the inward flow velocities exceed the velocity of sound [26]. The problem can be put then, in idealized form, as the calculation, first, of the motion and, then, of the surrounding pressure and velocity fields, in the case of the collapse of an empty spherical cavity in a compressible liquid [27].

The equations of compressive flow do not, in general, admit of explicit solutions. Gilmore [28, 29] however, has derived an equation relating the radius of the cavity explicitly to the velocity and acceleration of the wall of the cavity. In the special case of the empty cavity, Gilmore's differential equation is

$$R\ddot{R}(1 - \dot{R}/c_0) + \frac{3}{2}\dot{R}^2(1 - \dot{R}/3c_0) = -P_0/\rho_0, \quad (14)$$

which reduces to the corresponding form of (12) for vanishingly small values of $\dot{R}/c_0$. For the initial conditions considered by Rayleigh in the incompressive case, Gilmore gives

$$\left(\frac{R_1}{R}\right)^3 = \left(1 - \frac{\dot{R}}{3c_0}\right)^4 \cdot \left(1 + \frac{3\rho_0\dot{R}^2}{2P_0}\right), \quad (15)$$

which also reduces to the corresponding incompressive solution, Eq. (13), as $\dot{R}/c_0 \rightarrow 0$. Gilmore's results are derived from a hypothesis of Kirkwood and Bethe which states that in the spherically symmetric flow about the cavity the quantity $r(h + \frac{1}{2}u^2)$, which in isentropic flow is exactly equal to $r\dot{\phi}$, is propagated outward with variable velocity $(c + u)$. Here r is the radial coordinate; c is the local value of the velocity of sound and $\dot{\phi}$ is the time derivative of the velocity potential. The enthalpy h is defined in Fig. 9. Where the pressure in the fluid differs from the ambient value by only a few atmospheres, h is simply $(p - P_0)/\rho_0$, but where the pressures are not negligible in comparison with the modulus of compressibility, ρc^2 , of the liquid (about 21,000 atmos for water), the value of h depends also on the relation between the pressure and the density of the liquid. (In the present discussion, c and ρ must be considered variable; their values at ordinary pressures (near zero) will be denoted by c_0 and ρ_0). Here, p and u are the pressure and the radial flow velocity.

The sound pressure, p_s , at some large value of the radial coordinate is, according to the theory, determined as follows: The value of the propagated quantity

$r(h + \frac{1}{2}u^2)$ is easily evaluated at the wall of the cavity as $R\dot{R}^2/2$. The "outgoing path" of each of its successive values may be traced through the pressure and velocity field surrounding the collapsing cavity according to the rule given by the Kirkwood-Bethe hypothesis. The assumed behavior is illustrated qualitatively in the diagram, Figure 9. Several outgoing paths or "characteristics" are shown, each corresponding to a different value of the propagated quantity, the particular value being the value of

$R\dot{R}^2/2$ at the instant at which the path "left" the wall of the cavity. The slope of the line representing each such outgoing characteristic represents, at each point in

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Figure 10. Pressures and shocks radiated by a vapor cavity collapsing in a compressible liquid. (Values according to Mellen [30].)

Mellen [30] has made the indicated calculations, taking approximate values for $(c + u)$, with the result shown in Figure 10. Each solid line exhibiting a vertical rise shows the value of $r\phi$ (very nearly equal to rp_s/ρ) as a function of time for one selected value of a parameter involving the radial coordinate. In the case of the collapse of a cavity in water with external pressure P_0 equal to one atmosphere, the three radial distances indicated would be approximately 3.6, 360, and 36000 R_1 . The peak pressures at the "shock" front are, for the same case, 59, 0.42, and 0.0035 atmospheres. Mellen also obtained estimates of the peak pressures experimentally and found good agreement with the calculations.

From the pressures shown in Figure 10 as functions of the time, the corresponding frequency spectra might be computed. The spectrum shown in Figure 6 could be corrected, at high frequencies, for the compressive effects neglected in its original derivation. In view of the somewhat limited accuracy of the values shown and of a number of idealizations made tacitly in the brief treatment presented above, it is sufficient merely to indicate the high-frequency asymptote corresponding to the discontinuity in the sound pressure. The resulting computed spectra (Figure 11) do show a fair resemblance to observed spectra of cavitation noise. It appears, however, that the computed magnitude of the sudden rise in pressure at the shock front, relative to the parts of the wave which determine the low-frequency part of the spectrum, is higher than is really the case. This is not surprising. The computations assume no loss of energy in the propagated wave except that inherent in the Hugoniot conditions, whereas, in fact, other losses do occur in the propagation of the high-frequency components of sound waves. It is possible, also, that small amounts of

Figure 11. Spectrum of sound radiated by a growing and collapsing vapor cavity in a compressible liquid. The dash lines show the asymptotes corresponding to the shocks indicated in Fig. 10.

gas in the cavity affect the amplitude of the shock. A calculation would be necessary to ascertain the effect in a specific case.* A more important reason for the discrepancy, however, is the fact that real cavities may not collapse as spheres. Ample evidence, both theoretical and experimental, indicates that collapsing cavities undergo radical distortions in shape and may even break up into clouds of smaller bubbles. The cavities which produce the noise represented in Figure 8 resemble those shown in Figure 12 which is taken from Ref. [31]. A quantitative description of the motion and, especially, of the development of the shock for any case other than that of spherical symmetry would be very difficult, however. All that can be said is that, despite the uncertainties concerning details, the main features of the sound generated by cavitation seem to be accounted for.

IV. SURFACE DISTURBANCES

The airborne sound which accompanies the splash made by a droplet or a solid object falling onto the surface of water is well known. It is rather to be expected that concomitant sounds exist in the water below the surface and, indeed, such is the case. This is not to say that there is any resemblance or necessary relation between the two sound fields—only that disturbances of the surface which make sounds in air generally produce other sounds in the water.

* Dr. T. Brooke Benjamin has indicated that theoretical work on a related question, the development of the pressures about a collapsing cavity containing a significant amount of gas, is underway at King's College, Cambridge. Results indicate that a shock will develop in the vicinity of a gas-filled collapsing cavity if the pressure at maximum compression exceeds a value of about 2000 atmospheres. This pressure is slightly greater than that reached at the first collapse of the gas bubble formed by an underwater explosion.

Figure 12. Transient cavities in the vortex sheet behind a sharp-edged disk. The water flows from left to right. (From Ref. 31.)

Many other kinds of surface disturbances, natural and artificial, result in underwater sounds. In this section, we shall devote our attention principally to sounds made by impinging solid or liquid objects. One reason for this limitation is that scientific results available to us are concerned principally with such disturbances. Actually, however, the main features of sound production by various kinds of surface disturbances are the same. For example, it will be shown that the underwater sound of sprays and of rainfall is explained simply in terms of the sound produced by the impingement of a solitary droplet. Similarly the underwater sounds of breaking waves can be at least partially understood in the same light.

The study of splashes, cavities, and other visible aspects of water-entry phenomena has occupied the attention of a great many researchers, of whom a number have elucidated the object of their studies by some remarkable photographs. Special mention is due to the study made more than fifty years ago by Worthington [32]. He produced extensive series of photographs showing consecutive stages in the motion of the surface of the liquid during splashes produced by the impingement of liquid droplets and spherical pellets. In 1918, Mallock [33] studied similar splashes and adverted to the existence of cavities and to the relation of the size and shape of the latter to the characteristic pitch of the sounds heard in air. In recent years, a number of researchers have studied various aspects of water-entry phenomena in connection with the behavior of missiles, usually without reference to the sound. The photographs, Figure 13, were obtained by May [34] and show the cavity and the splash produced by the vertical entry of a steel sphere. Recently, Richardson [35] has called attention

Figure 13. High-speed motion pictures of a 1-inch steel sphere entering the water at a velocity of 35 ft/sec. The time interval between frames is 1.6 millise. (From Ref. 34.)

to the waterborne sounds produced by the impact, the splash, and the volume pulsations of air trapped below the surface. The only systematic attempt to relate the sounds quantitatively to the size and velocity of the impinging object seems to be that of Franz [36], some of whose results will be presented toward the end of this section.

In principle, the understanding of the generation of sounds by surface disturbances requires no theoretical considerations other than those which enter into the question of the origin of all sounds. Thus those relations which determine solutions of the wave equation in terms of the motions and pressures occurring at the bounding surface of the fluid are valid and, in principle, applicable to the determination of the sound field in water. Here, the "bounding surface" of the fluid, i.e. the water, includes the entire air-water interface as well as the boundary at which the water is in contact with any penetrating object. But the determination of the boundary conditions necessary to specify the sound field may, in a given practical situation, present a hydrodynamical problem whose solution is unattainable. It is therefore expedient to seek a more direct way of expressing the relation between the data characterizing the cause of the disturbance and the sound field which results.

The dominant boundary condition on the underwater sound field of sources at or near the free surface is, of course, the requirement that fluctuations in pressure be zero at the free surface. For sound sources which disturb the free surface only acoustically, the modification of the field resulting from the presence of the surface is easily described in terms of out-of-phase reflections at the surface. Where a radical disturbance in the surface itself constitutes the sound source, the situation is more complicated. For simplicity, we consider only disturbances which are symmetrical about a vertical axis. The sketch, Figure 14, indicates the vertical entry of an axially symmetric body of some specified density and shape. We can further specify the body and its motion in terms of a characteristic linear dimension, L , and the velocity, U , with which it strikes the surface (at time $t = 0$). The figure indicates the significance of the coordinates, r and θ . We are interested in the sound pressure, p_s , which, at sufficiently great distances from the disturbance, obeys the wave equation,

$$c^2 \nabla^2 p_s - \partial^2 p_s / \partial t^2 = 0. \quad (16)$$

The solution can be expressed in terms of spherical harmonics (Lamb [6], § 292). For our purpose, however, it is more useful to consider the sound field as that

Figure 14. Sketch illustrating surface disturbance caused by an entering body.

due to a system of axially symmetric multipole sources located at the origin of the disturbance.* The time-varying strengths of the multipole sources are such as to describe the actual flow velocities outside the immediate vicinity of the disturbance. At some specified stage of the entry, the strength of any such multipole, say of order m , must, from consideration of dimensions, be proportional to UL^{m+2} . But the sound pressure associated with a multipole source of order m is proportional to the $(m + 1)$ th time derivative of the strength of the source. These, with appropriate further dimensional considerations, allow the sound pressure to be expressed in the form

$$p_s(r, \theta, t) = \frac{\rho U^2 L}{r} \sum_{m=0}^{\infty} \left(\frac{U}{c} \right)^m A_m \left[\theta, \frac{U}{L} \left(t - \frac{r}{c} \right) \right] \quad (17)$$

The free-surface condition requires that A_0 (corresponding to a simple source) be zero, so that for small values of the Mach number (U/c) the dipole term ($m = 1$) can be expected to predominate. The dependence of the functions A_m upon the polar angle θ can be expressed very simply: In particular,

$$A_1(\theta, \tau) = Z(\tau) \cos \theta, \quad (18)$$

* The condition for which such a procedure is valid may be stated as follows: (1) all flow velocities involved are very small in comparison with the velocity of sound; and (2) the wavelength of the highest sound frequency of interest is large in comparison with the linear dimensions of the disturbance. Ordinarily this means simply: $(U/c) \ll 1$; $(fL/c) \ll 1$.

Figure 15. Splash, cavities, and bubbles accompanying entry of aluminum sphere. The sphere, 3.9 inches in diameter, struck the surface with a velocity of 29 ft/sec.

so that the dependence of the sound pressure upon ρ , c , U , L , r , θ , and t , can be expressed in the form

$$p_s(r, \theta, t) = \frac{\rho U^3 L \cos \theta}{rc} Z \left(\frac{U}{L} (t - r/c) \right) \quad (19)$$

It will be recognized that gravity and also various properties of the liquid and gaseous media will to some extent influence the course of the disturbance. Fig. 15 shows some further details of the complicated sequence of events accompanying the vertical entry of a solid sphere. Besides the primary flow implicit in the displacement of the water by the entering object, we can discern a number of more-or-less distinct features such as the spray, the entrance cavity, the cavity formed by the air trapped below the surface after closure of the entrance cavity, and the smaller air bubbles. The manner in which the last two phenomena radiate sound has been discussed in Sect. 2. But even without adverting to details, we recognize various physical parameters not considered in the derivation of [19] as relevant: the acceleration of gravity, g ; surface tension, T ; the viscosity of the liquid, μ ; and the pressure, P_0 , adiabatic compressibility modulus, γP_0 , and thermal diffusivity, D' , of the overlying atmosphere. Their influence may be indicated, in a formal way, simple by the inclusion of the appropriate dimensionless parameters as additional arguments in the function Z in Eq. (19), the essential results being unaffected:*

$$p_s = \frac{\rho U^3 L \cos \theta}{rc} Z \left(\frac{U}{L} (t - r/c), \frac{U^2}{gL}, \frac{\rho UL}{\mu}, \frac{\rho U^2 L}{T}, \frac{\rho U^2}{P_0}, \gamma, \frac{UL}{D'} \right). \quad (20)$$

* Sounds associated with mechanical vibrations of the entering object are outside the scope of the present discussion. They may be very significant, however.

But even this seemingly extensive list really includes only a few of the more obvious parameters. Eq. (20) is presented, therefore, not as a complete statement of the form of the relation involved but merely as an indication of its complexity.

An experimental investigation of the sound produced by the vertical entry of objects of just one shape and density (say, steel spheres) into water at ordinary conditions would require the empirical determination of a function involving at least three independent arguments. Moreover, even the most complete determinations might describe the actual sound only stochastically, since the details of the motions which result in sound are not necessarily reproducible. It is not surprising, therefore, that such data do not exist.

However, Franz [36] has shown experimentally that an important part of the sound produced by the impingement of water droplets and small solid objects is produced by the impact, so that only the density and compressibility of the water are involved. The term "impact" here refers not merely to the sudden contact of the surfaces but rather to the entire initial regime during the entry of a blunt or pointed object in which inertial reactions predominate. Fig. 16 shows his experimentally determined spectral distribution of the underwater sound energy radiated at the vertical impingement of water droplets. The size and velocity of the droplets were

Figure 16. Spectral distribution of underwater sound energy produced by the vertical impact of water droplets. (Figs. 16, 17, and 18 from Franz [36].)

varied over wide ranges. Nevertheless, the data define a single function when reduced according to the analysis leading to eq. (19). It is not feasible to distinguish each combination of droplet size and velocity in the plot, but the data show very little systematic dependence upon either variable. Figure 17, traced from an oscillograph record, shows the universal function describing the sound pressure. It will be observed that the major part of the sound energy is generated immediately after the initial contact and during an interval of time smaller than that required for the droplet to travel a distance equal to its own radius.

In addition to the sound of impact, droplets of certain combinations of size and velocity occasionally produced a damped sinusoidal pulse of sound pressure similar to that produced by the formation of an air bubble at a nozzle. The sound, when it occurs, is caused by a tiny air bubble trapped beneath the surface by the splash. Such a bubble is visible in the sequence of photographs in Fig. 18. This sequence is

Figure 17. Pressure pulse produced by the impact of a water droplet.

part of a high-speed motion picture which Franz made prior to his investigation; the appended oscillographic record does not correspond to the particular splash appearing in the photographs but shows, nevertheless, the considerable time interval between the impact sound and the bubble sound. Attention is directed to the fact that no measurable sound is produced during the greater part of the interval in which the fascinatingly varied undulations of the surface occur.

Franz also measured the underwater noise generated by steady showers of water droplets falling on an extended area. The results agreed with the measurements of the sound made by solitary droplets and, so far as available meteorological data permit comparison, with underwater noise levels measured during rainfall. (The sounds from a continuous succession of randomly spaced impacts add incoherently so that the total energy in each band of frequencies is conserved).

Similar, though less extensive, data were obtained for the entry of solid objects of various shapes. An intriguing theoretical problem is that of relating the time-dependent doublet strength, and hence the sound field, to the shape and motion of the entering body, or to its shape, mass, and entering velocity. It does not appear likely that the answer will be obtained in any simple form. A special case is the entry of a massive solid body having a conical nose. Here, considerations of similarity indicate that the sound pressure pulse must begin as a "ramp function", i.e. must begin as zero and increase linearly with the time.

The preceding discussion has emphasized the gaps in available concrete information concerning sound produced by surface disturbances. Perhaps, however, its pres-

Figure 18. Oscillogram of sound pressure and photographs of splash produced by impinging water droplet. The interval between frames is 13 millisecc. The impact sound occurs between Frames 1 and 2; bubble sound begins between Frames 13 and 14. The bubble can be seen in Frames 14 to 18. The mass of the droplet is 56 mg; its velocity at impact, 350 cm/sec.

entation will indicate something of the complexity of the subject and the vast opportunity for further research.

V. UNSTEADY FLOW

The generation of sound by an unsteady flow differs in an essential way from the mechanisms discussed in the previous sections. Whereas bubbles and splashes inherently require the presence of both liquid and gaseous states of matter, an unsteady flow can generate sound in a homogenous fluid, either liquid or gas alone.

The recognition of the contribution to the noise from high-speed aircraft made by turbulence and other forms of unsteadiness has led to many investigations of this form of noise. Comprehensive reviews of these investigations have been published recently by Fowell and Korbacher [37] and by Mawardi [38]. The terms "aero-dynamic noise" and "flow noise" have been applied to the phenomenon; but we prefer the term "unsteady-flow noise" in order to point up the essential feature causing the noise and distinguishing it from other kinds of fluid-dynamic noise.

Results obtained in the study of noise from air flows are applicable to the

analogous situation in water, since the mechanisms involved are fundamentally the same. However, because of the greater density and lower speeds encountered in water, the relative importance of various phenomena can be different. The present brief review will be limited to those aspects of the problem of greatest interest in connection with flowing water.

Recognition of the association of certain sounds with unsteady flow goes back to the nineteenth century. Rayleigh, for example, attributed the "aeolian tones" made by wind streaming past a wire to the vortex wake behind the wire; and the "jet-edge tones" of whistles were also believed to result from fluctuations in the jet. Richardson [39] has described the early investigations of these sounds. These investigations were concerned primarily with the frequency of the sound. No serious attempt seems to have been made to understand the factors controlling the intensity of the sound.

A theoretical basis for investigating the magnitude of the sound pressure associated with unsteady motion was provided by Lighthill [40] in 1950. He discarded the classical form of the acoustic wave equation, which is admittedly only a small-amplitude approximation, and rederived an exact wave equation from the exact equations of fluid motion. This exact wave equation is inhomogeneous and contains terms which represent sources of sound associated with fluctuating velocities. For a liquid, the essential terms in Lighthill's equation are

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = -\rho \frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j}, \quad (21)$$

where p is the pressure, c the velocity of sound, u_i the i -th component of particle velocity, x_i the i -th space coordinate, $\nabla^2 = \partial^2 / \partial x_i \partial x_i$ and a repeated index indicates summation in accordance with tensor notation. If the velocity components are sufficiently small, i.e., if $\rho u_i u_i \ll p_s$, then the right side of the equation is negligible and the classical homogeneous wave equation results. If, however, the velocities are not negligible, the right side represents sources of sound.

The fluctuating pressure in an unbounded fluid can be related to the fluctuating velocities by a volume integral over all space of the right side of Eq. (21). Thus

$$p_s(x, t) = - (\gamma_4 \pi) \int (1/r) q(x', t') dV, \quad (22)$$

where $p_s(x, t)$ is the instantaneous sound pressure at time t and position x in space (x without subscript indicates the three coordinates x_1, x_2, x_3). In the integral, r is the distance between the point x and the volume element dV at position x' , and $q(x', t')$ is the value of the right side of Eq. (21) at position x' and earlier time $t' = t - r/c$. In the application of this solution, the fluctuating velocities constituting $q(x, t)$ are considered to be independent variables whose values are given as part of the description of the flow.

In the usual unsteady flow, the fluctuating component of the velocity differs significantly from zero in only a limited portion of the unbounded space. The associated pressure fluctuations are then very much larger directly within the unsteady region than outside it.

At a point within the unsteady region, the fluctuating pressure depends primarily on the values of q at points in the immediate vicinity, corresponding to small values of r . In this vicinity, the difference r/c between the time t and earlier time t' is negligible. Since the sound velocity c does not appear in Eq. (22) in any other way, the pressure fluctuations are independent of the sound velocity. This implies that the fluctuations inside the unsteady region do not depend on the compressibility of the medium.

Far from the unsteady region, at distances large compared with the dimensions of the region, the situation is different. If the compressibility of the fluid is neglected in the calculation for these distances also, the magnitude of the pressure fluctuations is found to decrease very rapidly with increasing distance, as r^{-3} . If the difference between

t and t' is taken into account in the evaluation of the integral, however, a term results which is proportional to (ρ/rc^2) . The pressure fluctuations associated with this term vary inversely as the distance in the manner of ordinary spherical waves of sound. Indeed, these fluctuations correspond to the sound energy radiated by the flow, and this distant region is accordingly called the "radiation field."

At intermediate distances, neither far away nor directly within the unsteady region, the dependence or independence of the pressure fluctuations on the compressibility of the fluid is determined by the frequency of the fluctuations. At low frequencies, associated with sound of wavelength c/f much larger than the distance, the fluctuations do not depend on the compressibility. When it is desired to distinguish the essentially-incompressible pressure fluctuations existing within a wavelength of the unsteady region from the compressive fluctuations in the radiation field, the close-up distances are called the "near field." In the near field, the magnitude of the pressure fluctuations increases more rapidly with decreasing distance than in the radiation field.

Because the pressure fluctuations within and near the region of unsteady flow do not involve the compressibility of the medium, these fluctuations have been called "pseudosound" by Blokintzev. [41] However, a pressure-sensitive hydrophone responds to the pseudosound just as it responds to any sound pressure; the fact that these fluctuations do not involve propagated sound energy makes no difference.

Fluctuations within the unsteady region.—An estimate of the magnitude of the pressure fluctuations within an unsteady flow was made by Taylor [42] in 1936. He related the pressure to the fluctuating velocity by an expression like Eq. (21) but with the time-derivative term omitted; the omission is equivalent to the assumption that the fluid is incompressible. To obtain a tractable velocity field which nevertheless duplicated some of the characteristics of isotropic turbulence, the velocities were assumed to be distributed in space like standing waves of sound in a box. For this synthetic model of isotropic turbulence, the fluctuating pressure and velocity are related by $\bar{p} = 1.6\rho\bar{u}_1^2$, where u_1 is one component of the fluctuating velocity and the tilde ($\sim$) indicates rms values of the fluctuations.

More recently, calculations of the fluctuating pressure in isotropic turbulence were made independently by Heisenberg, [43] by Obukhov, [44] and by Batchelor. [45] Their calculations all involve equivalent assumptions concerning the statistical characteristics of the distribution function for the velocities. These more fundamental calculations lead to the result $\bar{p} = 0.6\rho\bar{u}_1^2$, the coefficient being lower than Taylor's original estimate. According to Uberoi, [46] however, the value of the numerical coefficient is quite sensitive to the exact form of the statistical distribution function of the velocities.

Batchelor also obtained a relation between the space correlation function of the fluctuating pressure and the correlation function of the velocity. For the specific case of a velocity correlation of the Heisenberg type, the calculated pressure correlation falls to zero more rapidly than the velocity correlation, and the longitudinal integral scale of the pressure is about half that of the velocity.

Ogura and Miyakoda [47] used Batchelor's relations to calculate the spectral density of the fluctuating pressure from simplified spectral functions for the fluctuating velocity. These calculations indicate that the spectral density of the pressure falls more rapidly with frequency, in the high-frequency region, than does the spectral density of the velocity.

The first attempt to measure the fluctuating pressure in an unsteady flow seems to have been reported by Rouse. [48] Within a turbulent jet of air discharging into free space, the fluctuating pressure and velocity were found to be related by $\bar{p} = 1.1\rho u_1^2$. Strasberg and Cooper [49] attempted some measurements of the fluctuating pressure in the turbulent wake behind a cylinder. At a point 24 cylinder diameters downstream, $\bar{p} = 1.7\rho\bar{u}_1^2$. They also determined the spectral densities of the fluctuating pressure and velocity. The spectra measured 24 diameters downstream are shown on Fig. 19,

Figure 19. The spectral densities of the fluctuating pressure and the fluctuating velocity in the wake of a cylinder, at a downstream distance of 24 diameters. The measurements were made in air.

where the normalized dimensionless spectral densities* of both the pressure and the velocity are plotted against the Strouhal number (fD/U_0) , based on cylinder diameter D and free-stream velocity U_0 . The two sets of points indicate that the measured spectra of the pressure are essentially the same as those of the velocity. This is an unexpected result, not at all what would be anticipated from the theoretical relations for isotropic turbulence.

However, any comparison between measured values of fluctuating pressure and the theoretical predictions based on isotropic turbulence is subject to two uncertainties:

1. The validity of the measurements is in some question. There is no conclusive evidence that the probes used to measure the pressure do actually indicate the fluctuating pressure that would have existed in the absence of the probe.
2. The measurements were performed in flows which were not isotropic. In this connection, Kraichnan [50] has estimated the pressure fluctuations in several simpli-

* The spectral density of a quantity is defined so that the integral of the spectral density on frequency is equal to the mean-square of the fluctuating component of the quantity. In particular, for the spectral density $G_p(f)$ of the fluctuating pressure, $\int_0^\infty G_p(f) df = (\bar{p}_s)^2 = \langle p_s^2 \rangle_{Av}$. Also, if $G_p(f)$ is normalized by division by $(\bar{p}_s)^2$ and non-dimensionalized by multiplication by (U_0/D) , the integral on Strouhal number is then unity.

fied models of anisotropic flow. For the case of a steady mean shear ($\partial U_1/\partial x_2$) with superimposed isotropic turbulent fluctuations of rms value $\bar{u}$ and correlation scale L , $\bar{p}_s \propto \rho \bar{u} L (\partial U_1/\partial x_2)$. This anisotropic model is probably closer than is isotropic turbulence alone to the actual flows in a turbulent jet or in the wake of a cylinder.

Mention should be made of the periodic fluctuations in pressure, of very large amplitude, existing in the wake close to a cylinder shedding vortices. Blokhintzev [41] has calculated the magnitude of these fluctuations, by assuming that the wake is a steady Kármán vortex street. The calculated pressure fluctuation along the centerline of the wake is given by $\bar{p} = 0.07 \rho U_0^2$; along this line the fluctuation is primarily at twice the shedding frequency. The calculated value agrees quite well with the value measured by Strasberg and Cooper [49] at this frequency for distances within 3 cylinder diameters of the cylinder. At the shedding frequency, the fluctuations are even larger; close to the cylinder, values were observed as large as $\bar{p} = 0.3 \rho U_0^2$. Further downstream, the measured values decrease as the periodic fluctuations degenerate into random turbulence.

The pressure fluctuations in a free turbulent region is a topic worthy of considerable additional attention. The practical importance of these pressure fluctuations can be indicated by an example of their magnitude. In the wake of a cylinder in water at a speed of 5 meters/sec (10 knots), the rms pressure fluctuation 24 cylinder diameters downstream is estimated to be about 5×10^3 dyne/cm². This pressure corresponds to a level more than 70 decibels above the ambient noise in the sea at sea state 2.

Pressure in the near field.—The fluctuating pressures outside a turbulent region are much smaller than those mentioned above. In the near field close to a turbulent region, however, their magnitudes can be significant. Jorgensen [21] has measured the pressure fluctuation outside a free turbulent jet of water. The spectral density of the pressure, measured 4 jet diameters off the axis and 4 diameters downstream from the mouth of the jet, is shown in Fig. 20. The bottom and left scales are non-dimensional, but dimensional scales have been put along the top and right side for a jet diameter of 0.6 inch at a speed of 30 knots. For these specific conditions, the fluctuations at 500 cy/sec are some 30 db above the ambient noise at sea state 2.

Radiated pressure fluctuations.—In the radiation field far from an unbounded region of unsteady flow, the pressure fluctuations are even smaller than in the near field. The sound radiated by an unsteady flow has received perhaps more attention than any other form of flow noise, because of its importance in connection with high-speed jet aircraft. At the speeds encountered in water, however, the radiated pressure fluctuations are completely negligible.

The negligible values of the sound pressure radiated by turbulence in water are illustrated by the case of the turbulent jet. Fitzpatrick and Lee's [51] measurements with air jets give for the rms sound pressure, in the direction of average intensity,

$$\bar{p}_s = 2 \times 10^{-3} \rho U^2 (U/c)^2 (D/r), \quad (23)$$

where D is the diameter of the orifice, and U the mean efflux velocity. At a speed of 15 meters/sec (30 knots) and a distance of 100 jet diameters, the rms sound pressure is only about 0.005 dyne/cm²; at least 40 decibels below the ambient noise in sea state 2.

The influence of boundaries.—The discussion to this point has been concerned with flows in an unbounded space. If boundaries are present, the sound field is of course modified by the boundaries. If the boundaries are within the unsteady region, sound radiation can be associated directly with the boundaries, and this boundary radiation can be much stronger than the radiation from the unsteady motion itself.

The complete relations describing the sound field generated by an unsteady

Figure 20. The spectral density of the fluctuating pressure in the near-field of a free jet of water. (From Jorgensen [21].)

flow in the presence of boundaries were developed by Curle [52] in 1955. If the boundaries are directly within the unsteady region, their principle contribution to the radiated sound pressure is given by a surface integral over the boundaries, of the pressure and the boundary acceleration, viz.,

$$p_s(x,t) = -\frac{\rho}{4\pi} \int \frac{\ddot{u}_j'}{r} \cos \theta_j dS + \frac{1}{4\pi} \frac{\partial}{\partial x_j} \int \frac{p'}{r} \cos \theta_j dS. \quad (24)$$

Here r is the distance between the point x and the surface element dS ; p' is the pressure and $\ddot{u}_i'$ the component of acceleration in the x_i direction, at the surface element dS at earlier time $t-r/c$; and θ_i is the angle between the x_i direction and the surface normal out of the fluid. It is of interest that the equation is of the same form as the classical solution of the homogeneous wave equation in terms of the pressure and acceleration at the boundary (cf. Lamb [6] §290); the same terms appear in Curle's result even when the motion near the boundaries is so large that the homogeneous wave equation is no longer applicable.

If the boundary is contained within a region small compared with the wavelength of the sound at the frequency of interest, the integrals in Eq. (24) may be evaluated without reference to the shape of the boundary. In the radiation field far from the boundary, the sound pressure is then given by the simple relation

$$p_s(t) = \frac{1}{4\pi r} \left[\rho \dot{V} + \frac{\rho V \dot{U}}{c} \cos \theta_u + \frac{\dot{F}}{c} \cos \theta_F \right], \quad (25)$$

where V is the instantaneous volume enclosed by the boundary; U the velocity of the centroid of the enclosed volume; F the total force exerted on the fluid by the boundary; and θ_u and θ_F the angles between the radius vector toward x , and the directions of $\dot{U}$ and $\dot{F}$ respectively. The dots indicate time derivatives. In this equation, the terms on the right have their values at earlier time $(t-r/c)$. The $\dot{V}$ term in the equation corresponds to the ordinary simple source associated with volume pulsation (cf. Eq. (4) of Section 2). The other two terms have a dipole-like directionality and are associated with translational oscillation of the boundary and with oscillating forces on the fluid.*

Even if the boundary is rigid and does not vibrate, sound can nevertheless result from oscillating forces acting on the fluid. Thus, the aeolian tones radiated by a cylinder have been explained in terms of the oscillating force, between the fluid and the cylinder, associated with periodic shedding of vortices. To calculate the magnitude of the sound, Etkin, Korbacher and Keefe [53] assume that the transverse oscillating force on the fluid is given by

$$F = 0.90 (\frac{1}{2}\rho U_0^2) LD \sin(2\pi f_1 t), \quad (26)$$

where U_0 is the free-stream velocity, D and L are here the diameter and length of the cylinder, and f_1 is the frequency of vortex shedding given by $f_1 = 0.2U_0/D$. Substituting Eq. (26) into (25), and assuming that the cylinder is rigid, the sound pressure is calculated as

$$p_s(t) = 0.18(\frac{1}{2}\rho U_0^2)(U_0/c)(L/r)(\cos 2\pi f_1 t) \cos \theta, \quad (27)$$

θ being the angle between the radius vector and the flow direction. The equation indicates that the sound pressure is proportional to the cube of the velocity.** This relation was verified by their experimental data, some of which are shown in Fig. 21.

Similar calculations and measurements are reported by Phillips. [54] He calculates the fluctuating force from measured values of the fluctuating velocity in the wake of the cylinder and obtains a coefficient 0.76 instead of the 0.9 in Eq. (26). The measured sound pressures of both Etkin et al and Phillips are lower than the value given by Eq. (27) by constant factors. The difference is explained by the fact that the vortex shedding is not correlated along the entire length of the cylinder, so that the phase of the oscillating force varies axially. If the correlation length is assumed to equal the cylinder diameter times a factor b , then the quantity $(bDL)^{1/2}$ should be substituted for L in Eq. (27). Phillips estimates that b is about 17 for the range of Reynolds number $U_0 D/\nu$ from 80 to 160, and about 3 for Reynolds numbers above 300; whereas Etkins estimates b as about 8.

The intensity of aeolian sounds has also been measured by Gerrard [55] over a wide range of Reynolds numbers. His data can be interpreted as indicating that the sound pressure varies with the cube of velocity, as required by Eq. (27), but that the numerical constant is larger by a factor of about 4 for Reynolds numbers below 300. This higher value may be due to the fact that, at low Reynolds numbers, the vortex shedding is correlated all along the cylinder axis.

Both Phillips and Etkin et al assume that the vibration of the cylinder does not radiate any significant sound. The latter, in fact, report measurements showing that the sound is independent of the elastic properties of the cylinder. However, it is well known that the intensity of the aeolian tones increases when the cylinder vibrates in resonance at the shedding frequency; this phenomenon was observed by Strouhal himself. Phillips' explanation is that the vibration of the cylinder "locks-in" or correlates

* High-order terms, with higher negative powers of c , have been omitted from Eq.(25); such terms are associated with higher moments of the boundary vibration and pressure distribution.

** Eq. (27), without the numerical coefficient, was predicted by Blokhintzev, *ibid*.

Figure 21. The sound pressure radiated by a cylinder shedding vortices in air, as a function of speed. (From Etkin et al [531].)

the vortex shedding along its entire length, causing the sound pressure to increase by a factor $(L/bD)^{1/2}$.

Although the sound generated by the vibration itself is neglected by both Etkin and Phillips, it should be noted that in a relatively dense liquid such as water, the amplitude of the sound may be increased by the vibration even in the absence of any modification of the pattern of vortex shedding. To indicate the magnitude of the sound that may result from vibration, consider a rigid cylinder held by compliant supports so that the cylinder vibrates transversely, when immersed in water, in free oscillation at a frequency f_r with logarithmic decrement $\pi\delta$. Assume, as a first approximation, that the amplitude of vibration is so small that the gross motion of the fluid is not at all affected. The cylinder vibration will result from a reaction to the force exerted by the cylinder on the fluid. In terms of the force exerted by a fixed cylinder, e.g., the force F given by Eq. (26), the reaction force acting on the cylinder is $-(F + M_a \dot{U})$, with M_a being the added mass due to the water. If the force and vibration are sinusoidal with frequency f_1 equal to the shedding frequency, so that $F = F^0 e^{2\pi i f_1 t}$ and

$\dot{U} = \dot{U}^0 e^{2\pi i f_1 t}$, the amplitudes F^0 and $\dot{U}^0$ are related by

$$\dot{U}^0 = [F^0 / (M_a + M_c)] \cdot [(f_r / f_1)^2 - 1 + i\delta]^{-1}, \quad (28)$$

M_c being the mass of the cylinder. The sound pressure amplitude p_o is then given by Eq. (25), with $\dot{U}$ and $\dot{F}$ replaced by $2\pi i f_1 \dot{U}^0$ and $2\pi i f_1 (F^0 + M_a \dot{U}^0)$, respectively, and $p_s(t)$ by p_o .

At resonance, $f_1 = f_r$; the sound pressure amplitude is then given by

$$p_o = \frac{f_1 F^0}{2rc} \left[i + \frac{M_a + \rho V}{\delta(M_a + M_c)} \right]. \quad (29)$$

The right term inside the brackets represents the direct contribution of the vibration to the radiated sound; if the displaced mass ρV is of the same order as the cylinder mass, the contribution can be considerable, especially since δ is often smaller than 0.01 for ordinary mechanical systems. The vibration can thus cause a large increase in the sound pressure in liquids, independently of any influence on the vortex shedding or other aspects of the gross flow. The vibration will, in fact, modify the gross flow and the associated fluctuating force. This interaction has not been studied in any detail.

These aeolian sounds are associated with fluctuations at a predominant frequency. However, the fluctuations usually contain a random component. In the flow past a cylinder, the fluctuations become almost completely turbulent at high Reynolds number, above about 10^5 . The radiated sound is then also random. The spectral distributions of the random force and vibration are related by an expression similar to Eq. (28); if the vibration contributes to the sound, the sound spectrum will be peaked around the resonant frequency of the system.

The foregoing discussion indicates that there is a satisfactory understanding of the sound generated by flow about a cylinder. However, this case is only the simplest example of the acoustic interaction of a surface with an unsteady flow. A more complicated situation involves the sound associated with a turbulent boundary layer. This is of practical interest in connection with sound generated at the skin of an aircraft or at the hull of a ship.

Noise from a boundary layer.—The sound generated by the fluctuations in the boundary layer on a rigid bounding surface have been investigated theoretically by Phillips. [56] His results indicate that the sound from a plane boundary layer is small, except perhaps near the transition region. The radiated sound pressure is proportional to $\rho U_0^2 (U_0/c)$ if the flow maintains similarity. Accordingly, it is likely that only negligible sound will be radiated at the speeds encountered in water.

If the boundary surface is relatively flexible, however, flexural vibration of the surface can result in significant sound. This surface vibration is excited by the local pressure fluctuations within the boundary layer; the resulting sound pressure can be much larger than that radiated by the pressure fluctuations themselves. The vibrating boundary acts like a sounding board and substitutes a simple source with no (U_0/c) dependence for the dipole-like source of a rigid boundary.

A calculation of the radiated sound can be made with the following sequence of steps:

1. The pressure fluctuations at the boundary are estimated from a knowledge of the flow.
2. The flexural vibration of the boundary in response to the pressure fluctuations is determined.
3. The radiated sound pressure associated with the vibrating surface is calculated.

This procedure assumes that the motion resulting from the vibration is too small to modify the grosser flow, so that the pressure fluctuations at the boundary can be treated as an independent variable.

Once the boundary pressure is known, steps (2) and (3) involve the application of well-known acoustical equations which give the sound pressure radiated by a membrane or a plate in terms of an arbitrary distribution of pressure on its surface. Because the fluctuations in the boundary layer are random in both time and space, the calculations are relatively complicated, but no essential difficulty is introduced by the randomness. It is necessary to know the statistics of the fluctuations, viz., the special density and the space correlation of the pressure fluctuations. The evaluation of these quantities, however, which is essentially the first step listed above, has not been accomplished in a completely satisfactory manner as of this writing. A theoretical estimate of the characteristics of the pressure fluctuations can be attempted by following the procedure outlined by Batchelor and already discussed in connection with the local fluctuations within a turbulent region. However, too little is known about the fluctuations within the boundary layer to permit the calculation to be carried out with any rigour.

Some preliminary measurements of the fluctuating pressure in a turbulent boundary layer at the bottom of an open channel have been described by Einstein and Li. [57] Willmarth [58] has reported some measurements of the fluctuating pressure at the boundary of a pipe. He estimates that $\bar{p} = 3.5 \times 10^{-3} (\frac{1}{2} \rho U_0^2)$, where U_0 is the free-stream velocity, but the measurements may be in error because the fluctuations may not be correlated over the entire area of the pressure-sensing device. Certainly, additional measurements are required to make possible an estimate of the space correlation and the spectral density of the pressure fluctuations, as required by step (1) above.

Theoretical calculations of the boundary-layer noise have been attempted by Ribner, [59] and more recently by Corcos and Liepmann, [60] and by Kraichnan, [61] all using certain assumed characteristics of the pressure fluctuations in the layer. The first two of these papers deal primarily with a boundary surface which is "floating" i.e., unconstrained. Kraichnan, on the other hand, treats a square plate constrained at its periphery; he also goes into greater detail concerning the character of the fluctuations in the boundary layer. Which of these calculations is closer to the actual situation cannot be determined until some measurements have been made of this form of noise.

Other unsteady flows.—Before this section is concluded, mention should be made of several other types of sound associated with unsteady flow which have received some attention in recent years:

(1) "Jet-edge tones," which are generated when a jet impinges on a thin plate or wedge. The sound contains a predominant frequency determined by a periodic undulation of the jet, the frequency depending primarily on the velocity of the jet and the separation between the mouth of the jet and the edge. Richardson [39, 62] has described his own and several other investigations of the phenomenon. Other discussions have been published by Curle, [63] by Powell, [64] and by Bouyoucos and Nyborg. [65] Two alternative mechanisms have been suggested. Richardson and Curle believe the undulations are caused by instability of the flow itself, whereas the others believe that the sound reacts on the jet and "triggers" the instability. In either case, the sound is presumably generated by a fluctuating lift force on the edge.

(2) "Orifice-pipe tones," generated when a fluid discharges through a sharp-edged orifice at the end of a tube. The frequency of the sound has been determined for a wide range of conditions by Anderson. [66] The sound, in this case, is probably a simple source associated with periodic fluctuation in the rate of efflux from the orifice.

(3) Resonant cavities excited by external flow past the mouth of the cavity. The frequency of these sounds has been investigated by Blokhintzev [67] and by Harrington. [68] There seems to be some interaction between the cavity and the unsteady flow past the mouth, because the predominant frequency of the sound may differ somewhat from the frequency at which the cavity resonates in the absence of flow. The sound can be considered to be a simple source associated with alternating flow in the mouth of the cavity.

The investigations of jet-edge tones, orifice tones, and resonant cavities have been concerned primarily with the frequency of the sounds, without any attempt to determine their amplitudes. Accordingly, it is not possible to say whether any of them are significant sources of underwater noise. However, the uncertainty results from the lack of adequate descriptions of these flows. Once suitable descriptions of the flows become available, the associated sound pressure can be calculated by the methods described in this section.

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DISCUSSION

S. Byard

I would just like to describe in a very few words the results of some simple laboratory experiments which have been carried out by Mr. Hey at the Admiralty Research Laboratory, and which seem to illustrate some of the characteristics of propeller cavitation noise.

These experiments have been made with quite small hydrofoils, the smallest being less than one inch by one inch, either mounted in a miniature water tunnel with transparent sides, or mounted at the periphery of a disc which is rotated under water. By suitably orienting the foil, a tip vortex cavity can be formed at the trailing edge, or alternatively blade cavitation induced at the leading edge, and the noise spectra of the two forms of cavitation are markedly different.

In these model tests the noise spectrum associated with the tip vortex cavity is localized around 2 to 3 kilocycles/second, whereas the spectrum of the noise due to blade cavitation is more generally distributed, extending to the higher frequencies.

In the case of a ship's propeller, where cavitation is well established, both forms of cavitation are normally present and give rise to a continuous noise spectrum with a marked peak.

Some simple scaling experiments in which the effect of doubling the size of the hydrofoil was observed, showed an approximate doubling of the diameter of the tip vortex, and a lowering of the frequency of the peak in the noise spectrum. It was also possible to show experimentally that the source of the noise associated with the tip vortices is located close to the hydrofoil, and does not extend along the vortices.

It is thought that experiments along these lines, which can be regarded as supplementing and extending the work on the single collapsing cavity which has been carried out by Kendrick some time ago at the Admiralty Research Laboratory, can help us understand the mechanism of propeller cavitation noise.

G. K. Batchelor

I should like to ask a question of Mr. Fitzpatrick. He remarked that, in his experiments, no noise generated by the turbulence in jets could be detected. Such noise

would presumably vary as the 8th power of Mach number, and I think a few rough calculations suggest that the level would be so small that one couldn't expect to be able to detect it. I should like to ask if he has done any experiments in which there was a lifting surface somewhere in the flow. There is a connection between the fluctuations in the lift on a solid body, and the amount of surface noise generated. The intensity of the surface noise varies as the 6th power of the Mach number, and might therefore be greater than that of the volume noise at low Mach numbers.

I suppose the most favorable circumstances for generation of surface noise might be some kind of flat plate edge-on to a stream.

H. M. Fitzpatrick

I know of no such experiments involving a jet and lifting surface except, of course, the "jet-edge" studies mentioned.

M. Strasberg

When you try to do such an experiment you run into a complication. If you put a plate in a jet—you are, I presume, thinking of a rigid plate which would not oscillate in the jet, so that the radiation would be the kind which is associated with the lift forces on the plate. What happens in practice when you try to do that is that the plate itself cannot be made infinitely rigid, and begins to oscillate and respond to local pressures the jet exerts. As a result of the oscillation of the plate, sound will be radiated, which very much complicates any analysis of the experimental results.

That sound which is radiated by a vibrating plate, excited into vibration by the fluctuations of pressure on it, is a very important type of sound. A theoretical analysis has been given by Dr. Kraichnan, who is, I see, a couple of rows behind you there.

M. J. Lighthill

I could make comments on this point. But I would like to say first how much, how very much I enjoyed Dr. Fitzpatrick's lecture. He tried to say some things I would disagree with, but he wasn't able to. I find myself in complete agreement with everything he said. I think it was extremely desirable that a survey of that kind should have been made.

On the question raised by Dr. Batchelor, whether hydrodynamic sound of a one-phase character (involving no bubbles of any kind) could be detectable, I suppose that the dipole radiation due to moving a circular cylinder about in a fluid is one of the most promising kinds. A lot of information about the sound radiated in this case is available. Phillips (J. Fluid Mech. 1 (1956), p. 607) has recently correlated extensive measurements by Holle and Gerrard in the range of Reynolds numbers from 400 to 40000, by means of a formula which gives

$$P = 0.006 \frac{\rho U^3 l d}{a^3}$$

where P is the total acoustic power radiated, ρ the density, U the velocity of the cylinder, l and d its length and diameter, and a the velocity of sound; and he gives reasons why the constant should be independent of Reynolds number in the range in which the wake is irregular but the boundary layer is laminar at separation.

In water, if $U = 30$ knots, $l = 10$ ft. and $d = \frac{1}{2}$ in., then the Reynolds number is in this range (it is about 10^5), and the formula gives $P = 10^{-3}$ watt.

C. A. Gongwer

I concur, of course, the aeolian tone will be generated, but it only occurs in the fairly narrow range of Reynolds number associated with the Karman vortex street.

I noticed Dr. Fitzpatrick's curve of the sound from the underwater jets, which

were made so carefully, that the sound peaks at the Strouhal number, which is usually associated with the vortex street. This suggests some kind of regular pattern of vortices associated with submerged jets, the value .18 or .2, suggests the vortex street.

The purely hydrodynamic noise is, of course, hard to find, unless you go to a submerged hydrodynamic oscillator in which there may be no free interfaces whatsoever. In this case one can generate pure hydrodynamic noise by exciting a resonant system in relatively incompressible liquid.

There are old cases of controversies of this type, about whether you should streamline hydrophones, where people have said that there is no hydrodynamic noise of significance, but experimental tests have shown some anomalies. For example, pointing directional hydrophones at submerged bumps on objects, one will get a noise coming from this bump, under conditions in which cavitation really could not exist at all. There is a real mystery there, and if one can throw some light on this subject, it would be well worth while.

There may be the question of compound interferences from roughness, and so forth, putting the interface into the system somewhere, or tiny bubbles, but in the absence of the interface, it is really hard to explain some of these noises.

M. Strasberg

Dr. Gongwer's comment about one of our spectrum curves indicates that the curve requires some clarification. I believe he stated that our curve of the spectrum of the radiated sound from a jet had a peak at a Strouhal number of about 0.2, suggesting that some regular vortex pattern might be associated with the jet. However, on the curve referred to we did not use the Strouhal number, but rather a dimensionless frequency based on the sound velocity.

We use two types of dimensionless frequency. For the sound close to the jet, where the behavior is the same as in an incompressible fluid, we do use the Strouhal number, frequency times diameter divided by flow velocity, as the dimensionless frequency parameter. On the other hand, for the sound radiated far from the jet, we think it is more appropriate to use a dimensionless frequency based on the sound velocity, that is, frequency times diameter divided by sound velocity.

There is some question whether either frequency parameter by itself is adequate. Actually, both parameters may influence the radiated sound. But we think that the radiated sound depends more on the parameter fD/c than on fD/U_0 .

So it was just an accident that the peak occurred at a value of 0.2, the same as the Strouhal number for vortex shedding.

T. B. Benjamin

I wish only to emphasize the importance of one aspect of underwater noise touched upon in Strasberg and Fitzpatrick's excellent survey. This is the formation of real shock waves, akin to underwater blast waves, by collapsing cavitation bubbles. To call to mind this effect, it may be remembered that the pressure pulse produced at a distance by a cavitation collapse can have an effective duration of about 10 microseconds; but under some circumstances the pulse may develop a shock front whose transit past a fixed station may occupy considerably less than one microsecond, that is, very much less than that of the whole pressure wave.

There appear to be two respects in which the presence of shocks may be specially important, the first being their role in cavitation damage. In writings on this subject the term "shock wave" is used extensively, and the importance of the brief duration of cavitation pressures is fully appreciated; however, the distinction between very short yet continuous pulses, such as would always occur if water were strictly incompressible, and real shock waves is not always recognized. Nevertheless, one is naturally led to give weight to this distinction, since it is known that the stresses developed in a solid boundary by the incidence of shock waves are much greater than those due to continuous pressure pulses of the same amplitude. Some experiments done at

Cambridge have revealed great difficulties in trying to detect whether or not shock waves are present in cavitation pressure fluctuations, but left no doubt that they do arise at least sometimes. It does not seem untimely to express a hope that in the hands of able experimenters such as Professor Knapp and Dr. Ellis, this aspect of cavitation damage will soon be fully elucidated.

The second reason justifying attention to the matter of shock waves concerns the cavitation noise spectrum, as is found by making frequency analyses of hydrophone outputs. It has been pointed out before now that if a finite pulse possesses a discontinuity of the sort represented by the mathematical property $f(t^+) - f(t^-) \neq 0$ (i.e., a finite "jump" as at a shock front), its energy spectrum is asymptotically proportional to the inverse square of frequency; in other words, it decreases at a rate of 6 decibels per octave. Moreover, no other kind of pulse has a spectrum with this asymptotic property. This fact may possibly have a bearing on some experimental measurements of noise spectra; but a probably more useful consideration is as follows. If a pulse having the property described above is applied to a resonant system, the frequency response decreases asymptotically (i.e., well above the resonance frequency) as the inverse fourth power of frequency. Again, no other kind of pulse produces this response. Thus the response at very high frequencies from a practical hydrophone, which is bound to be affected by self-resonances at a number of frequencies, should be sustained at a slope of -12 decibels per octave if shocks are present.

The theoretical problem of shock formation by collapsing bubbles is clearly a very difficult one. The allied but evidently simpler problem where a spherical shock arises as the result of conditions *prescribed* over a spherical boundary, such as Sir Geoffrey Taylor, Dr. Whitham and many others have treated, seems formidable enough; but the cavitation problem presents the great additional difficulty that the motion of the inner boundary (the bubble surface) is not initially known. Dr. Gilmore, for instance, has made an important contribution towards the understanding of bubble motion in compressible fluids, and his method of treatment was the first to give useful results for cases where the fluid velocities are of the order of the sound velocity; but apparently much remains to be done to clear up the question of strong cavitation where a shock wave probably forms very close to the bubble surface and has a profound effect on the bubble motion.

I understand that Mr. Fitzpatrick has made some calculations, not mentioned explicitly during his talk, which enabled him to estimate the least severe conditions of cavitation which should give rise to shock waves, and independently I have also attempted this. My own work stopped short after considering a weak shock formed far from the cavity centre, but I hope eventually to hear that Mr. Fitzpatrick has proceeded closer to the "heart of the matter."

These remarks are intended only to reaffirm the desirability of regarding water as a compressible fluid as far as cavitation collapse is concerned, and so to commend compressibility effects in cavitation as an interesting and useful field of study. Recognition of the theoretical difficulties need not reflect a pessimistic view of progress in this subject, for the signs are that a great deal will be achieved towards a complete solution within the next few years.

F. R. Gilmore

The authors have given a very interesting and well organized review of the problem of hydrodynamic noise. I wish to comment only on the section dealing with the noise produced by a collapsing cavity. According to the theoretical work of Lord Rayleigh, as a spherical bubble in an incompressible liquid collapses to infinitesimal size, velocities and pressures in the neighborhood of the bubble approach infinity, provided that the pressure of any vapor or gas in the bubble either remains constant or at least does not rise rapidly enough to prevent complete collapse. Recent theoretical work yields similar results even when the compressibility of the liquid is

BUBBLE WALL VELOCITY FOR A COLLAPSING BUBBLE IN WATER, WITH $\Delta P = 1 \text{ ATM}$ AND ($P + 3000 \text{ ATM}$) $\rho^7 = \text{CONSTANT}$

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taken into account (see my comments on M. S. Plesset's paper). For actual cavitation bubbles, however, there must be a point in the collapse after which these simple theories no longer apply, either because the bubble is no longer spherical, or because the pressure of the vapor inside starts to increase rapidly as the collapse becomes too rapid to permit vapor pressure equilibrium, or for any of a number of other possible reasons. Such complicating factors, which are very difficult to treat theoretically, provide an effective "cut-off" to the infinite pressure peak given by theory. However, if one is interested in the pressure pulse propagated to some finite distance from the bubble, there is another cut-off which may be more amenable to theoretical treatment. This arises from the well-known tendency of finite compression waves to become steeper as they propagate. In a compression wave having a sharp peak the peak will move faster than the rest of the wave, until a vertical front (shock wave) is formed. Thereafter, the peak will gradually advance into the shock wave and be effectively "lost." The height of the pressure peak is thus significantly reduced as it propagates (in addition to the geometric reduction in the spherical situation), and the pulse at some distance from the bubble may be independent of the very last stages of the bubble collapse. This possibility deserves careful theoretical investigation, using perhaps the methods developed for underwater explosion shocks during World War II. Since I am presently occupied with Air Force instead of Naval problems, I would particularly like to encourage someone else with the appropriate theoretical background to undertake such an analysis.

XI

FREE-STREAMLINE THEORY AND STEADY-STATE CAVITATION

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The following is a brief review of several aspects of free streamline theory, with particular reference to steady state cavitation [1].

CAVITY MODELS

The Helmholtz concept of free streamline long had its principal applications in the theory of wakes and jets. The well-known Kirchhoff model of the infinite wake, in which free streamlines detach from a body and enclose a constant pressure stagnation region, provided a theory of fluid resistance within the framework of classical hydrodynamics, but was recognized, practically from inception, to be unrealistic in its essential features [2]. It has recently become apparent that the natural application of free streamline theory is to the phenomena of cavitation rather than wakes.

In steady state cavitation, as observed in water tunnel studies, a vapor-filled cavity forms behind a body past which the fluid is moving at sufficiently high speed, the cavity being essentially at the vapor pressure of the liquid and its boundary a constant pressure free surface. The conditions here differ from those of the classical wake theory in that the cavity pressure p_c is below rather than equal to the static value p_∞ . The pressure difference is usually measured in non-dimensional units by the *cavitation parameter*

$$\sigma = \frac{p_\infty - p_c}{\frac{1}{2}\rho u^2}, \quad (1)$$

where u is the free stream velocity and ρ is the fluid density. σ is the basic similarity parameter of the theory of cavitation, and in the absence of gravity, viscosity, etc., all non-dimensional quantities are functions of it alone.

Observed cavities have positive cavitation number and of course finite dimensions. It is the function of the theory to describe these flows and the associated physical quantities. The limiting case $\sigma = 0$ —which yields an infinite cavity corresponding to the classical wake—often proves useful in studying the finite cavity. Mathematically, the essential difference between the problems of cavitation and the traditional problems of hydrodynamics is that in the former the shape of the flow region is unspecified and has to be determined from the condition that the pressure, and hence flow speed, is constant on the free surface.

In attempting to describe a finite cavity within classical hydrodynamics—that is, under the assumptions of steady, gravity free, irrotational flow—we run at once into the paradox that such a flow cannot be realized physically. To be specific, let us consider the behavior of the free streamlines that detach from a body in cavitation flow in an unbounded stream. Because the minimum pressure, and therefore the maximum speed, must occur on the free streamlines (the cavitation hypothesis), it follows that these curves are convex to the flow [3]. This allows only the following possibilities: *i.* The free streamlines may extend to infinity downstream without intersecting one another; this can be shown impossible except when $\sigma = 0$. *ii.* They may intersect,

which obviously contradicts the continuity of the flow (and can also be shown mathematically untenable if they are considered to overlap). *iii.* The streamlines turn back, forming a re-entrant jet which, if continued, must pass through the rear of the body or through the cavity wall (Fig. 1). The last alternative is the only one mathematically consistent with the theory but is of course non-physical. Thus, under the present assumptions, it is impossible in principle to satisfy the physical requirements of steady state cavitation.

Water tunnel and water entry experiments reveal clearly the existence of the re-entrant jet. However, the jet is greatly weakened by turbulent mixing in the stagnation region, and in larger cavities it can be seen striking the cavity walls and then being swept away by the flow. To all intents and purposes these flows may be considered stationary. In smaller cavities, or in flows past obstacles with large afterbodies, the jet can be observed (in high speed photographs) distorting the cavity and giving rise to an unsteady pulsating motion.

Another approach to the problem of the finite cavity is through the artificial but conceptually simpler Riabouchinsky model. This avoids the non-physical doubly-covered flow plane by introducing the mirror image of the obstacle as a streamline of the flow (Fig. 2). As far as quantitative results are concerned it is a matter of indifference which cavity model is adopted, for the Riabouchinsky and re-entrant jet flows yield almost identical values of drag and cavity dimensions. From this it appears that conditions at the rear of the cavity have but little effect on gross flow quantities.

In the two-dimensional case the calculation of these flows is reduced by the hodograph method to a simple exercise in conformal mapping. Consider first the symmetric re-entrant jet flow past a flat plate, shown in Fig. 1 with its image planes. Let $f = \varphi + i\psi$ be the complex potential and let the cavity speed be unity. Then one finds:

$$f'(s) = M \frac{s(s^2 - b^2)}{s^2 - h^2}, \quad M \text{ real}, \quad (2)$$

$$f'(z) = \left(\frac{s+1}{s-1} \right)^{\frac{1}{2}} \left(\frac{b-s}{b+s} \right), \quad (3)$$

where $b = -2h^2 - h + 2$, and the cavitation parameter is given by

$$\sigma = \frac{-4h^3 + h^2 + 4h - 2}{(h^2 + h + 1)^2}. \quad (4)$$

The expression for the drag coefficient obtained from (2) and (3) by integrating the pressure difference $p - p_c$ over the front face is

$$C_D(\sigma) = (1 + \sigma) \left[1 - \frac{\int_{-1}^0 \left(\frac{1+s}{1-s} \right)^{\frac{1}{2}} \left(\frac{s-b}{s^2-h^2} \right)^2 s ds}{\int_{-1}^0 \left(\frac{1-s}{1+s} \right)^{\frac{1}{2}} \left(\frac{s+b}{s^2-h^2} \right)^2 s ds} \right]. \quad (5)$$

This gives the asymptotic formula, valid for small σ ,

$$C_D(\sigma) \sim \frac{2\pi}{4 + \pi} (1 + \sigma). \quad (6)$$

FIG. 1A

FIG. 1B

FIG. 1C

A more precise estimate is

$$C_D(\sigma) \sim \frac{2\pi}{4 + \pi} \left[1 + \sigma + \frac{\sigma^2}{8(\pi + 4)} \right]. \quad (7)$$

The formula (6) underestimates within .8 percent at $\sigma = 1$ and improves in accuracy as $\sigma \rightarrow 0$. If the flat plate is replaced by a symmetric wedge of semiangle $\alpha\pi$ the exponent $\frac{1}{2}$ in (3) and (5) is replaced by α .

In asymmetric flow without circulation past an arbitrary obstacle of diameter l , the drag and lift coefficients are related to the jet width J and direction γ by the formulas

$$C_D(\sigma) = 2 \frac{J}{l} \sqrt{1 + \sigma} (1 - \sqrt{1 + \sigma} \cos \gamma), \quad C_L(\sigma) = - \frac{2J}{l} \sin \gamma. \quad (8)$$

Calculations of asymmetric re-entrant jet flows have not yet been carried out in specific instances.

Turning to the Riabouchinsky flow past the plate (Fig. 2), and assuming the flow so normalized that $f = -1, +1$ at the front and rear stagnation points, one finds

FIG. 2

$$\frac{dz}{df} = \frac{\sqrt{f^2 - k^2} + f\sqrt{1 - k^2}}{k\sqrt{f^2 - 1}} \quad (9)$$

where

$$\sigma = \frac{2k'}{1 - k'}, \quad \text{and} \quad k' = \sqrt{1 - k^2}. \quad (10)$$

The various flow quantities can be expressed in terms of the complete elliptic integrals of first and second kinds, denoted in the usual notation by $E = E(k)$, $E' = E(k')$, $K = K(k)$, $K' = K(k')$. Thus, we have: the ratio of cavity width d to plate length l ,

$$\frac{d}{l} = \frac{k' + E' - k^2 K'}{k'^2 + E' - k^2 K'}; \quad (11)$$

the ratio of cavity length h to plate length,

$$\frac{h}{l} = \frac{E - k'^2 K}{k'^2 + E' - k^2 K'}; \quad (12)$$

the drag coefficient,

$$C_D = 2(1 + \sigma) \left(\frac{E' - k^2 K'}{k'^2 + E' - k^2 K'} \right). \quad (13)$$

From the asymptotic properties of elliptic integrals as $k' \rightarrow 0$ one obtains (6) and (7) again, and also

$$\frac{d}{l} \sim \frac{4}{4 + \pi} \left(\frac{2 + \sigma}{\sigma} + \frac{\pi}{4} \right) \quad (14)$$

$$\frac{h}{l} \sim \frac{4}{4 + \pi} \left[\left(\frac{2 + \sigma}{\sigma} \right)^2 - \frac{1}{2} \log 4 \left(\frac{2 + \sigma}{\sigma} \right) - \frac{1}{4} \right]. \quad (15)$$

These values are accurate within .7 percent at $\sigma = 1$ and improve rapidly for smaller σ . (6) is a special case of the formula

$$C_D(\sigma) \sim C_D(0)(1 + \sigma), \quad (16)$$

which seems to be generally valid for both plane and axially symmetric cavities having fixed separation points [4], and is in good agreement with observation. The error term in (16) grows more rapidly with σ in the axially symmetric case. Experimental results indicate that the formula $C_D(\sigma) \sim C_D(0)(1 + \alpha\sigma)$ should replace (16) in flows past bodies having variable points of separation (such as the cylinder and sphere).

Formulas analogous to (14) and (15) for the axially symmetric Riabouchinsky flow past a body of diameter l are

$$\frac{d}{l} \sim \sqrt{\frac{C_D(\sigma)}{\sigma}} \quad \text{and} \quad \frac{h}{l} \sim \sqrt{\frac{C_D(\sigma)}{\sigma^2}} \log \frac{1}{\sigma}. \quad (17)$$

From these it is apparent that the plane cavity has the larger length-width ratio for fixed cavitation number, whereas the axially symmetric cavity is the flatter of the two (as measured by d/l). The derivation of (17) is based on a convincing but heuristic comparison between cavities and flows past ellipsoids [5].

In the limiting case $\sigma = 0$, when the cavity is infinite, its asymptotic shape is given by

$$\begin{aligned} y &\sim Cx^{\frac{1}{2}} && \text{in plane flow;} \\ y &\sim \frac{Cx^{\frac{1}{2}}}{(\log x)^{\frac{1}{2}}} && \text{in axially symmetric flow [6].} \end{aligned} \quad (18)$$

The drag is proportional to C^2 in the former case and to $C^{\frac{1}{2}}$ in the latter.

The theory of cavitation flow past smooth obstacles—that is, obstacles without corners or sharp edges—still has important gaps. The problem here is that the separation points of the free streamlines are not known a priori and must be determined as part of the solution; at the same time, it is not entirely clear what conditions are needed to determine a unique and physically acceptable solution. It is well known that a free streamline at detachment has either infinite curvature or the same curvature as the body from which it separates. The latter type of detachment will be called *smooth*. A flow will be considered *physically acceptable* if the free streamlines do not intersect the body or themselves and if the maximum speed occurs on the free boundary (as required by cavitation). Since a free streamline in cavitating flow is convex, its detachment (from a smooth body) must evidently be smooth if the flow is to be physically acceptable, but this condition is certainly not sufficient, as one sees by counterexample.

It is known [7] that within the class of bodies for which physically acceptable infinite cavity flows always exist are those having non-decreasing curvature on both sides of the forward stagnation point (Fig. 3). However, it is a simple matter to exhibit other convex bodies for which physically acceptable solutions of this type do not exist. In this connection the flow past an asymmetric wedge is instructive. (The wedge can be looked upon as an approximation to a convex obstacle with large curvature at the vertex.) The various possibilities when $\sigma = 0$ are the following: *i*. The

Figure 3.

dividing streamline meets the body at the vertex, and free streamlines detach from the ends forming an infinite cavity; in this case the ratio of the two lengths cannot be arbitrary and hence the flow is exceptional. *ii.* The free streamlines again detach from the ends, but the dividing streamline meets the wedge at a point other than the vertex; in which case the velocity is infinite at the corner and flow is physically unacceptable. *iii.* One of the free streamlines detaches from the vertex, determining a solution that is simply the flow past an inclined flat plate; this flow is not always acceptable since the free streamline may intersect the second side of the wedge. A way out of this difficulty is to allow the free streamline detaching from the vertex to reverse direction, forming a re-entrant jet (Fig. 4), while the portion of the flow not included in the jet bounds an infinite cavity whose free streamlines separate from the ends. Except for the feature of the two-sheeted flow plan, the latter model satisfies the physical requirements and, furthermore, is observed experimentally [8]. Its basic idea can most likely be extended to arbitrary curved obstacles to provide physically acceptable solutions which are otherwise lacking without the jet. In these cases the flow would contain multiple cavities. Such a theory is still unexplored.

CALCULATIONS OF AXIALLY SYMMETRIC FLOWS

Developments in the theory of axially symmetric cavitation flow reverse the customary order of things in that the general theory is already well advanced and boasts numerous qualitative results on existence, uniqueness, geometric behavior of the free streamlines, extremal properties, etc., while, on the other hand, useful explicit solutions are still unknown. (There do exist explicit solutions but without practical

Figure 4.

significance.) This may be in the nature of things and it is quite possible that such solutions are simply unattainable.

It is therefore natural to fall back on numerical methods, and with the increasing emergence of high speed computing machines it is appropriate that methods be developed for the calculation of cavity flows. At the present time, the labor in these calculations is formidable, and most questions of convergence and accuracy remain unanswered.

An interesting and valuable illustration is provided by the problem of the *vena contracta*, which is perhaps the best studied of the axially symmetric free surface problems. The experimental value for the contraction coefficient of a jet issuing from a circular orifice in a plane wall is in the neighborhood of .61. This figure is of particular interest because it coincides with the known theoretical value $\pi/(\pi + 2) \approx .611$ of the contraction coefficient of a plane jet issuing from a slot. This agreement has stimulated the conjecture that the theoretical values of corresponding plane and axially symmetric contraction coefficients are indeed the same. Trefftz [9], in calculations we shall not discuss here, first gave strong support to the conjecture by arriving at the value .61 for the axially symmetric jet, and later several calculations by the relaxation method yielded the same result [10]. However, Garabedian [11], using an altogether different method, recently obtained the value .58. This difference is large enough to raise serious doubts concerning the conjecture and to spur inquiry as to the sources of the discrepancy.

Garabedian's point of departure is the concept of axially symmetric flow in a space of $\epsilon + 2$ dimensions. The stream function of such a flow obeys the equation

$$\psi_{xx} + \psi_{yy} - \frac{\epsilon}{y} \psi_y = 0, \quad (19)$$

Figure 5.

where x is the axial and y the radial coordinate in the meridian plane. One can now formulate the problem of jet flow in terms of ε ; namely, a curve $\Gamma(\varepsilon)$ (Fig. 5) and a stream function $\psi(x, y; \varepsilon)$ are sought such that $\psi = 0$ on the axis of symmetry, $\psi = \text{const.}$ on the bounding streamline $C + \Gamma$, and $\frac{1}{y^\varepsilon} \frac{\partial \psi}{\partial n} = 1$ on Γ . Of course, $\varepsilon = 1$ corresponds to axially symmetric flow and is the case of immediate physical interest. The contraction coefficient as a function of ε is $C_c(\varepsilon) = (\bar{Y}(\varepsilon)/Y)^{1+\varepsilon}$ where Y and $\bar{Y}$ are the orifice and jet radii (cf. Fig. 5). If it were indeed true that $C_c(1) = C_c(0) = \pi/(\pi + 2)$, one might conjecture that $C_c(\varepsilon)$ remains constant, and hence that $C'_c(\varepsilon)_{\varepsilon=0} = 0$. However, this quantity can be accurately computed and is equal to $-.057$. Garabedian arrives at his estimate $C_c(1) = .58$ by cubic interpolation, using the exact values of $r = \bar{Y}/Y$ at $\varepsilon = -1, 0$, and ∞ where $r = 0, \pi/(\pi + 2)$, and 1 , respectively, and the computed quantity $\frac{d}{d\varepsilon}(\bar{Y}/Y)_{\varepsilon=0} = .243$. Even without the latter figure, interpolation gives $C_c(1) = .586$, an indication that $.58$ is probably accurate.

Turning to the relaxation method, we recall that the procedure here is to cover the flow region with a rectangular mesh and to replace the differential equation by an appropriate difference equation. In the problem at hand, the free streamline must be chosen so that the velocity computed by finite differences is constant on the curve. This requires a trial and error procedure which finally yields a curve on which the computed speed variation is considered sufficiently small. Unfortunately, it is not entirely clear what should be considered "sufficiently small" in a free boundary problem, especially since the computed velocity fluctuation can be insensitive to relatively large displacements of the free streamline. That this should be so can be anticipated for theoretical reasons, but is also confirmed by specific calculations which compare an exact solution for plane Riabouchinsky flow with one computed by the finite difference method [12]. The speed variations on the free streamline in the latter calculation were

considerably smaller percentagewise than the error in the computed height above the separation point. The principal source of error in finite difference calculations is probably the infinite curvature of the free streamline at separation, for small errors in shape here are reflected in relatively large errors in the downstream position of the free streamline—without necessarily incurring large velocity fluctuations downstream. Good accuracy at detachment requires careful refinement of the mesh or special handling of the flow at separation. The importance of truncation error has still to be clarified.

Although several approximate methods have been applied to cavity flows with axial symmetry [13], calculations based on exact theory are quite rare. Recently, using the scheme of dimensional interpolation outlined above, Garabedian obtained the value $C_D = .827$ for the drag coefficient of the circular disk in infinite cavity flow. The extrapolated experimental data favors a value between .80 and .81, but its spread includes .827 at the upper end. In his treatment of the Riabouchinsky cavity, for which the preceding method is unsuitable, Garabedian [14] has developed an iteration procedure based on linearization of the original boundary value problem. Namely, he shows that on any curve Γ which is displaced normally by an amount δn from the free streamline, the correct stream function satisfies the condition

$$\frac{1}{y} \frac{\partial \psi}{\partial n} + \frac{\kappa}{y} \psi = 1 \quad (20)$$

within an error of the order $(\delta n)^2$, where κ is the curvature of Γ . An approximate solution of the flow problem is thus provided by a stream function ψ satisfying (20) on the starting approximation Γ ; the next approximation to the free streamline—and the starting point of the next cycle—is the curve on which $\psi = 0$. In the calculation of flow past a disk, Garabedian finds $C_D(\sigma)/(1 + \sigma) = .865$ when $\sigma = .2235$. This is a few percent higher than the experimental value, but confirms the observed increase in slope of the $C_D(\sigma)$ curve. His figure 2.30 for the ratio of cavity to disk diameter at this value of σ is in close agreement with observation.

COMPARISON METHODS

A fruitful source of qualitative results has been the *comparison method* that was initiated by Lavrentieff and later extended by Gilbarg and Serrin [15]. This method features simple geometric arguments in the physical plane, which are often equally

Figure 6.

Figure 7.

effective in axially symmetric and compressible flow problems. Basic to the approach are certain comparison theorems, of which the following is an important example.

Let D and $\bar{D}$ be flow regions of two (plane or axially symmetric) flows having uniform velocities $\underline{q}_0, \bar{q}_0$ at infinity, where $\bar{q}_0 \geq q_0$. Let D and $\bar{D}$ be bounded by the streamlines γ and $\bar{\gamma}$, extending to infinity in both directions. If $D \subset \bar{D}$, and if γ and $\bar{\gamma}$ have a point P in common (Fig. 6), then the respective flow speeds q and $\bar{q}$ satisfy at P the inequality $\bar{q}(P) \geq q(P)$, and furthermore, when $\bar{q}(P) \neq 0$ the equality holds if and only if $D = \bar{D}$ and the two flows are identical.

As illustrative consequences that are useful in cavitation we mention the following:

1. In plane symmetric or axially symmetric infinite cavity flow, let T_1 and T_2 be obstacles in the upper half plane having the same endpoint A , and suppose that T_1 lies above (or touches) T_2 ; then the relative positions of the corresponding free streamlines detaching from A are reversed. This implies that the shape factor C in (18) is larger for T_2 than for T_1 and hence the cavity drag of T_2 exceeds that of T_1 . Because of the formula (16) we may infer a similar inequality for small positive σ . The same idea can be used to obtain simple bounds on the drag of an obstacle. Indeed, let the latter be a curve denoted by C extending from $(0, 0)$ to (a, b) . It is assumed that the infinite cavity flow, which may detach from any point on C , is physically acceptable in the sense of the previous discussion. Then the drag on C is bounded from below by the drag on E_1 , and from above by that on E_2 , where E_1 and E_2 are the following

Figure 8.

curves: E_1 consists of a vertical segment L erected at the origin and of an arc of the free streamline in the cavity behind it, the length of L being such that the endpoint of the arc is at (a, b) ; E_2 is the vertical segment $x = a$, $0 \leq y \leq b$.

2. In plane or axially symmetric Riabouchinsky flow past a convex obstacle, the length of the cavity and the ratio of length to width are monotonically decreasing functions of the cavitation number. This result is a simple consequence of the above comparison theorem, and for sake of illustration of the technique we carry out the details of proof.

Let Fig. 7a indicate a typical configuration of two Riabouchinsky flows with the same incident velocity past the obstacle C , in which C_1 , C_2 and Γ_1 , Γ_2 are the respective image obstacles and free streamlines. If Γ_1 lies below Γ_2 then direct application of the comparison theorem at the separation point P shows that the smaller of the two cavities has the larger cavitation number, as required. On the other hand, if Γ_1 intersects Γ_2 , as in Fig. 7a, a suitable similarity contraction takes $C + \Gamma_1 + C_1$ into a curve $C' + \Gamma'_1 + C'_1$ lying within the cavity $C + \Gamma_2 + C_2$ and such that Γ'_1 has a point of contact with Γ_2 (Fig. 7b). Under this contraction the flow speed at corresponding points can be preserved. It follows by the comparison theorem that the speed on Γ'_1 , and hence on Γ_1 , exceeds that on Γ_2 . This proves the monotonicity of the cavity length. The monotonicity of the length-width ratio is proved similarly. Let the cavity with the smaller cavitation number be contracted to have the same length as the other. It suffices now to prove its width smaller. Suppose this were not so; then a typical configuration of the two flows would be as in Fig. 8a. Another contraction of the flow with the smaller cavitation number would bring it into a position where its free streamline lies below the other, and at one point is in contact with it (Fig. 8b). The comparison theorem is now applicable, and provides an evident contradiction, thereby establishing the required monotonicity property.

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13. Cf. R. N. Cox and J. W. Maccoll, Recent contributions to basic hydroballistics. in this volume, pp. 215-233.
14. P. Garabedian, The mathematical theory of three-dimensional cavities and jets, *Bull. Amer. Math. Soc.* 62, 219-245 (1956).
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DISCUSSION

J. D. Nicolaides

I wonder if the speaker would care to comment on the efforts that I mentioned yesterday, of attempting to determine the transient development of the cavity, using the NORC computer?

D. Gilbarg

No. I think—well, I will say something. Because of the difficulties around the singular point, I would say the finite difference method would give a useful answer only if you are not too concerned with accuracy.

However, if they are pressed to their limits of capability, these machines may

give pretty good answers. The likelihood is, from what we know, you will get useful information, which is not too accurate, whatever that happens to mean.

C. J. Cohen

I would just comment that one could do this matter by systematically reducing the size of the mesh, so one would at least get a bound-under answer and you would know where you stand.

D. Gilbarg

Yes, I think that should be done, definitely.

C. J. Cohen and L. D. Gates, Jr.

Professor Gilbarg stated in his paper that the relaxation solution of the Riabouchinsky problem at Dahlgren indicated that the method was ill-suited for cavitation flow problems because although the spread in the computed velocities along the free boundary was of the order of one per cent, the error in the maximum cavity width was of the order of 10 per cent.

During the Symposium, one of us, (C. J. Cohen) commented from the floor that to his recollection the 10 per cent error in maximum boundary width had been mostly eliminated by applying corrections at the outer boundaries of the flow. (These outer boundaries had been introduced to allow a difference system solution, and the outer boundary values for the stream function were at first taken to be those for undisturbed flow.)

Dr. Cohen's recollection was in error although we feel that the large error in cavity width noted by Professor Gilbarg was in fact due to the original crude conditions imposed on the outer boundaries.

Perhaps a more serious failure of the difference method was revealed when a trial boundary for plane flow computed from the analytical plane flow solution was tested using corrections to the outer boundary conditions. The observed velocity variation was of the order of 5 per cent, which was felt to be rather large and due to truncation error. The average velocity was about 2 per cent above the true velocity. This latter figure represented an improvement over previous data obtained without the outer boundary correction. This was what prompted Dr. Cohen's remark.

M. D. Van Dyke

I should like to ask whether there is any doubt that locally, just at the edge of the disk, the singularity has the same nature as in the two-dimensional free streamline flow.

D. Gilbarg

No, there is no doubt about that.

T. Y. Wu

In Professor Gilbarg's lecture the re-entrant jet model seems to be preferred as the "only non-artificial model" for the irrotational cavity flows. Actually there are many other mathematical models, e.g., Riabouchinsky's image model and Roshko's dissipation model; and they are all, at least in my opinion, more or less artificial. I like to discuss these models by pointing out their physical basis and certain limitations, hoping that a better understanding of these models may help to clarify some unsolved problems.

In the real fluid flow past a bluff body with a separated flow region in the wake, experimental observations indicate that the discontinuous surfaces in the flow, or free streamlines, are actually thin shear layers, into which the vorticity is fed from the boundary layer in front of the separation point. The shear layers in general do not

continue smoothly far downstream, but roll up to form vortices which in turn diffuse and eventually are dissipated in the wake. Consequently, when the hydrodynamic force exerted on the body is calculated by integrating the local pressure over the body, the flow may be assumed to be irrotational provided the dissipative wake can be represented by an equivalent, properly chosen, potential flow model. This argument, I believe, should provide the physical justification of any workable mathematical model for cavity flows. A model is satisfactory as long as it gives a good description of the flow near the body; and it is artificial in the sense that it fails to describe the wake far downstream.

Consider a two-dimensional flat plate set normal to a uniform stream of velocity U . The image model introduces an image plate, of the same size as the real plate, to be put at a certain distance downstream of the real plate, the pair of plates being joined by free streamlines. From the potential theory, the force exerted on the image is then equal and opposite to the drag D on the real plate. In a coordinate system where the fluid at infinity is at rest, the force holding the moving image plate does a negative work equal to $(-DU)$, which should be removed from the flow field if the resulting flow is assumed to be potential. An estimate of this amount of work corresponds approximately to the energy dissipated in the wake. Along the same line of reasoning, the jet momentum of the reentrant jet model carried away from the first mathematical sheet should be associated physically with the wake dissipation. For even though the jet can often be observed, it is quite weakened by the turbulent mixing; certainly its observed width is much smaller than its theoretical value $(0.22(1 + \sigma) \times \text{plate width})$. In the dissipation model the viscous dissipation is assumed to take place within a strip in the wake, aligned parallel to the free stream; the detailed mechanism as to how the flow is dissipated is rather immaterial. All these models have in their mathematical formulations one free parameter (such as the location of the image, the jet width, or the starting point of the dissipation zone), which can be expressed in terms of the cavitation number σ . Perhaps for this reason they yield about the same value of C_D . For small values of σ , the first two models give

$$C_D(\sigma) \approx \frac{2\pi}{\pi + 4} \left\{ 1 + \sigma + \frac{\sigma^2}{8(\pi + 4)} \right\} \quad (21)$$

whereas the coefficient of σ^2 is $[6(\pi + 4)]^{-1}$ for the dissipation model. Other than this one parameter, no freedom is left elsewhere to adjust the models. For example, there is no solution of the potential flow when the image plate is different in size from the real one.

Furthermore, the application of these models to more general cases, such as a cavitating lifting surface, must be carried out with care. For instance, the image of a lifting surface could be made symmetric with respect to either a central plane or only a certain point. To make sure that the formulation of any model is correct and compatible with the potential theory, I would believe that nothing else could serve as a better guide than the physical significance underlying the model itself. Compared with this point, the relative simplicity of the mathematical details involved in the analysis of each model is rather a minor matter. To name one example, the cavitating lifting surface with the cavity length equal approximately to the body length presents a difficult mathematical problem; apparently the difficulties could not be removed without a thorough understanding of the physical features of the flow near the rear of the cavity. Therefore it seems fruitful to devote some effort to the investigation of the physical basis of the cavity models; such an investigation would undoubtedly benefit future research in this field.

M. P. Tulin

I know that time prevented Dr. Gilbarg from discussing aspects of this rich subject which now have important implications for the design engineer, so permit

me to supplement his fine presentation with a brief discussion of certain currently important cavity flow problems.

Flows involving lifting surfaces of one configuration or another are now especially of interest. For two-dimensional foils with sharp edges (which fix the cavity separation), adequate, although approximate, theory for the prediction of lift, drag, pitching moment, and cavity shape exist. In practice, however, supercavitating or ventilating foils will often be operating with time varying angles of attack, as wings of finite aspect ratio, and often in the proximity of free surfaces; implied is the possible use of supercavitating foils for propellers, for alighting gear on water based aircraft, and for sustention of high speed hydrofoil boats.

In an excellent earlier talk, Mr. John Parkinson of the NACA has given us some idea of trends in water based aircraft alighting gear development. We can surmise from his remarks that designers of water based aircraft in addition to hydrofoil boat designers would like to know how the loads on a ventilated (or supercavitating) foil fluctuate during operations in a seaway, and whether, even in smooth water, these foils might flutter; they would also like to be able to predict effects of finite span and free surface proximity. Ability to design high speed foils which won't suffer large changes in lift due to free surface effects is obviously essential.

Water tunnel research people, working to provide needed experimental information, would like to know how the presence of solid walls or a free jet influence the flow over a supercavitating foil, particularly for low cavitation numbers when the cavity becomes very large. The usefulness of their information depends upon adequate knowledge of interference effects.

The current problems indicated above together with those mentioned by Dr. Maccoll, Mr. Parkinson and Mr. Nicolaides in their earlier talks and discussions are few in number compared to those which designers will have to face in the future as speeds of ships, aircraft, and underwater ordnance vehicles increase. The demands of technology in the field of cavity flows deserve to be met and influenced by pertinent mathematical treatments.

I. Imai

I would like to point out one thing about the determination of the free-streamline separation point on the body. Professor Gilbarg told us that there are three possibilities for the free streamline curvature at the separation point, that is, negative (inward) curvature, positive (outward) infinite curvature, and finite curvature. He ruled out the negative curvature from the geometrical viewpoint and the positive infinite curvature by an intuitive argument to avoid "infinity", concluding that the finite curvature is the only possibility.

However, I think that the same conclusion could be drawn most naturally by considering the separation phenomenon on the basis of the boundary layer theory. If we assume that the streamline leaves the body at a certain point with infinite curvature, it can be proved by the inviscid potential flow theory that the pressure gradient there should be infinite. This in turn means on the boundary layer theory, that the separation should have occurred earlier somewhere upstream of that assumed separation point. This is a contradiction. Therefore, on the basis of the boundary layer theory we can determine the separation point as such a point that the free streamline leaves the body with finite curvature. It can also be shown by the inviscid flow theory that the condition of finite streamline curvature necessarily leads to the conclusion that the finite curvature of the free streamline should be equal to the curvature of the body at the separation point.

XII

PHYSICAL EFFECTS IN CAVITATION AND BOILING

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INTRODUCTION

The problem to be considered here is the growth of a vapor bubble in a superheated liquid. When the vapor pressure of the liquid exceeds the ambient pressure, it becomes possible for a vapor bubble to grow from a small "nucleus" in the liquid. This nucleus is a region of nonliquid phase and presumably consists of a gas or vapor phase stabilized on a solid particle. The factors which affect the rate of bubble growth are the surface tension, the liquid inertia, and the difference between the pressure within the bubble and the ambient or external pressure. We shall suppose that the bubble begins its growth near the condition of equilibrium of the forces acting upon it. The initial growth will be slow, but it is accelerated with increase in size because of the reduction in surface tension. When the rate of bubble growth becomes appreciable, however, the temperature, and hence the pressure within the bubble, drops and the rate of growth is decreased. One therefore might expect a maximum in the velocity of the bubble wall. The reduction of the temperature within the bubble is a consequence of the latent heat requirement of the evaporation which takes place at the vapor-liquid interface as the bubble grows.

FORMULATION OF THE PROBLEM

For a quantitative solution of the problem, some simplifying assumptions may be made. These simplifications will be presented for the example of a vapor bubble growing in moderately superheated water. It will be assumed that the bubble is spherical throughout its growth. One can show that the spherical shape is stable for a growing bubble in a spherically symmetric pressure field [1]. We are clearly excluding from consideration the asymmetric buoyant force of gravity which becomes important if the bubble growth is followed for so long a time that a significant translational velocity is acquired. A translational velocity of the bubble as a whole not only causes a deformation of shape but also increases the rate of heat inflow over that used in the present analysis. In water superheated by about 10°C no great error is introduced by the buoyant force provided the bubble growth is not followed beyond a radius of 1 mm or for a time longer than a few milliseconds.

With a superheat of the order of 10°C at one atmosphere pressure, a vapor bubble grows from its initial microscopic size of approximately 10^{-3} mm to a radius of 1 mm in a time of the order of 10 millisecc. The corresponding average radial velocity is 10 cm/sec. The maximum radial velocity is about 10 times this value. We see that the velocities of concern in the bubble growth are very small compared with the sound velocity in either the liquid or the vapor. The hydrodynamic equation of motion which contains the effect of liquid inertia is thus rather simple: the liquid motion of concern is incompressible and spherically symmetric. With the limitation to irrotational flow one has the velocity potential

$$\varphi = \frac{\dot{R}R^2}{r}, \quad (1)$$

where $R(t)$ is the liquid interface or bubble wall. Further, the liquid equation of motion gives the Bernoulli integral

$$-\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 = -\frac{p(r,t)}{\rho} + C(t). \quad (2)$$

Evaluation of this equation at $r = R$ gives the equation of motion of the bubble wall

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p(R) - P_0}{\rho} \quad (3)$$

where ρ is the liquid density, $p(R)$ is the pressure in the liquid at the bubble boundary, and P_0 is the external pressure in the liquid or the pressure at infinity. The energy equation for the liquid may be written as

$$\rho c_v \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \Delta T + \dot{q} + \mu \Delta v^2 \quad (4)$$

where μ is the viscosity, k the thermal conductivity of the liquid, and $\dot{q}$ is the heat source per unit volume per unit time. With the velocity field $\vec{v} = R^2 \dot{R} / r^2$, the viscous heat generated is maximum at the bubble wall and has the value

$$(\mu \Delta v^2)_{r=R} = 12\mu \frac{\dot{R}^2}{R^2}. \quad (5)$$

In a typical bubble growth situation (103°C , $\dot{R}_{\max} = 32 \text{ cm/sec}$, $R = 3 \times 10^{-3} \text{ cm}$) this term is of the order of $0.1 \text{ cal}/(\text{cm}^3\text{sec})$. The rate of heat loss by conduction at the bubble wall $\left| \rho c_v \frac{dT(R)}{dt} \right|$, is of the order of $10^4 \text{ cal}/(\text{cm}^3\text{sec})$. Clearly the viscous heat generation is negligible.

In considering the conditions within the bubble, one has the simplification that the vapor density is so small compared with the liquid density that inertia effects in the vapor may be neglected. The vapor density is about 10^{-3} of the liquid density. Actually the pressure gradients in the vapor are an additional order of magnitude smaller than 10^{-3} of the liquid pressure gradients. The reason for the additional reduction is that the velocity and variation in velocity in the vapor are at least an order of magnitude smaller than the liquid velocity. Evaporation at the liquid interface fills the bubble as it grows and there is negligible motion of the vapor. We may therefore take the pressure in the bubble as uniform and, since the bubble wall moves at a velocity small compared with the sound velocity in the vapor, we may say further that the pressure within the vapor follows practically instantaneously its value at the bubble wall. We now argue that the uniform pressure within the bubble is the equilibrium vapor pressure of the liquid. When the liquid interface moves with a velocity $\dot{R}$, then the pressure in the vapor, p_{vap} , is related to the equilibrium vapor pressure p_{eq} (the vapor pressure for a stationary interface) by [2]

$$\frac{p_{vap}}{p_{eq}} = \frac{\rho_{vap}}{\rho_{eq}} = \frac{c^*}{c^* + \dot{R}} \quad (6)$$

where c^* is connected with the sound velocity c by

$$c^* = \frac{\alpha c}{\sqrt{2\pi\gamma}}. \quad (7)$$

In Eq. (7), α is the accommodation coefficient and γ is the ratio of specific heats. α is not known at the temperatures of present interest; it has been measured to be 0.04 for water at 0°C. This would give $c^* = 8 \text{ m/sec}$. There is some basis for believing that α is larger than 0.04 at the temperatures of concern here, but in any case the pressure of the vapor is essentially the same as the equilibrium vapor pressure.

We may consider finally the temperature within the bubble. In general the temperature of the vapor would vary with position as well as with time. The thermal diffusivity of the vapor, D' , is significantly larger than the thermal diffusivity of the liquid, and the characteristic diffusion length in the vapor, $\sqrt{D't}$, is large compared with the bubble radius. We may therefore make the approximation that the temperature is uniform in the bubble.

In summary, the physical model upon which the analysis is based consists of a spherical vapor bubble which has uniform temperature and pressure, the temperature of the vapor is that of the liquid at the bubble wall, and the pressure is the equilibrium vapor pressure for that temperature. In addition, the effects of viscosity and compressibility may be neglected.

THE EQUATION OF MOTION AND THE BOUNDARY CONDITIONS

We have seen that the viscous effect is negligible in the energy equation and we shall show that it is also negligible in our boundary condition. In the equation of motion for the bubble (Eq. (3)), one needs the pressure $p(R)$ in the liquid at the bubble wall; this pressure is related to the pressure in the vapor p_v by

$$p(R) = p_v(T) - \frac{2\sigma}{R} - 4\mu \frac{\dot{R}}{R}. \quad (8)$$

The effect of viscosity is to increase the surface tension by $2\mu\dot{R}$, and this increase is about 0.5 dyne/cm and is therefore negligible. We shall therefore write

$$p(R) = p_v(T) - \frac{2\sigma}{R}. \quad (9)$$

The boundary condition for the heat equation

$$\rho c_v \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = k \Delta T + \dot{q} \quad (10)$$

may be simply deduced in the following way. The heat $\dot{Q}$ which must be supplied to the bubble per unit time is

$$\dot{Q} = \frac{4\pi}{3} L \frac{d}{dt} (R^3 \rho') \quad (11)$$

where L is the latent heat of evaporation per unit mass and ρ' is the vapor density. This heat is supplied by conduction into the bubble so that

$$\dot{Q} = 4\pi R^2 k \left(\frac{\partial T}{\partial r} \right)_{r=R} \quad (12)$$

where k is the thermal conductivity of the liquid. Hence

$$\left(\frac{\partial T}{\partial r} \right)_R = \frac{L}{3k} \frac{1}{R^2} \frac{d}{dt} (R^3 \rho'). \quad (13)$$

The small variations of L and k with temperature are neglected. It will be assumed that the temperature of the liquid is initially uniform

$$T(r,0) = T_0. \quad (14)$$

The bubble growth is supposed to begin at $t = 0$; at a distance from the bubble the temperature will be uniform, undisturbed by the cooling produced by the bubble growth. We have, then,

$$T_\infty = T(\infty, t) = T_0 + \frac{D}{k} q(t). \quad (15)$$

We shall admit, for the present, heat sources which correspond to relatively slow temperature changes. For example, a temperature rise of $1^\circ\text{C}/\text{sec}$ would correspond to an insignificant change in T_∞ in the time of a few milliseconds over which we follow the bubble growth. The initial superheat temperature T_0 will be essentially the liquid superheat throughout the bubble growth. It is convenient to introduce a radius R_0 defined by the relation

$$\frac{2\sigma}{R_0} = p_v(T_0) - P_0. \quad (16)$$

Physically, R_0 is the effective initial radius of the bubble; it represents an extrapolation of the free spherical bubble down to the equilibrium radius for the given initial conditions. It should be noted that a bubble at rest with radius R_0 is in unstable equilibrium. The actual nucleus from which the bubble grows is not necessarily spherical and its surface energy may be less than $4\pi\sigma R_0^2$ (it may be zero); however, such a nucleus and the free spherical bubble of radius R_0 will have the same growth behavior when their radii are two or three times R_0 .

In terms of the parameter R_0 , the equation of motion (3) may be written

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p_v(T) - p_v(T_0) + (2\sigma/R_0)(1 - R_0/R)}{\rho}; \quad (17)$$

or

$$\frac{1}{2R^2\dot{R}} \frac{d}{dt} (R^3\dot{R}^2) = \frac{p_v(T) - p_v(T_0) + (2\sigma/R_0)(1 - R_0/R)}{\rho}. \quad (18)$$

If the cooling effect of evaporation is disregarded so that $p_v(T) = p_v(T_0)$, Eq. (18) may be integrated to give

$$\dot{R}^2 = \frac{R_0^3}{R^3} \dot{R}_0^2 + \frac{4\sigma}{3\rho R_0} \left(1 - \frac{R_0^3}{R^3} \right) - \frac{2\sigma}{\rho R} \left(1 - \frac{R_0^2}{R^2} \right). \quad (19)$$

This solution will be called the Rayleigh solution. For $R \gg R_0$, Eq. (19) gives

$$\dot{R}^2 \approx \frac{4\sigma}{3\rho R_0} = \frac{2}{3\rho} [p_v(T_0) - P_0]. \quad (20)$$

The actual motion may be expected to deviate markedly from the Rayleigh solution because of the cooling effect.

Let us denote the temperature at which the vapor pressure equals the external pressure P_0 by T_b , which is thus the "boiling temperature,"

$$p_v(T_b) = P_0. \quad (21)$$

Then, for superheats not too far above the boiling temperature, the vapor pressure may be approximated by a linear function of the temperature so that we may write

$$\frac{p_v(T) - P_0}{\rho} = \frac{p_v(T) - p_v(T_b)}{\rho} = A(T - T_b). \quad (22)$$

While a more accurate linear approximation to $p_v(T)$ has been used over the interval T_0 - T_b , A is very nearly the slope of the vapor pressure, $p_v(T)$, at T_0 divided by the liquid density. The equation of motion now becomes

$$\frac{1}{2R^2\dot{R}} \frac{d}{dt} (R^3\dot{R}^2) = A(T - T_b) - \frac{2\sigma}{R}. \quad (23)$$

When T is given as a function of R , the solution of the problem is of course determined.

To find this relation, it is necessary to solve the heat transport problem at the moving boundary of the bubble wall. Because of the boundary condition at the moving bubble wall it is convenient to transform the heat flow equation to Lagrangian coordinates. It does not appear possible to solve this heat equation in closed form; however, physical considerations suggest an approximation which gives an analytic solution. The thermal diffusivity of the liquid is so small that the characteristic diffusion length $\sqrt{D t}$ is small compared with the bubble radius $R(t)$. In other words, the liquid layer over which the temperature changes from its value at infinity to its value at the bubble wall may be taken to be small. This "thermal boundary layer" assumption leads to an analytic solution in successive approximations [3]. The first and second approximations have been found; the second approximation has been used to demonstrate that the first approximation has acceptable accuracy. This first approximation to the temperature field in the liquid gives for the temperature at the bubble wall

$$T(R) = T_\infty - \frac{L}{3k} \frac{\int_0^t \frac{d}{d\tau} (R^3 \rho') d\tau}{[\int_0^t R^4(\epsilon) d\epsilon]^{\frac{1}{2}}}. \quad (24)$$

THE BEHAVIOR OF THE SOLUTION FOR BUBBLE GROWTH

The bubble growth problem is now determined, and we may consider the behavior of the solution for vapor bubbles growing in moderately superheated water. The initial conditions considered are specifically that the bubble is spherical of radius R_0 and at rest for $t = 0$. Such a bubble is unstable. The growth is initiated by the rising temperature of the bulk liquid for which we have

$$T_\infty = T_0 + \frac{D}{k} q(t). \quad (25)$$

The heat source is supposed to give slow changes in this external temperature. As we have already remarked, for ordinary heating rates the bulk temperature of the liquid will remain essentially unchanged during the few milliseconds of bubble growth which will be followed here. The external temperature rise of the liquid is really only a device for initiating the growth. It may be emphasized that the important range of bubble growth is not appreciably affected by the initial conditions. Thus, one might suppose that the growth was initiated by expansion of a permanent gas in the bubble. The permanent gas content of the nucleus will soon become unimportant and the subsequent behavior will be essentially that of the pure vapor bubble described here. Another possibility is that the initial surface tension condition is not that of the free bubble, but

again, so long as the surface tension is normal when the bubble is two or three times its initial size, the bubble growth will subsequently be that given by the present analysis.

The bubble growth as found for our initial conditions may be divided into three phases: A delay period in which the growth of the bubble from equilibrium is very slow being initiated by the slow rise in the liquid temperature; there is essentially no temperature drop in the bubble vapor during this "delay" period. When the bubble has reached approximately twice its initial radius, it has acquired a small velocity and the surface tension effect is reduced. For superheats of the order of 10°C the time t_d required for this delay period is about 0.01 millisecc; for a superheat of 2°C, t_d is 0.07 millisecc. The bubble growth appears to begin abruptly near $t = t_d$. In the second or early phase of growth, the bubble velocity increases rapidly and continues to increase until the cooling effect becomes significant. The growth velocity goes through a maximum and thereafter continues to decrease as the third, or asymptotic, phase of growth is reached. At the beginning of the early or second phase of the growth, the heat source term which initiates the growth is negligible and makes no contribution to the subsequent bubble behavior. From a physical standpoint, this means that the observable bubble behavior is independent of the particular conditions initiating its growth.

It is clear that liquid inertia effects are important in determining the bubble growth in the second phase, during which the bubble experiences its maximum radial acceleration. After this phase, however, the rate of bubble growth is controlled by a balance between the rate of evaporation and the rate of cooling it produces. It is evident that the bubble must continue to grow since a stationary bubble would be at the temperature of the superheated liquid and therefore would have an excess internal pressure. Hence, the temperature at the bubble wall must continue to drop because of evaporation. But the temperature cannot drop below the boiling point and still maintain the pressure difference necessary for growth. It follows that the temperature of the bubble wall must approach the limit T_b as $t \rightarrow \infty$, and this fact is sufficient to characterize the asymptotic phase of bubble growth.

We may approximate the leading term in the asymptotic phase by simple physical arguments as follows. At a time when $R \gg R_0$, the difference between the temperature in the liquid at the bubble wall and that in the liquid at a distance is only slightly less than $T_0 - T_b$. This temperature drop takes place principally in a liquid layer around the bubble of approximate thickness given by the diffusion length $\sqrt{Dt}$. The heat flow into the bubble per unit time is therefore given roughly by

$$\dot{Q} \approx \frac{k(T_0 - T_b)}{\sqrt{Dt}} 4\pi R^2. \quad (26)$$

The heat requirement per unit time for evaporation, on the other hand, is

$$\dot{Q} = L \frac{d}{dt} \left(\frac{4\pi}{3} R^3 \rho' \right) \approx 4\pi R^2 \dot{R} L \rho'. \quad (27)$$

The approximation indicated in Eq. (27) is based on the fact that the heat flow requirement because of the volume change is considerably greater than that arising from the relatively small change in vapor density. When (27) is equated to (26), there results

$$\dot{R} \approx \frac{k(T_0 - T_b)}{L \rho' \sqrt{Dt}}. \quad (28)$$

Analytically one finds that the leading term in the asymptotic region is

$$\dot{R} \approx \sqrt{\frac{3}{\pi}} \frac{k(T_0 - T_b)}{L \rho' \sqrt{Dt}}. \quad (29)$$

This leading term in the expansion for $\dot{R}$ is in error by only 10 to 30 per cent in the range of easy experimental measurement.

The results of the analysis for bubble growth in water at 1 atmosphere pressure and 103°C are shown in Figs. 1-4. The radius-time behavior for several superheat temperatures is shown in Fig. 5. Comparisons with experimental observations [4], are shown in Figs. 6-8.

Figure 1. Theoretical radius vs time curve for the growth of a spherical vapor bubble in water at 1 atm external pressure, superheated to 103°C. The bubble is originally (at time $t = 0$) in unstable equilibrium. The growth is initiated by the introduction of a uniform heat source in the liquid (e.g. by irradiation), such as to produce a temperature rise in the liquid of 1°C in 100 sec. The curve shows the slow initial rate of growth.

Figure 2. Theoretical radius and radial velocity curves for the growth of a pure vapor bubble in water at 1 atm external pressure, superheated to 103°C.

Figure 3. Theoretical radius and bubble wall temperature curves for the 103° vapor bubble of Figure 2.

Figure 4. The theoretical radius vs time curve is shown under the same conditions as in Figure 1. The delay time at the beginning of the growth is practically undetectable.

Figure 5. Asymptotic radius versus time curves calculated for water at 1 atm external pressure and the indicated superheat temperature.

Figure 6. Comparison of theoretical bubble radius-time values with experimental values for water at 1 atm external pressure, superheated to 103.1°C. The solid curve is the Rayleigh growth curve, obtained by neglecting heat transfer effects; the dashed curve is that predicted by the asymptotic solution of the text, which takes heat transfer into account.

Figure 7. Comparison of theoretical radius-time values with three sets of experimental values obtained in water at 1 atm external pressure, superheated to 104.5°C.

Figure 8. Comparison of theoretical radius-time values with two sets of experimental values obtained for water at 1 atm external pressure, superheated to 105.3°C.

EFFECT OF EXTERNAL PRESSURE ON VAPOR BUBBLE GROWTH

Some general remarks may be made concerning the growth of vapor bubbles under various superheat conditions and for different ambient pressures. We are still confining our interest to bulk liquid temperatures which do not change appreciably in times of bubble growth of the order of some milliseconds. A comparison will be made of the bubble growth in superheated water at 1 atmosphere with that in superheated water first under high pressure and secondly under low, or even negative pressures.

If water at 1 atmosphere pressure is superheated, for example, to 106°C, the Rayleigh velocity is 408 cm/sec. Because of the cooling effect, the maximum bubble growth velocity is 160 cm/sec. If we now consider water with a boiling point of 210°C at approximately 19 atmospheres pressure and if we take the same superheat of 6°C, the Rayleigh velocity is 1.39×10^3 cm/sec. The maximum bubble growth velocity is only 16.9 cm/sec. Rayleigh velocities and maximum bubble growth velocities are given in Table I for both 6°C and 3°C superheats for 1 atmosphere and 19 atmospheres-pressure. The striking difference in velocity behavior between the high pressure situation and the 1 atmosphere case is easily explained physically. The heat inflow to the bubble is directly dependent upon the magnitude of the vapor density ρ' , and the vapor density

Figure 9. These curves give the variations in the equilibrium water vapor density and water vapor pressure with temperature.

is 16 times greater at 210°C than at 100°C (cf. Fig. 9 and Table II). One must therefore expect a larger cooling effect in the high temperature, high pressure condition. With this larger cooling effect goes also a larger decrease in the driving pressure

$$p_v(T) - P_0 \approx \rho A(T - T_b) \tag{30}$$

since the slope of the vapor pressure curve increases with temperature (cf. Fig. 10). For the specific case considered here, one has

$$\frac{A(210^\circ \text{C})}{A(100^\circ \text{C})} \approx 12. \tag{31}$$

Figure 10. These curves give the variations of the surface tension of water and of the vapor pressure derivative with temperature.

We may consider the vapor bubble growth in incipient cavitation with the same arguments. Vapor bubble growth in incipient cavitation at, for example, a water temperature of 15°C is the same phenomenon as that which we have been considering here. A difference to be remarked is that the ambient pressure is often significantly less than zero, but the fact that one is dealing with tensions associated with negative external

pressure does not alter the physical description of the bubble growth. It may be observed (Table II) that

$$\frac{\rho'(15^\circ \text{C})}{\rho'(100^\circ \text{C})} \approx 0.2, \quad (32)$$

and further that

$$\frac{A(15^\circ \text{C})}{A(100^\circ \text{C})} \approx 0.03. \quad (33)$$

The consequence of these values is clear: the thermal effects in the growth of cavitation bubbles are unimportant; only inertia effects are of significance. The Rayleigh velocity will be the growth velocity for such bubbles [5].

EFFECT OF RAPID INCREASE OF LIQUID TEMPERATURE

It has already been remarked that for the heating rates ordinarily encountered the temperature rise in the bulk liquid is unimportant when the bubble growth is followed for times of the order of 10 or 100 millise. The only parameter which is affected by the rate of temperature rise is the delay time and the values which have been given above correspond to a rise in liquid temperature $a = 0.01^\circ\text{C}/\text{sec}$. Very large rates of liquid temperature rise may be encountered in a power surge in a liquid-cooled or liquid-moderated nuclear reactor. For this reason solutions of the bubble growth equations have been carried through for the temperature increase rates given by $a = 200^\circ\text{C}/\text{sec}$ and $a = 2,000^\circ\text{C}/\text{sec}$; the former value corresponds roughly to temperature rise rates observed in some reactor safety tests [6].

The results of the calculations at 1 atmosphere pressure are shown in Figs. 11 and 12 for 3°C superheat, and in Figs. 13 and 14 for 6°C superheat. The results at approximately 19 atmospheres pressure are shown in Figs. 15 and 16 for 3°C superheat, and in Figs. 17 and 18 for 6°C superheat. the effect of increasing the rate of temperature rise, a , is easily understood. The delay time is decreased noticeably, but the value of $\dot{R}_{\text{max}}$ is essentially unchanged. The maximum growth velocity occurs so early in the bubble history that the bulk liquid temperature is almost unchanged from its initial value. As the bubble growth goes into the asymptotic phase, the radius-time curves are very nearly parallel and the asymptotic behavior is given as before by $R \sim t^{1/2}$. This variation of bubble radius with time cannot persist for too long a time for large values of a . It may be shown that the bubble radius changes from a $t^{1/2}$ variation to a $t^{3/2}$ variation at later times. The details of this analysis will be presented elsewhere.

Figure 11. Bubble growth velocities are shown as functions of time for an initial temperature of 103°C ; the external pressure is 1 atm. The rate of bulk liquid temperature rise is given by α .

Figure 12. Bubble radius as a function of time is shown for 103°C initial temperature and 1 atm external pressure.

Figure 13. Bubble growth velocities are shown as functions of time for an initial temperature of 106°C ; the external pressure is 1 atm.

Figure 14. Bubble radius as a function of time is shown for 106°C initial temperature and 1 atm pressure.

Figure 15. Bubble growth velocities are shown as functions of time for an initial temperature of 213°C ; the external pressure is approximately 19 atmos (boiling point 210°C).

Figure 16. Bubble radius is shown as a function of time for the conditions of Fig. 15.

Figure 17. Bubble growth velocities are shown as functions of time for an initial temperature of 216°C ; the external pressure is approximately 19 atmos (boiling point 210°C).

Figure 18. Bubble radius is shown as a function of time for the conditions of Fig. 17.

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TABLE I

T_0 °C	216	213	106	103
Rayleigh velocity cm/sec	13.9×10^2	9.62×10^2	4.08×10^2	2.82×10^2
$\dot{R}_{max}$ cm/sec	16.9	2.28	159	32.7
$P_0 \times 10^{-6}$ dynes/cm ²	19.1	19.1	1.01	1.01
R_0 cm	2.96×10^{-5}	6.12×10^{-5}	4.83×10^{-4}	1.02×10^{-3}

TABLE II
PHYSICAL CONSTANTS FOR WATER VAPOR AND WATER

$T^{\circ}\text{C}$	15	100	210
<i>Water Vapor</i>			
$\mu \times 10^4 \text{ gm/cm} \cdot \text{sec}$	.920	1.26	1.66
$k \times 10^{-3} \text{ erg/cm} \cdot \text{sec} \cdot ^{\circ}\text{C}$		2.4	3.2
$c_v \times 10^{-7} \text{ erg/gm} \cdot ^{\circ}\text{C}$	1.4	1.4	1.4
$\rho_{eq} \times 10^4 \text{ gm/cm}^3$	.129	5.98	95.9
$D = k/\rho_{eq} c_v \text{ cm}^2/\text{sec}$		.29	.024
Gas constant/gm of water vapor:		$B = 4.61 \times 10^6 \text{ erg/gm} \cdot ^{\circ}\text{C}$	
<i>Water</i>			
$\mu \times 10^3 \text{ gm/cm} \cdot \text{sec}$	11.4	2.84	1.30
$k \times 10^{-1} \text{ erg/cm} \cdot \text{sec} \cdot ^{\circ}\text{C}$	5.87	6.80	6.59
$c_v \times 10^{-7} \text{ erg/gm} \cdot ^{\circ}\text{C}$	4.17	3.76	3.29
$L \times 10^{-10} \text{ erg/gm}$	2.46	2.26	1.90
$\rho \text{ gm/cm}^3$	.999	.958	.853
$P_{v_{eq}} \times 10^{-6} \text{ dynes/cm}^2$	.0170	1.01	19.1
$\sigma \text{ dynes/cm}$	73.5	58.8	~36
$D \times 10^3 \text{ cm}^2/\text{sec}$	1.41	1.89	2.35
$A \times 10^{-3} = \frac{dP_v/dT}{\rho} \frac{\text{dyne cm}}{\text{gm} \cdot ^{\circ}\text{C}}$	1.10	37.7	447.

DISCUSSION

R. W. L. Gawn

I hope you won't think I have gone too far from the strictly physical aspects of Professor Plesset's excellent lecture if I talk about naval architecture and the physical cavitation on ships.

You probably know the physical effects of cavitation as far as a ship is concerned are objectionable in every sense: speed is lost, propeller blades vibrate or sing, noise is emitted, and the material of the propeller is eaten away.

Now despite many years of endeavor and all sorts of research, these defects are still manifest in different degrees depending on the class of the ship and its speed.

The propeller designer is guided in the suppression or limitation of the cavitation by model experiments, but the value of the information thus obtained depends on the extent to which the tests can be relied upon as representative of conditions on the ship.

In the past, models have been tested in axial flow water tunnels at various speeds and pressures, but there is a growing recognition of the necessity of reproducing the velocity field of the ship in magnitude and direction, and facilities are being provided accordingly, one aim being the test of an actual hull-propeller combination. And this we have been doing at Haslar.

This may prove a valuable step forward, although similitude between model and ship conditions will not be completely realized.

Generally, velocity, pressure and density are correlated in the parameter known as the cavitation number. Other physical qualities are involved, such as gas and solids in solution and in suspension, surface tension and viscosity in sea water. I am not sure we know much about the surface tension of sea water. We know a little bit about viscosity.

These may not all be sufficiently significant to call for close matching, but there is at present little information on this, and research is essential for clarification.

Notable researches have been carried out with fresh water and with pure water, but ocean water has received scant attention. Until the relevant physical properties of

the sea are known, the ship model experimenter will not be aware of the basic conditions that should be simulated in his propeller cavitation tests.

The malignant disease known as propeller erosion, first recognized about 60 years ago, is still with us despite a good deal of improvement in propellers as regards hydrodynamic design, metallurgy, and the properties of the material.

The root of the propeller is the region most frequently affected, and close attention is directed toward shaping the root sections, root fillets, and propeller boss, accordingly.

Collapse pressure of bubbles and corrosion, separately or in combination, have been adduced as causes and with different emphasis, according to the particular investigator concerned. Perhaps the author would comment on this, and on the recent claims to the effect that the life of the propeller can be prolonged by local cathodic protection.

Most of the bubbles in the boiling water around a ship's propellers are so minute as to provoke the thought that it should be possible to devise means for their elimination. An ad hoc attempt by myself to achieve this by injecting silicones into the water to reduce the surface tensions was not at all encouraging. Perhaps the author would remark on the possibility of such direct attack on the bubbles to remove them as soon as they are formed.

K. F. Herzfeld

I would like to make some comments on the nuclei which are necessary as origin for a small bubble which will then grow, either in boiling or cavitation. The first question is whether bubbles start preferentially at a smooth wall, as compared to the interior of the liquid. While there does not, to my knowledge, exist any systematic investigation of the subject, I have an impression that a smooth wall is not a favorite origin for bubbles. On the other hand, some photographs by Dr. Snay seem to contradict that; here cavitation bubbles form on the smooth lucite wall, either upon reflection of the shock wave or in the negative phase of the bubble pulse. On the other hand, a wad of glass wool is a favorite starting place [1], and it is known that some rough pieces initiate boiling.

Next, what is the nature of the nuclei in the liquid? Fox and I had proposed [2] that they are microscopic gas bubbles, covered by an organic skin, which prevents them from dissolving. This could explain why the previous application of pressure prevents the start of cavitation, since it would crush the skin. However if this were so, there should be a lower limiting pressure—that necessary to crush the skin—below which previously applied pressure would have no effect on cavitation. This disagrees with experiment. According to Strasberg [3] there is no such lower limit, but the tension necessary for cavitation to set in under influence of ultrasonic waves increases smoothly with the amount of pressure previously applied. These and other experiments seem to exclude the assumption of gas bubbles with organic cover as necessary cavitation nuclei.

This leaves as the other alternative solid particles. These must be non-wetted by the liquid and carry undissolved gas, probably in a crack, which acts as the starter of a growing bubble. This assumption has been mainly proposed by E. Meyer at the University of Goettingen and experiments carried out with iron oxide seem to confirm it.

A few theoretical remarks should be added on the origin of cavitation damage. On principle a collapsing bubble produces a very high pressure and a high temperature. If the shock wave produced in the metal were plane I doubt that it would produce pitting; a purely compressional wave does not produce such damage. On the other hand, due to the local concentration, there is probably some shear involved, which might be responsible for damage.

If the mechanism is due to high temperature, it might be threefold: direct, by softening or melting; by sudden liberation of gases previously dissolved in the metal;

by chemical reaction, which is in line with Mr. Gawn's remark, since according to Griffing [4], free radicals are produced in the hot gas of a collapsing bubble.

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F. R. Gilmore

The behavior of expanding vapor bubbles seems to be very well understood, thanks to the good work of Professor Plesset and his students. I would like only to make a few remarks on the related problem of collapsing vapor bubbles. Several years ago I developed a theory which tries to take into account the compressibility of the liquid, a factor which is important when the inward bubble wall velocity approaches or exceeds the velocity of sound in the liquid [1]. This analysis is based on an hypothesis due to Kirkwood and Bethe [2], the accuracy of which is very difficult to assess. To check the validity of this analysis, I have integrated numerically the partial differential equations for a bubble in water, collapsing under a constant pressure difference of one atmosphere, using the "method of characteristics" and an IBM 604 computer. The results are shown in Fig. 1.* The calculated bubble-wall velocity agrees within 6 per cent with that given by the analytic theory, over the calculated range of 0.2 to 4.9 times the velocity of sound in water. The inward velocity thus appears to increase like the radius to the minus one-half power as the radius becomes small, instead of the minus first power given by acoustic theory, or the minus three-halves power given by incompressible theory.

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H. S. Preiser

I would like to amplify the remarks of one of the previous speakers on the possibility of cathodic protection relieving cavitation damage.

Several experiments to this end were carried on in Italy during 1949 by Professor G. Petracchi. Small metal specimens were suspended in the turbulent throat of the venturi section of a water tunnel apparatus. Cavitation bubble collapse was produced on the specimens. A method was incorporated into the experiment whereby the specimens could be made the cathode of a simple electrolytic cell. Pit depth and weight-loss measurements were taken on uncoupled specimens exposed to similar environments. Those specimens under cathodic influence exhibited material resistance to destruction when compared to similar specimens not under cathodic protection.

There is ample opportunity to investigate the influence cathodic protection has on cavitation damage by suitable experimental techniques. In fact, the Bureau of Ships has set up such a program at the Boston Naval Shipyard. A model propeller has been designed specially to produce excessive cavitation on the blades when rotating at appropriate speeds. The cavitation pattern on the flat blades (bevelled edges) of the propeller is controlled by variation of the pitch ratio and rotational speed. The pro-

* Ed note: This figure appears on page 280, in Gilmore's discussion of the paper by Fitzpatrick and Strasberg.

propeller operates in the open sea water of the Boston Harbor from a barge which serves as a supporting platform. The propeller rig is fitted with suitable brush arrangements to permit cathodic protection to be applied to individual blades. A silver-silver chloride reference electrode is built into the plastic hub cap.

The purpose of these Boston experiments is to separate the electro-chemical aspects of cavitation damage from the purely mechanical effect. It is assumed that the total damage in a cavitating environment is the combination of corrosion and impingement attack where the total damage is greater than the sum of each individual component of the damage. It is felt that by arresting the corrosion damage by means of cathodic protection, the severity of mechanical damage can be reduced. So far due to operating deficiencies in the drive mechanism, no data are available. Progress is going forward at present and some information should be reported within the ensuing year.

Another phenomenon which occurs on a metal surface as a result of a cathodic reaction is the occlusion of hydrogen. The hydrogen atoms adhere to the surface of the metal until removed by oxygen depolarization, mechanical washing or overvoltage in which the atomic hydrogen combines to the molecular state and bubbles off as a gas. One theory worthy of investigation is whether the vapor bubble produced by cavitation boiling can combine with the hydrogen bubble when impinging on a cathodic surface to produce a cushioning effect because of its partial compressibility. ONR is sponsoring some exploratory research on the cavitation damage under cathodic protection conditions at the Hydrodynamic Laboratory of the California Institute of Technology.

One other point worthy of mention in a discussion of this nature. Corrosion reactions are comparatively slow and therefore accelerated cavitation testing techniques such as the ultrasonic device used by Professor A. T. Ellis may prevent corrosion reactions from having any influence on the total damage produced. Perhaps exposing specimens under cavitation pulses followed by short intervals of quiescence will reveal the more significant data regarding the role of corrosion in a cavitating environment.

P. Eisenberg

Mr. Preiser has referred to some suggestions that I have made concerning the nature of cathodic protection against cavitation damage. I should like here to elaborate somewhat these ideas and thus give the background for the experiments referred to as being ONR sponsored, i.e., work by Dr. A. T. Ellis at the California Institute of Technology.

As the basis for his experiments on cathodic protection, Petracchi (*La Metallurgia Italiana*, Vol. 41, No. 1, Jan.-Feb., 1949) postulated that cavitation damage is largely attributable to anodic corrosion associated with currents set up between adjacent crystals of the material, such currents resulting from deformation caused by the mechanical stresses produced by collapsing cavities. Actually, similar ideas had been suggested prior to Petracchi's paper—thermal stressing being assumed responsible for such currents. Petracchi applied cathodic protection to a number of specimens in sea water and obtained marked reductions in damage even with very small current densities. On the other hand, experiments can be cited in which very large currents were needed to show any protection. Furthermore, there is the question of what happens in non-conducting materials which are nevertheless susceptible to cavitation damage. Thus, it is necessary to determine the mechanism of cathodic protection in detail—the practical importance is evident.

The idea of protection produced by a blanket of hydrogen evolved at the cathode (in much the same way as large quantities of air introduced into cavitating turbines) suggests itself immediately. However, for such protection it is likely that extremely large current densities are required to produce sufficient quantities of gas to cushion the cavity pressures. In looking for another mechanism in which the current densities would be more commensurate with those of Petracchi's experiments, one

can postulate a highly efficient protection system if the hydrogen is introduced directly into the cavities to form a cushion. This may actually occur if hydrogen bubbles (even though microscopic in size) are introduced into the low pressure region since cavities will tend to form preferentially about such bubbles acting as the required nuclei.

Determination of the mechanism of such cathodic protection is complicated by the protection provided against electrochemical corrosion in a corrosive medium, and Petracchi's experiments are not definitive in this respect. It seems clear that mechanical damage will be reduced if attack occurs only on non-corroded material as it would be in a cathodically-protected system.

To attempt to separate the effects of cathodic protection for mechanical vs. electrochemical corrosion and to determine the validity of the mechanism suggested in the foregoing is the goal of work at the California Institute of Technology.

G. K. Batchelor

I have a question, sir. I am not sure whether it should be addressed to the old cock or the the young cock. Perhaps I can address it to the two of them collectively.

The question is this. Can they give us an authoritative statement on the importance of radiation damping during the collapse of a vapour-filled cavity? I refer to the conversion of some of the kinetic energy of the water into sound waves, which are lost from the neighborhood of the cavity.

As a layman to the literature, I have got the impression it is believed to be a small effect. As against that, some calculations by Proudman at Cambridge concerning the collapse of a cavity containing some permanent gas have shown that acoustic radiation might account for as much as 50 per cent of the total kinetic energy in one cycle, under conditions which would be typical for cavities occurring in sea water in the neighborhood of propellers.

M. S. Plesset

I should like to refer Professor Batchelor's question to Dr. Gilmore. He has given this problem some attention.

Concerning the problem of cavitation damage which has been discussed by Mr. Gawn, this is indeed a large subject as Mr. Gawn has mentioned. I appreciate his comments as well as those of Mr. Eisenberg and Mr. Preiser.

I shall now defer to Dr. Gilmore for a discussion of the question posed by Professor Batchelor.

F. R. Gilmore

I really do not know what fraction of the bubble's energy is radiated away acoustically during this period, but I suspect it is not a large fraction. There are, however, basic difficulties in defining the acoustic energy radiated during such a period, because most of the flow energy is still in the neighborhood of the bubble, where amplitudes are so large that the acoustic relations do not apply.

Whether this flow energy is ultimately radiated away acoustically or remains to re-expand the bubble may well depend upon processes occurring near the very end of collapse. For example, experimental evidence indicates that a bubble containing no permanent gases may collapse completely and not rebound at all. In such a case, one might say that at the time of complete collapse all of the bubble energy has been radiated away acoustically and/or dissipated viscously (for example, by a shock wave), unless one prefers to say that the subsequent production of a pressure pulse by the liquid which is still moving inward at the collapse time represents acoustic energy produced *after* the bubble has vanished. On the other hand, evidence indicates that presence of only a small amount of air in a bubble can cause the bubble to rebound to a considerable fraction of its original size. Moreover, only a part of this loss in energy may be due to acoustic radiation, since dissipation in the compressed gas and

in the liquid near the wall (which becomes very irregular and nonspherical during the rebound) may be important.

M. Strasberg

I believe there is a paper by Dr. Kennard in which a statement is made of the amount of acoustic energy radiated from a pulsating explosive bubble.

I don't remember what you arrived at, Dr. Kennard.

E. H. Kennard

I don't either, but if anybody writes me I shall send a copy of the paper. The statement about radiated energy was incidental.

XIII

HYDRODYNAMICS OF UNDERWATER EXPLOSIONS

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ABSTRACT

A survey is given of the various hydrodynamic aspects of underwater explosions. The topics selected deal with shockwave phenomena, the behavior of the bubble, surface phenomena and hydrodynamic problems of damage to structures.

In particular, the following items will be discussed:

a. The shockwave formation by a detonating charge. The long distance propagation of shockwaves. The anomalous surface reflection.

b. The change of shape of migrating bubbles. The energy dissipation at the bubble minimum. The energy partition of non-migrating and of migrating bubbles. The theoretical treatment of pulsating bubbles. Migration caused by rigid or free boundaries.

c. The spray dome. The gas break-through of shallow explosions. The surface phenomena of venting and non-venting underwater explosions. The plume formation. The rise of the bubble from deep explosions.

d. The reloading of an air-backed plate subjected to an underwater explosion. The response of cylindrical shells to a shockwave. The whipping of ships.

INTRODUCTION

When an explosive charge is fired under water, a sequence of complex events is started. Out of the great number of phenomena which occur, those have been selected which seem to be of greatest interest from the hydrodynamic point of view. Many of the problems which will be discussed are not completely solved today and more work is necessary until a satisfactory understanding is obtained.

SUMMARY OF THE BASIC UNDERWATER EXPLOSION PHENOMENA

The detonation of the charge converts the solid explosive material into gaseous reaction products which have an exceedingly high pressure. This pressure is transmitted to the surrounding water and propagates as a shockwave in all directions.

Fig. 1 illustrates the pressure-time history which is observed in the water at a fixed distance from the point of explosion. Upon arrival of the shockwave the pressure rises practically instantaneously to the peak value. Subsequently the pressure decreases steadily but at a very fast rate. It takes only a few milliseconds until the pressure has decreased to $1/e$ or 36.8% of its maximum value. The shockwave peak pressure and the decay constant depend on the charge weight and in the distance of the point of observation. Empirical equations for these as well as other shockwave parameters can be found in the literature [1].

Fig. 1 shows that subsequent to the shockwave other pressure pulses occur. These pulses arise from a much slower phenomenon, namely the pulsating of the gas bubble which contains the gaseous products of the explosion. The high pressure of the gas causes an initially rapid expansion of the bubble and the inertia of the outward

Figure 1. Pressure waves and bubble phenomena of underwater explosions. The upper part shows a pressure-time plot, the lower, the position and size of the bubble for specific moments which correspond to the curve above as indicated by the vertical lines.

moving water carries it far beyond the point of pressure equilibrium. The outward motion stops only after the gas pressure has fallen substantially below the ambient pressure. Now the higher surrounding pressure reverses the motion. Again the flow overshoots the equilibrium and when the bubble reaches its minimum size the gas is recompressed to a pressure up to several hundred atmospheres. At this point we have effectively a second explosion and the whole process is repeated. The bubble oscillates in this way several times.

In Fig. 1, the position and the size of the bubble is shown for a few specific moments which correspond to the pressure-time curve as indicated above. The pressure-time history reflects the low gas pressure during the phases where the bubble is large and it shows the pressure pulses which are emitted from the bubble near its minimum.

The period of the first pulsation [1] is more than one half second for common charges. This is a long time when compared with the extremely fast processes occurring with explosions. In particular, this duration is long enough for gravity to become effective. Such a bubble has great buoyancy and, therefore, migrates upward. However, it does not float up like a balloon, but shoots up in jumps.

In Fig. 1, the dotted curve represents the position of the bubble center as a function of time. This curve shows that the rate of rise is largest when the bubble is near its minimum, but is almost zero when the bubble is large. (Note that Fig. 1 is a time plot of bubble position and size. It must not be interpreted that the bubble moves sideways. Actually, the bubble rises vertically upward.)

The interaction of the pressure waves and the pulsating bubble with the water surface, the bottom of the sea and the target structures produces a great variety of interesting phenomena and effects. Some of these will be discussed in this paper.

SHOCKWAVE PHENOMENA

Shockwave Formation

Fig. 2 illustrates the formation of the shockwave by a detonating charge. The graph shows the pressure-distance distribution at two moments t_1 and t_2 . We assume a spherical charge which is ignited at the center. A spherical detonation wave proceeds outward causing the reaction of the explosive in concentric shells one after the other. At the moment $t = t_1$ this detonation wave has reached the charge surface and the explosion is completed. The graph shows the pressure distribution inside of the gas globe for this instant, curve labeled t_1 . The maximum pressure occurs at the surface of the gas globe. The pressure-distance curve decreases rapidly—in fact, discontinuously—at this point when going toward the center. Proceeding further in this direction, the pressure levels off and becomes constant at a distance of about one-half of the radius from the center of the gas sphere. The infinitely sharp pressure spike at the surface of the gas sphere is a consequence of the spherical propagation of detonation waves (G. I. Taylor, Doering [2]).

The pressure transmission from such a high pressure gas ball to the ambient medium has been calculated by Wecken [3] as well as Holt and Berry [3]. They have studied the problem of a detonating charge in air. Fig. 2 shows an adaptation of their results to an explosion in water, curve labeled t_2 . The gas globe has expanded as indicated by the dashed circle. Ahead of the bubble the shockwave has moved into the water and a rarefaction wave runs back into the gas. There is a small second shock which propagates toward the center. At a later moment this shock will be reflected at the center and will be finally transmitted into the water. Experimentally observed shockwaves show a small but reproducible irregularity in their tails. It is believed that this caused by the "second shock."

When the maximum shockwave pressure is traced back to the gas globe, one finds that the initial shockwave pressure in water is about one-half of the detonation pressure. This means the gas expands only partially when the shockwave is formed and only a fraction of the total energy of detonation is transmitted into the shockwave; the remainder produces the expansion of the gas globe and the bubble pulsations. It turns out [4] that about 50% of the total energy is radiated in the shockwave and that the other 50% constitutes the bubble energy.

Although the problem of the shockwave formation by a detonating charge is basically understood, no numerical calculations applicable to underwater explosions have been made so far. The same holds for the subsequent propagation of the high-amplitude shockwave through the water. It might be mentioned that there are several approximate treatments (Kirkwood-Bethe, Kirkwood-Brinkley, Snay-Mathias [5]) which describe these phenomena accurately enough for many practical purposes. However,

Figure 2. Shockwave formation by a detonating spherical charge. The vertical dashed line in the pressure plot designates the position of the gas-water interface at $t = t_2$.

Figure 3. Comparison of the asymptotic pressure decay relation with experiments. The curve which represents the equation noted is adjusted to a good fit with experimental points near the 10^3 psi pressure level. Experimental data are due to ARONS, reference 7.

a rigorous treatment of the shockwave phenomena produced by a detonating charge has not been made. It would require the use of an electronic computer.*

Long Distance Propagation

Since the shockwave pressure decreases when it propagates through the water, one might assume that there will be a point where the amplitude has decreased so far that the wave has lost its high-amplitude character and can be treated as a low amplitude acoustic wave. Several workers in this field [6] have stated independently and almost at the same time that a shockwave never becomes exactly an acoustic wave. The asymptotic relation is noted on Fig. 3. A spherical acoustic wave decreases inversely proportional with distance, but a weak shockwave decreases somewhat more rapidly due to the factor $\sqrt{\log R}$ in the denominator.

If we compare this asymptotic law with experimental evidence, Fig. 3, we find

* Such a calculation has been recently reported by BRODE.

poor agreement for tests carried out with one half pound charges, but somewhat better agreement for the tests with 25 lb. and 50 lb. charges.

The reason for this disagreement may be either inaccurate experimental data or an unrealistic assumption for the derivation of this formula.

The asymptotic law was obtained from the Rankine-Hugoniot conditions for the shockfront and the equations of *inviscid* fluid motion behind the shockfront. This means that outside the discontinuity all dissipative effects are ignored and that the asymptotic equation is based on a wave profile as illustrated in Fig. 3 by the solidly drawn curve. Actually, weak shockwaves have a profile as sketched in Fig. 3 by the dotted curve: The front shows a steep but steady rise and the peak is rounded off by the effect of viscosity. The question has been raised [7, 8] as to whether the equation shown in Fig. 3 would still hold for such conditions. Arons [7] has attacked this problem by means of an approximate calculation and found a decay stronger than that given by this equation. In this calculation an acoustic treatment is used to account for the viscous effects and afterwards a correction is applied for the cumulative effects of the non-acoustic contributions. Although the results are probably qualitatively correct, it is not clear whether the accuracy of the calculation is sufficient to demonstrate a valid deviation from the asymptotic law.* On the other hand Aron's calculations agree very well with the experimental points shown in Fig. 3. This brings us to the question of the accuracy of the experimental results. To measure such low pressures with electronic equipment, very sensitive and therefore very large gages are necessary. The presence of such a large gage in the water distorts the incident wave so that a true pressure measurement is in principal not possible.

The data of the graphs in Fig. 3 are corrected for finite gage size assuming that the response-time of the gage is equal to the transit-time of the wave; hence, the hydrodynamic distortion of the pressure field is neglected. It is not known today how large this effect is. Unfinished and unconfirmed studies of Slifko at NOL have resulted in response times which are about twice the transit time. The use of such a large response time would move the experimental points in Fig. 3—in particular those for the $\frac{1}{2}$ lb. charges—upward. This would improve the agreement with the asymptotic relation.

Surface Reflection

When the shockwave emitted from an explosion (Fig. 4), hits the free water surface, the pressure in the water at this interface must be equal to that of the air. It is well known that this leads to the formation of a rarefaction wave, which—if we use the acoustic approximation—is seemingly emitted from the image of the point of the explosion. This yields the dashed pressure-time-curve for the point P. Considering the high pressures of shockwaves, it appears that the negative phase shown will not occur in actuality. Only highly purified water can sustain a substantial tension without beginning to cavitate. Sea water contains numerous cavitation nuclei and cavitates as soon as the pressure drops below a certain level usually somewhat above the vapor pressure [7]. This results in the solidly drawn pressure curve:—the tail of the wave is cut off. At the pressure scale of such a record no distinction between cavitation pressure and the pressure of the undisturbed medium can be made.

This cavitation of an elastic liquid is called bulk cavitation. Kennard [9] has given the basic laws of this phenomenon, but very little more has been done. Kennard [10] and Arons [7] have calculated the extent of the cavitated region. It is a surprisingly thin layer which extends far to the side, Fig. 4. Nobody has so far attempted to

* In a recent publication [23] LIDTHILL has derived the asymptotic formula, Fig. 3, from the "Burgers" equation which includes dissipation at all points of the wave profile. This theory will go far towards dispersing the above mentioned objections against the asymptotic equation.

Figure 4. Reflection of a shockwave from the water surface.

calculate the fluid motion within such a cavitated region or to understand when and how the closure of this cavitation occurs.

The acoustic approximation seems to describe the surface effect reasonably well. But, if we give the problem a closer look, we find again that we are dealing with high amplitude waves and that the acoustic approximation can be misleading. The first evidence is seen in Fig. 4 and it is inconspicuous enough to be overlooked. The surface signal arrives earlier than calculated by the acoustic theory, also there is not a clean cut-off,—but a gradual decrease of pressure. Obviously this faster-than-acoustic-propagation is a finite amplitude effect. This fore-running of the surface signal becomes more and more pronounced the shallower the point of observation is, Fig. 5. Finally, the surface signal arrives so early that it reaches the shockfront itself. This point lies on the boundary of the anomalous surface reflection (Penny and Keil [11]). Beyond this point the rarefaction from the surface engulfs the whole wave and changes drastically the profile of the wave. Rosenbaum and Snay [12] have treated this problem theoretically by using the approach of the pseudo-stationary fluid motion. The agreement of these calculations with experimental evidence is good. However, we do not have a precise mathematical description of this phenomenon yet.

BUBBLE PHENOMENA

The Migrating Gas Bubble

Fig. 6 shows a migrating pulsating bubble. The pictures are from a model test where the air pressure above the water surface was reduced. By this way it is possible to scale the effects of gravity and to simulate realistically the bubble migration. By adjusting the air pressure to a certain value, it is possible to scale any desired charge weight. For instance, this test corresponds to a 354 lb. TNT charge detonated in a depth of 80.8 feet.

The change of the shape of the bubble is interesting. At the moment of the maximum expansion the bubble is an almost perfect sphere. But, when the bubble con-

Figure 5. Anomalous surface reflection. The pressure-time curves refer to the points indicated by the arrows. The dashed curves show the shape of the wave from acoustic treatment.

tracts, the sphere is distorted. The straight bottom seen on the second frame is due to the fact that the bubble surface is inverted into the inside. The actual lower bubble surface is faintly visible in the interior of the bubble.

This change of shape is caused by the difference of the hydrostatic pressure between bubble top and bottom. Therefore, upon contracting, the bottom of the bubble is pushed inward faster than the top and the kidney-like shape seen in the second frame of Fig. 6 is formed. A short instant later the two interfaces impinge on each other and the bubble becomes a torus (third frame). The gas is near the periphery and there is water all the way through the bubble near the center line. As in a vortex ring there is a strong upward velocity near the centerline and this carries the bubble upward when it re-expands.

The last frame shows the moment of the second bubble maximum. The bubble is almost spherical again. The black region below the bubble is water which is darkened by the solid products of the explosion.

The inversion of the bubble is apparently the mechanism by which such a migrating bubble dissipates its translational energy or in other words it is the mechanism by which the drag of such a moving bubble is originated. The analogy to the re-entrant jet of fast moving bodies under water is apparent. This impinging of the two interfaces produces a water-hammer effect and dissipates energy. It also introduces vorticity into the sofar irrotational fluid motion.

Kolodner and Keller [13] and, earlier, Penny and Price [1] have attacked the problem of the bubble contraction in a gravitational field and carried the calculations to a point shortly before the collision of the interfaces. These calculations describe the change of shape very well. No attempt has been made so far to follow up the later phases, for instance to treat the bubble as an expanding vortex ring.

The Bubble Pulse

Fig. 7 shows the pressure-time record of a bubble pulse. This record has been obtained from a 1600 lb. TNT charge exploded at a depth of 100 ft. In this case,

Figure 6. Gas Bubble of an Underwater Explosion.

Frame 1: Bubble maximum

Frame 2: About 6msec before bubble minimum

Frame 3: About 6msec after bubble minimum

Frame 4: Second bubble maximum

(times quoted refer to the condition scaled in this model test: 354 lbs TNT in 80.8 ft. depth)

Figure 7. Shape of the bubble pulse. The record shows a later portion of the pressure-time history. The shockwave is omitted. T means the duration of the first bubble oscillation.

there is a strong bubble migration and therefore also an inversion of the bubble. The spike or pressure jump is probably caused by the water hammer effect of the colliding bubble surfaces. (The oscillations which suddenly appear shortly after $t = 0.5T$ are caused by the reflection of the shockwave from the bottom of the sea.)

If we explode a small charge in a very great depth (say 1 lb. in 500 ft.) the period of bubble oscillation is so small that the gravity or buoyancy cannot produce any significant bubble migration. The shape of the bubble pulse from such a non-migrating bubble is indicated by the dotted line in Fig. 7. It is smooth, without any irregularities and shows a somewhat higher pressure. Apparently no inversion of the bubble takes place here and the bubble contracts to a smaller volume. Underwater photographs have shown indeed that a non-migrating bubble does not change its shape but remains spherical over many pulsations until it finally dissolves into many little gas bubbles.

Figure 8. Energy partition of an underwater explosion. The upper portion refers to a non-migrating bubble. The energy distribution of the 2nd cycle is that found after the first bubble minimum. This distribution is affected by the bubble migration as shown in the graph. ΔZ is the migration between the first and second bubble maximum, Z , the total hydrostatic head at the depth of explosion.

Energy Partition

An analysis of the oscillations of a non-migrating bubble by Arons and Yennie [4] has shown that despite the fact that the bubble remains spherical a strong energy dissipation takes place near the bubble minima. For instance, at the first bubble minimum 25 per cent of the original bubble energy is acoustically radiated by the bubble pulse. Another 37.5 per cent is dissipated by a mechanism which is not clearly understood at the present time. This leaves only 37.5 per cent of the original bubble energy for the next pulsation. This is illustrated in Fig. 8.

This energy dissipation of a non-migrating bubble is somewhat mysterious. It is not caused by a collision of the bubble surfaces as we have seen it for the migrating bubble. It may have something to do with the stability of the gas-water interface. According to G. I. Taylor's [16] criterion, this interface is highly unstable near the bubble minimum, but is stable for the major portion of the oscillation. This instability may produce a spray of water which is projected into the compressed gas and reduces the internal energy of the gas by cooling.

For migrating bubbles [14] the energy radiated by the bubble pulse decreases with increasing migration Fig. 8. The amount of the dissipated energy (called "unaccounted for" in the figure) remains about the same, although the mechanism probably changes. Therefore, the bubble energy (of the second cycle) increases with increasing bubble migration as shown in Fig. 8.

Theoretical Treatment for Non-Migrating Bubbles

There is a great number of publications on the theory of the oscillating gas bubble [1, 15]. Most of these assume that the bubble pulsates in an incompressible medium. Since the bubble pressure is low for a great portion of the oscillation, such approximate theories describe the movements of the bubble very well, except for short moments near the bubble minimum. Of course, these theories cannot describe the early bubble expansion shortly after the detonation and the energy losses at the minima. An acoustic approximation or improved version of it can account for the energy radiation by the bubble pulse, but not for the additional energy dissipation at the minimum. Therefore, all these calculations are applicable only to one cycle of the oscillation. For each of these cycles, the parameters of the theory must be redetermined on the basis of the empirical evidence on the energy redistribution at the bubble minimum. It is possible to adjust the parameters of the acoustic theory in such a way that the observed radius-time curve is reproduced over several cycles. Such calculations would yield too large amplitudes for the bubble pulse and, therefore, are in poor agreement with pressure measurements.

Migration Caused by Bounding Surfaces

Besides gravity other factors may produce a migration of the bubble. For instance, a rigid surface attracts the pulsating bubble and a free surface such as the water surface repels it. This phenomenon is analogous to the classical Bjerknes spheres. Bjerknes demonstrated long ago through the use of air-filled rubber bags in water that bubbles attract each other when their oscillation is in phase and that they repel each other when pulsating out of phase. A bubble that oscillates near a rigid surface can be represented by the bubble and its image pulsating in phase. Therefore, the bubble and its image attract each other. At a free surface, the image of a sink is a source and vice versa. Hence, the image of the bubble pulsates out of phase and the bubble is repelled from the free water surface.

For an explosion in shallow water these two effects oppose the upward migra-

Figure 9. Bubble Behavior of an Explosion on a Rigid Surface. The charge was fired on a steel plate.
The water surface is visible near the top of each frame.

tion due to gravity. Fig. 9 shows an explosion on the bottom. The attractive action of the bottom is clearly visible. The bubble seems to cling to the bottom, but the effect of gravity is stronger and finally tears the bubble away. The repulsive effect of the surface is not noticeable here, because of the strong upward momentum of the bubble.

In the test shown in Fig. 10 a rigid cylinder has been mounted under water. Frame 1 shows the bubble maximum. In frames 2 and 3 the attraction force of the cylinder becomes apparent. Frame 4 shows that the bubble has not migrated straight upward but toward the solid body in the water.

SURFACE PHENOMENA

The Spray Dome

When the shockwave strikes the water surface, the air-water interface becomes highly unstable. It breaks up into many spikes or needles which quickly disintegrate into droplets. According to the mathematical treatment of G. I. Taylor [16] the original irregularities grow exponentially with time. This is nicely seen in Fig. 11. (Test made by Young and Goertner, NOL.) The jets spring up predominantly from the crests of the ripples which were there before the explosion. This phenomenon is of particular interest, since one can determine the peak pressure of the shockwave photographically by measuring the initial velocity of the spray dome. However, the spray which is emitted from the peak of an irregularity moves faster than the average and this may

Figure 10. Bubble Pulsation Near a Rigid Cylinder.

Figure 11. Formation of the Spray Dome from a Rippled Water Surface. The jets emerge from the crests of the ripples.

Figure 12. Shallow Explosion with Gas Blow-Out. Charge: 4200 lbs TNT. Depth: 2.1 charge radii.

be one reason why this method did not always give consistent results. There is a theory by Keller and Kolodner [17] on this spray formation. A quantitative correlation of their calculations with experimental evidence has not been made.

The Water Column

Fig. 12 illustrates the development of the water column from a very shallow explosion. (Test made by Young and Willey, NOL.) The gaseous reaction products of the explosion blow out into the atmosphere and form a dark cloud of smoke and water. The column below is practically cylindrical and it is hollow inside.

Rosenbaum and Snay [18] have theoretically studied the growth of such a water column. They calculated the fluid motion which is produced by an expanding cylindrical piston. Of course, this analysis cannot explain why this water column has a cylindrical shape. This has been studied by Hudson (see his discussion at the end of the paper).

Fig. 13 shows a somewhat deeper * explosion. There is very little or no gas

* The tests discussed here were not carried out with the same charge weight. Therefore, the terms "shallow depth" or "great depth" must be understood as relative measures that account not only for the actual depth of the explosions, but also for the weight of the charge. A suitable way of doing this is to measure depth either in charge radii or in maximum bubble radii. This parameter assures geometric scaling and the qualitative appearance of the surface phenomena depends largely on it. Some quantitative effects, in particular the height of the column, depend on a further parameter which is essentially the absolute weight of the charge. This also holds for phenomena induced by the gravity migration of the bubble, for instance the formation of the plumes.

Figure 13. Water Column of an Explosion with Venting Bubble. Charge: 2.75 lbs TNT and 3 grams KC10₁/A1 50/50. Depth: 0.4 maximum bubble radii.

blow-out in this case. The small scale test shown on the right hand side concerns the same conditions as shown by the larger explosion on the left. The water surface is at the little triangle. The black lines are steel bars which support the windows of the tank. In this model test an explosive is used which yields highly luminous products. This makes it possible to study the motion of the gas. No blow-out is visible here. On the contrary, there are indications that air is sucked into the bubble, because, after reaching its maximum, the bubble does not contract again. Hence the pressure inside the bubble must have been in equilibrium with the ambient pressure. This venting of the bubble occurs through the column and probably causes the contraction near the base.

The Case of the Non-Venting Bubble

If the depth of explosion is increased further, the appearance of the water column is radically changed, Fig. 14. A thin water jet shoots upward which is surrounded at the base by a lower and broader spray formation. The bubble does not vent in this case. The first frame of the model test which scales the same test geometry shows the bubble shortly after the maximum in the stage of contraction. There is clear water above the bubble which proves that there was no communication between the bubble and the atmosphere.

The model test in Fig. 14 scales a small charge weight (1 gram). Consequently, the effect of gravity is small and the above mentioned repulsion of the pulsating bubble from the water surface becomes dominant: Frames 2 and 3 show that the bubble migrates downward.

Figure 14. Surface Phenomena for the Case of a Non-Venting Bubble. Charge: 2.75 lbs TNT and 1 gram KC10₁/A1 50/50. Depth: 0.9 maximum bubble radii.

The Plumes

Placing the charge still deeper, the so-called plumes occur, Fig. 15. The first frame shows the spray dome which is caused by the shockwave. Somewhat later (second frame), a jet suddenly appears above the spray dome and bursts upward with great violence. Such a jet is commonly called a "plume". If a distance-time plot is made of this plume and if it is extrapolated back to zero distance from the water surface, it is found that the plume started to move at the moment of the bubble pulse. Thus, it appears that this plume is formed by the impact of the bubble pulse. Later (fourth frame) more plumes spring up when the gas bubble breaks through the water surface. Very little is known about the mechanism which leads to the formation of the plumes. It seems, that a strongly accelerated water surface breaks up either into a spray if the surface was originally smooth, or into plumes if the surface was already grossly disturbed, for instance by the prior impact of the shockwave. Small charges (below say ten pounds) do not produce such conspicuous plumes as are shown in Fig. 15.

Fig. 16 shows the water column of the nuclear underwater explosion at Bikini. Here, also, plumes are formed, but in an entirely different situation from that shown in Fig. 15. They emerge horizontally from the water column and later fall downward.

Deep Explosion

In Fig. 17, we see an explosion of a small charge in great depth. The spray dome is minute and disappears quickly. After a while the gaseous reaction products appear in form of a compact ball on the water surface. In the second frame, the

Figure 15. Plume Formation. Charge: 275 lbs TNT. Depth: 1.8 maximum bubble radii.

Figure 16. Plumes on the Bikini Column.

Figure 17. Rise of the Bubble from a Deep Explosion. Charge: 1.1 lbs TNT. Depth: 4-8 maximum bubble radii.

bubble can already be seen through the water. At this time the bubble has virtually ceased to oscillate and had probably the shape of a mushroom head. It is a relatively rare incident that the bubble stays together in this manner. Usually, it dissolves into many tiny bubbles and a sort of soda water appears on the surface. We have here again an interesting hydrodynamic problem, namely the motion and the stability of a large gas-volume under water. (See contribution of Lane and Green in reference [23].)

SOME HYDRODYNAMIC PROBLEMS OF THE DAMAGE TO STRUCTURES

We proceed now to the problem of damage done by underwater explosions to idealized structures. We will not discuss military applications of underwater explosions against actual ships or submarines, but will restrict ourselves to hydrodynamic aspects.

The Reloading of an Air-Backed Plate

It was mentioned that about 50 per cent of the total chemical energy of the charge is found in the energy of the bubble. Therefore, the question arises as to whether this large energy term contributes to the damage, for instance to the deflection or rupture of a plate. Schauer [19] has treated this question and he pointed out that, according to experiments, the initial deflection of an air backed plate takes place in two steps, as illustrated in Fig. 18. The second blow is neither caused by the bubble pulse which comes much later and produces a further deflection, nor by the shockwave, because at this time the shockwave has long disappeared. Also, the total deformation energy of the plate is larger than the energy flux of the shockwave impinging on the plate. It turns out that this second blow is caused by the expanding bubble by means of the following mechanism:

Figure 18. Reloading of an airbacked plate.

The impact of the shockwave accelerates the plate to a high velocity. This takes place within a very short time, i.e., near the origin of the plot in Fig. 18. The pressure in the water subsequently decreases rapidly with time and cavitation sets in near the plate. At this moment, the plate has a higher velocity than the water particles which follow;—each cavitated particle being slower than that ahead of it.

When the plate is retarded by the elastic forces of the material, the cavitated water particles catch up. They transmit their momentum to the plate and form a layer of uncavitated water which moves with the plate. This so called spray reloading occurs during the first phase of the deflection. The energy of this reloading stems from the shockwave.

When the plate has lost its velocity (first maximum of the deflection curve in Fig. 18) that portion of water which moved with the plate has also come to rest. But the water farther away from the plate is still moving and is pushed ahead by the expanding bubble. These two masses of water are separated by gaps, namely the cavitation bubbles. These are quickly closed and the moving mass impinges on the resting one. This produces a closure shock and the second reloading. Its energy stems from the bubble energy. By this means the bubble may transmit up to twice as much energy to the plate as the shockwave.

This type of reloading occurs only with thin plates. It is not expected to occur with more rigid structures, such as a submarine hull, because no or only weak cavitation [10, 20] will occur there, and cavitation is necessary to liberate the bubble energy for the reloading.

The Response of Cylindrical Shells to a Shockwave

The response of an elastic cylindrical shell to a pressure wave has been treated by Mindlin, Bleich [21] and others, Fig. 19. Here the acoustic approximation and the neglect of bubble effects is quite realistic. Hydrodynamically seen this is a straight

Figure 19. The response of a cylindrical shell to a pressure wave. The arrows indicate velocities. On the surface of the shell, the velocity of the plating must be equal to that of the water particles. The dashed curve indicates the wave reflected from the shell. The translation of the cylinder originates another pressure wave which is not shown in this sketch.

Figure 20. The whipping of a ship.

forward acoustic problem. The boundary conditions for the wave equation must be satisfied at the surface of the shell taking into account the translation, dilatation and the higher order flexural vibrations of the shell. Although the hydrodynamic concept seems simple, the analysis as well as the questions regarding the behavior of the material and the failure are not simple by any means and offer many challenging problems.

The Whipping of Ships

When an underwater explosion takes place under the keel of a ship, violent and dangerous transverse vibrations of the ship as a whole are excited, as is indicated by the bent line in Fig. 20. The girder strength of many ships is not great enough to withstand such a loading and often ships have been broken in two by such an explosive attack.

Chertock [22] has published an analysis of this phenomena. He assumed incompressible fluid motion, namely that the velocity potential has to satisfy the Laplace-Equation and the boundary conditions on the surface of the gas bubble and on the elastic beam which represents the ship.

The assumption of incompressible fluid motion seems at first glance very crude, since the shockwave and the bubble pulses which are compression waves are certainly important factors in producing these transverse vibrations. But such a variable velocity potential produces pressures in an incompressible fluid similar to those of the shockwave and bubble pulses. Of course the pressure-distance relation is somewhat different; one difference being that there is no energy dissipation in an incompressible fluid such as occurs at the shockfront of high amplitude waves. However, all that must be done to make this analysis quite realistic, is to adjust the velocity potential in such a way that it produces the same pressures near the target which the shockwave and the bubble pulses would have produced at this place.

Then the only discrepancy is the time lag which a compression wave has due to its finite propagation velocity, whereas in an incompressible medium the "wave" arrives at all points at the same time. Therefore, it becomes plausible that this seemingly crude approximation is not bad, as long as the period of the oscillations of the beam is large in comparison with the transit time of the wave. In most practical cases this condition is satisfied.

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DISCUSSION

G. Chertock

I think that Dr. Snay has given us a very interesting survey of the hydrodynamics of underwater explosions. I am particularly impressed by how well he managed to avoid all the confidential and classified material. That is quite a difficult job.

I noticed a couple of points that were new to me, but were very interesting. One was the reason why the low shock wave pressures do not match the same curves as the high shock wave pressures.

This is something that we have known for some time, but I was interested in his suggestion that this may be an instrument effect, and I think we will look into it ourselves.

I was also interested in the photograph that we saw of the bubble welling up and not forming plumes—it just seemed to come up as a lazy ball, with water spilling around it. As Dr. Snay remarked, this is probably due to a bubble rising without pulsating. This is important to our work, because the pressures that are due to such a bubble are largely due to the pulsations. A nonpulsating bubble would have a very different pressure field, and a very different damage potential.

At the Model Basin most of our work has been in the interaction of the pressure wave with structures. The most difficult problems have not been the hydrodynamics but rather techniques for handling the structural reactions.

For example, Dr. Snay mentioned three cases of structural response. One of them is the deformation of a diaphragm. We don't know how to treat the mechanics of the diaphragm too well. The diaphragm material is expanding under some yield stress which is a function of the strain rates, and yielding is often delayed because of the delay time we know so little about. And the whole problem is nonlinear. In fact, there are so many parameters that you are often free to pick and choose, and make it fit any theory you want.

The second thing he discussed was the reaction of the infinite cylinder to a shock wave. You may have noticed the fact that the shock wave he took was a plane shock wave. In practice, of course, it almost always results from the point charge and we have a spherically divergent wave. We have difficulty treating that problem with any precision.

The third problem he mentioned is the whipping problem. There we made quite a few measurements on highly idealized models and we have had very good success in comparing the measurements with theoretical predictions, but it was models where we have knowledge of the structural characteristics of the model.

S. G. Reed

I hope it will not be an undue distortion to expand a bit on one or two points out of the wealth of interesting material presented by Dr. Snay.

Dr. Snay mentioned briefly the Bjerknes fields when talking about the attraction or repulsion of the explosion bubble near surfaces. The early discussion of Bjerknes is very suggestive . . . at least to one interested in the physics of the phenomena. I would like to give an example.

There is an analogy to electrostatics which has been used by Dr. Kennard and others in connection with the attraction or repulsion: the pulsating bubble far from surfaces can be replaced in first order by a "charge" Q proportional to the rate of change of its volume:

$$Q = \frac{1}{4\pi} \frac{dv}{dt}$$

The analogy is that the forces between pulsating volumes are proportional, to first order, to the product of their respective charges, but with sign reversed to the

electrostatic case. Thus in a free surface the induced charge is of opposite sign, so that there is repulsion; and in a rigid surface the induced charge is of the same sign, and there is attraction.

The Bjerknes' developed the analogy and brought in the notion of polarization, and this suggests a way of interpolating between free and rigid boundaries . . . within the realm of inviscid, incompressible hydrodynamics. Recently Birkhoff has used the idea of polarization and stressed many other analogies in an article in the von Mises memorial volume. The interpolation can be made in analogy to the behaviour of a homogeneous dielectric. What corresponds to the dielectric constant ϵ is the inverse of the density: ρ^{-1} . As an example, consider the model of an explosion bubble as a charge Q in water above a "bottom" taken to be a fluid of density ρ . The fluid boundary is assumed to remain plane. The square of velocity is neglected, otherwise there is no straightforward electrostatic analogy. The boundary conditions are continuity of velocity, and pressure continuity. These can be met by using images just as in the electrostatic problem of a point charge and a plane dielectric: in the water the potential is (r from bubble; r' from image)

$$\varphi_1 = Q/r + \left(\frac{\rho - 1}{\rho + 1} \right) Q/r'$$

and in the bottom fluid

$$\varphi_2 = \frac{2Q}{\rho + 1} \cdot \frac{1}{r}$$

In the appropriate limits this agrees with the well known cases. This may be useful to improve calculation of bottom effects on explosion bubbles well away from the bottom.

One can make a somewhat more realistic model of the bottom, imagining it to be a mixture of two fluids of mean density ρ , and then proceed in analogy with the microscopic physics of dielectrics.

The other remark has to do with the behaviour of a rising bubble. Calculations on Herring's theory give consistently larger rates of rise than are observed. Herring's theory constrains the bubble to remain spherical. Work by Kolodner and Keller, in which deformations of the bubble are taken into account, improves the agreement with experiment. The trend is evident from a simple consideration: that the bubble flattens owing to underpressure at its sides, once moving, and as it flattens the virtual mass increases. For a torus, for example, the virtual mass is roughly twice that of a sphere of equal volume. One would then expect the bubble to slow down appreciably compared with a sphere.

G. E. Hudson

Dr. Snay has commented on the studies of water columns arising from shallow explosions, and he showed us a picture of the Bikini test Baker water column as an illustration. Now this column was vertical, or nearly so, and I should like to point out that this is by no means always the case. In fact Dr. E. Swift and Mr. George Young of the Naval Ordnance Laboratory made a series of experiments in which the ambient atmospheric pressure was varied above a shallow layer of water in which a small charge was detonated at mid-depth. I have spent some time in analyzing the data from these experiments.

The most striking effect noticeable in high speed motion pictures of the resulting water columns is the variation in slope of the column generators as the pressure changes. The accompanying Figures I A, B, C, D show sequences of frames of the motion pictures which illustrate this. The depth d of the water layer was also varied, being 0.39 inches in Figure I A, 0.78 inches in Figure I B, 1.17 inches in Figure I C and 1.56 inches in Figure I D.

Film Number	Nom. Press (in Hg.)	Fr:					19	49	99
		0	1	2	3	4			
132	1								
157	5								
153	10								
114	15								
112	20								
110	24								
108	30								

Figure 1A. Water Depth: 0.39 inches.

Film Number	Nom. Press (in Hg.)	Fr: 0 1 2 3 4 19 49 99									
133	1
167	5
144	10
127	15
124	20
123	24
121	30

Figure 1B. Water Depth: 0.78 inches.

Figure 1C. Water Depth: 1.17 inches.

Figure 1D. Water Depth: 1.56 inches.

The generators of the water columns tend to become straight in many cases with time; their ultimate inclinations α from the vertical turn out to be a linear function of the pressure. In fact we may write this empirical relation in the form

$$\alpha = mp - \alpha_0 \quad (1)$$

where α_0 is nearly independent of depth and pressure and m is a function of depth only. For these experiments, we had

$$\alpha_0 = 20^\circ \quad (\text{average})$$

and

d (inches)	m (degrees/in. Hg)
0.39	1.51
0.78	1.24
1.17	3.11
1.56	4.60

Another empirical rule which seems to hold roughly over a large part of the column motion is that the ratio of the diameters w_1 and w_2 of a column at two different heights, z_1 and z_2 , above the initial water surface is a constant in time. Hence the motion is in a kind of normal mode, at least for large enough times. For a wide variety of cases the profile motion may be expressed by

$$w(z, t) = w_0 \ln \frac{t}{t_0}, \quad (2)$$

in which w_0 is a slowly varying function of z , and in fact may often be treated as constant. Not much is known empirically concerning t_0 however. Since

$$\frac{\partial w_0}{\partial z} \approx 0, \quad (3)$$

we can infer that the angular slope of a column generator is approximately

$$\alpha = -\tan^{-1} \frac{\partial w}{\partial z} = \tan^{-1} w_0 \frac{\partial \ln t_0}{\partial z}. \quad (4)$$

In this way we may relate $w_0 \ln t_0$ to z , $m(d)$, and p .

XIV

ON THE STABILITY OF THE LAMINAR BOUNDARY LAYER

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I. INTRODUCTION

This paper was originally intended to contain only a discussion of a new treatment of the central mathematical problem arising in the theory of stability of parallel flows. However, at the suggestion of the organizers of the Symposium, there is also included a brief survey of other aspects of the stability of the boundary layer and its transition to turbulence.

As is well known, when the speed of fluid motion is sufficiently high, it in general becomes fluctuating and turbulent even when the external conditions are steady. This is to be expected if one regards a body of fluid as a mechanical system with an infinite number of degrees of freedom. It is only when frictional forces are large, i.e., when the Reynolds number is sufficiently low that most of the possible modes of motion are damped out, resulting in the laminar flow.

The practical engineer is deeply interested in the problem of transition between laminar and turbulent flows because, among other things, turbulent flows are associated with much larger energy losses, and it is often desirable to try to avoid it (although in chemical engineering, it is sometimes desirable to promote turbulent mixing).

Unfortunately, with the present status of our knowledge, it is impossible to predict, with any great degree of certainty, the point of transition between laminar and turbulent flow in a boundary layer (say) under a given set of flow conditions. One reason for this is the large number of factors that enter into the determination of transition; e.g.,

1. turbulence and noise in the free stream,
2. roughness of the surface,
3. curvature of the surface,
4. pressure distribution along the surface,
5. temperature of the surface.

However, through theoretical and experimental investigations, it has often been found possible to define proper dimensionless parameters for describing the influence of the various factors. Still, it would be very difficult to devise any comprehensive experimental program to include all these factors, and many others, e.g., compressibility, surface suction, etc. It thus appears that a sensible approach is to study the effect of varying the individual factors when other conditions are kept unchanged, and, at the same time, to try to understand the basic mechanism of transition. In this way, we may hope to be better prepared to cope with a given situation.

In Section 5, I shall return to the discussion of the effect of some of the above factors on transition. For the moment, let us consider briefly the inherent instability of the laminar flow and the growth of the small disturbances. Such processes are certainly of primary importance if we wish to understand the basic mechanism of transition.

2. INSTABILITY OF LAMINAR FLOW

It is now known that the instability of the laminar flow with respect to small disturbances does not always play the dominant role in a given transition phenomenon.

Its effect is often obscured by that of the other factors mentioned above. However, since the amplification of small disturbances does provide a definite mechanism of transition in available experiments and since it is an inherent instability for which definite mathematical theories can be developed, it is an important phase of the study of the mechanism of transition. Indiscriminant application of the theory to practical problems should, however, be avoided.

From the theoretical point of view, the study of the stability of simple exact solutions, such as the Couette flow and the plane Poiseuille flow, has merits over the discussion of the boundary layer. (Indeed, Taylor [1] questioned Tollmien's theory [2] of the instability of the laminar boundary layer, because the growth of its thickness in the direction of the flow is neglected with only a mathematical justification.) Fortunately, the two cases of exact solutions mentioned above are indeed typical cases illustrating the two basic types of instability. As is well known, the theory developed for such flows can be—and has been—applied to study the stability of boundary layers.

The first successful investigation of the instability of laminar flow was Taylor's work on the motion between rotating cylinders [3]. He obtained experimental confirmation of his theoretical calculations without great difficulty. The cause of instability is also not hard to find. Indeed, when the inner cylinder is rotating and the outer cylinder is at rest, the centrifugal forces tend to produce a secondary flow. The effect of viscosity in this case is then to produce the unstable basic flow, and to provide some damping effect on the secondary motion. On the other hand, when the outer cylinder is rotating and the inner one is at rest, the flow is expected to be stable.

A similar situation may be expected to hold in the case of the boundary layer over a concave wall. Here, the fast moving particles are closer to the center of curvature and the flow is expected to be unstable when the viscosity coefficient is too small to provide sufficient damping. The instability of such flows was first predicted by Görtler, [4] and Liepmann's experiments [5] generally support his conclusions.

The more interesting case is that of parallel flows. Here, one is concerned with wavy disturbances. Lord Rayleigh predicted the instability of the flow when the velocity profile has a point of inflection. There has been ample evidence to support the theory. However, in the case of flow through a channel, where the curvature of the profile is all of the same sign, Heisenberg [6] also predicted instability of the flow caused by the effect of viscosity. The theory was criticized by Noether [7] on mathematical grounds. To some of the mathematical issues, we shall return in Section 4. Tollmien [2] studied the case of the boundary layer and made the theory more convincing. But general acceptance of the theory came only after the calculations of Tollmien and Schlichting [8] were largely verified by the experimental measurements of Schubauer and Skramstad [9]. Figure 1 shows the comparison between theory and experiment. The dashed curves were obtained by Schlichting following Tollmien's approach. The solid curves were obtained by Shen [10] by using a method developed by Lin [11] on the basis of Heisenberg's approach. It is seen that the latter results agree better with experiments. Substantially the same neutral curve was earlier obtained by Tollmien and by Lin.

What is the mechanism of the instability of parallel flows? When there is a point of inflection in the velocity profile, the cause may be traced to a mechanism of vorticity redistribution in a parallel flow. There is a maximum *magnitude* of vorticity at the point of inflection. If we imagine a disturbance giving an exchange of particles on the two sides of the maximum (remembering that such an exchange must be made without change of the vorticity in a *perfect* fluid), we see that a small disturbance can exist without an essential change of the vorticity distribution of the main flow: Thus, there is no "restoring force" to resist the disturbance. On the other hand, with a monotone increase or decrease of vorticity, an exchange of two fluid elements results in an excess and a defect of vorticity; and it is known that such elements will tend to migrate back to the layer where it came from. Thus, there is stability of the basic motion.

Instability of a parallel flow without a point of inflection in the velocity profile

Figure 1. From Shen (10), by permission of the Journal of Aeronautical Sciences.

must therefore be *caused* by viscosity. The viscous forces act as follows. Consider a travelling wave over a solid boundary. It can be shown that the viscous forces at the wall produce a phase shift of the oscillation which in turn produces a Reynolds stress. The Reynolds stress τ is found to be given by the formula [12]

$$\frac{\tau}{\rho v^2} = \frac{1}{3\alpha\delta_0}, \quad (1)$$

where

α = wave number

δ_0 = thickness of the Stokes boundary layer

$$= \sqrt{\frac{2\nu}{\omega}}, \quad \frac{\omega}{2\pi} = \text{frequency}$$

$\overline{v^2}$ = mean square of the component of oscillation perpendicular to the wall.

Thus, there is a Reynolds stress acting in the same direction as the shear of the mean motion, if the wave is moving in the direction of the basic flow. At large distances, this stress must be zero. The transition occurs at the critical layer where the wave speed is equal to the flow speed and the effect of viscosity there is again dominant. In a region essentially free from the effect of viscosity, the Reynolds stress remains constant [13]. This Reynolds stress converts energy from the basic flow into the small oscillations. If this energy conversion is sufficiently large to overcome the damping effect of viscosity, the motion will be unstable. Thus, viscosity plays a dual role. It is at the same time the indirect cause of instability and the suppressor of the oscillations. The relative importance of these effects decides whether the flow is stable or not. This is why disturbances of a given frequency are stable both for large and for small Reynolds numbers; because in the one case there is too much damping; and in the other case, there is insufficient excitation.

It has been found possible [12] to pursue this line of reasoning to derive, in a

simple manner, quantitative relations between the Reynolds number of the basic flow and other characteristic parameters of the oscillations. These relations could be obtained otherwise only by following very complicated mathematical analysis of the secular determinant of the characteristic value problem.

3. OSCILLATIONS OF FINITE AMPLITUDES

The unstable oscillations predicted by the instability theory are still far from turbulent flow. In fact, experimental observations and theoretical considerations suggest the following five stages of the phenomenon of transition from laminar to turbulent flow:

1. instability of laminar flow
2. amplification of small disturbances
3. amplification of finite disturbances
4. breakdown of finite disturbances
5. formation of turbulence
 - (a) formation of turbulent spots
 - (b) spread of turbulent spots
 - (c) fully developed turbulence.

One of the current issues is whether step 4 is basically a two-dimensional or three-dimensional phenomena. I shall not elaborate on this point, as Mr. Klebanoff will no doubt tell you something more about it in his discussion. I shall now turn to a discussion of the space structure of these oscillations.

The space structure of the oscillations indicates clearly that they are far from turbulent motions, since their scales are far too large. In the direction of the flow, they are of the order of the boundary layer thickness δ ; even its scale across the boundary layer is only of the order of $\delta R_\delta^{-1/2}$ close to the boundary. On the other hand, turbulent motions have a vorticity scale of the order of $\delta R_\delta^{-1/2}$ which is still smaller by a factor $R_\delta^{1/6}$. Thus, we must have a further breakdown of the motion into smaller eddies. This is in line with our general concepts of turbulent motion where energy is fed into motions of smaller and smaller scales, and is eventually dissipated into heat. Either the small scale motions are produced as harmonics of the oscillation when its amplitude becomes large, or they are made possible by distortion of the basic flow. In either case, it is necessary to study oscillations of finite amplitudes. By considering the distortion of the basic flow Meksyn and Stuart [14] have made some interesting calculations of such oscillations. More recently, Stuart [15] calculated the development of such motions in the course of time.

The study of higher harmonics, on the other hand, requires a more complete knowledge of the theory of small oscillations. It is known that a perturbation theory based on non-linear oscillations must be based on a complete set of normal modes in the linear theory. In the usual calculations, only *one* of the modes is calculated. To obtain the other modes,* an improved mathematical theory (such as that outlined in Section 4) is desirable.

I shall therefore refrain from discussing in detail the possible mechanisms leading from regular oscillations to the threshold of transition,** except to make the following two observations. First, there is a general concept*** originally suggested by Landau [17] who regards the final state of turbulent motion as reached through the successive instability of the flow system with respect to more and more modes as the basic flow is being distorted by the occurrence of the *lower* modes. Secondly, I wish to point out that a more extensive study of the linear theory, particularly of the amplitude distribution of these oscillations, does show that damped oscillations have a scale $\delta R_\delta^{-1/2}$ even

* Some calculations of higher modes were recently made by Grohne.

**Cf. Dryden [22] p. 57 for some discussions of this point.

*** Also independently set forth by J. T. Stuart.

away from the boundary. The picture is then possible that the unstable oscillations, when their amplitudes are large, might feed energy into the oscillations basically at the scale of turbulent flow. The amplitude of the latter oscillations also show a change-over from large to small scales like the intermittency phenomenon in fully developed turbulence or the phenomena of turbulent spots in transition.

The above discussions (and other experimental evidence at the Bureau of Standards) suggest the importance of a thorough study of the amplitude distribution of the oscillations, especially that of the damped modes. We therefore turn to the discussion of the improved mathematical theory of such oscillations.

4. IMPROVED MATHEMATICAL THEORY

The small two-dimensional disturbances in a parallel flow are assumed to be given by a perturbation stream function of the wavy type

$$\psi' = \phi(y)e^{i\alpha(x-ct)} \quad (2)$$

such that

$$u' = Re\left(\frac{\partial\psi'}{\partial y}\right), \quad v' = Re\left(-\frac{\partial\psi'}{\partial x}\right) \quad (3)$$

The linearized Navier-Stokes equations then lead to the Orr-Sommerfeld equation

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R [(w - c)(\phi'' - \alpha^2\phi) - w''\phi] \quad (4)$$

where $w(y)$ is the basic velocity profile, R is the Reynolds number, and all the variables involved are dimensionless. The conditions at the solid boundary are

$$\phi(0) = \phi'(0) = 0; \quad (5)$$

and those for the outer edge of the boundary layer are

$$\phi, \phi' \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (6)$$

This leads to a characteristic value problem with a secular determinant

$$F(\alpha, R, c) = 0. \quad (7)$$

Instability of the flow obtains when $c_i > 0$.

As is well-known, the solution of the problem is usually based on the fact that αR is large. However, the usual asymptotic solutions used for the calculation of the stability characteristics are valid either in the immediate neighborhood of the critical point where $w = c$, or away from the critical point. Strictly speaking, it is not justifiable to apply both types of solutions at the same time. It is due to a fortunate coincidence of circumstances that the old process can be justified from a numerical point of view in certain important special cases—including the Blasius flow and the plane Poiseuille flow. Actual difficulties arise in other important cases; for example, in the extension of the theory to supersonic boundary layers.*

We are therefore led to seek asymptotic solutions which are uniformly valid in a finite region containing the critical point. Such solutions are desired for at least two other reasons mentioned in Section 3; namely: (1) In trying to study the physical nature of such oscillations, the calculation of the amplitude distribution is essential. Such a calculation cannot be conveniently and accurately made by using the old solutions. (2) In an attempt to study oscillations of finite amplitude, the development of the non-

* The most illustrative example is the calculation made by Bloom who followed the older theory faithfully and arrived at incorrect results. Solutions adequate for a large number of cases have been found recently by Dunn and Lin (See Ref. 18, p. 72)

linear theory must be based on a well-developed linear theory, including the amplitude distribution.

While the difficulty has been generally recognized by workers in this field, Tollmien [19] was the first to attempt a solution to the problem. His solution is restricted in three ways: (a) only the case of neutral oscillations is treated; (b) only the first approximation is given and there is no obvious algorithm to obtain the higher approximations; (c) only the function and its first derivative are considered. Wasow [20] treated the general complex case including damped and amplified oscillations, but his treatment was not free from the restrictions (b) and (c). There is also an obvious difficulty in reconciling his "uniformly valid solution" with his own solution away from the singular point. This difficulty can, however, be easily removed, e.g., by using the transformation [15] used in the method to be described. I shall not go into further discussions of these issues, because Dr. Wasow is here to present a comparison of the various solutions including another one due to Langer. I wish merely to state that the solution to be presented appears to be simple, complete, and most readily applicable.

a. Older methods

Let us begin by considering the older methods. For αR large, it is known that formal asymptotic solutions of (4) may be found as follows:

$$\phi = \phi^{(0)}(y) + \frac{1}{\alpha R} \phi^{(1)}(y) + \dots, \quad (8)$$

$$\phi = e^{\pm \sqrt{\alpha R} Q(y)} \left[f^{(0)}(y) \pm \frac{1}{\sqrt{\alpha R}} f^{(1)}(y) + \dots \right], \quad (9)$$

where

$$Q(y) = \int_{y_c}^y [i(w - c)]^{\frac{1}{2}} dy, \quad f^{(0)}(y) = (w - c)^{-5/4}, \dots \quad (10)$$

and y_c is the critical point where $w(y) = c$. One of the formal asymptotic solutions of the form (8) (commonly denoted by ϕ_1) is regular, while another (commonly denoted by ϕ_2) exhibits a logarithmic behaviour at the critical point $y = y_c$. The singularity in this asymptotic solution and the solutions of the form (9) shows that they can be valid only when the immediate neighborhood of the critical point y_c is excluded. The multiple-valued nature of the singularity shows further that the region of validity must be restricted to certain sectors in the complex plane in the neighborhood of the point y_c . One has therefore to study very carefully the nature of the solution in that neighborhood. In the older theory, this is accomplished by a change of scale, i.e., by the introduction of the new variable

$$\eta = (y - y_c)/\epsilon, \quad (11)$$

followed by expansion of the solution of (4) in the form

$$\phi(y) = \chi(\eta) = \chi^{(0)}(\eta) + \epsilon \chi^{(1)}(\eta) + \dots \quad (12)$$

Such expansions are valid for finite values of η . It has been shown, [11] however, that, for large η , the asymptotic expansion of (12) can be formally identified with the solutions (8) and (9) after regrouping of terms.

Two things become clear from such an investigation. First, to obtain $\phi^{(0)}(y)$, we need *all* the terms in (12) showing clearly that a few terms from (12) will not be accurate enough for $y - y_c$ finite. Similarly, a few terms from (8) will not yield accurate results for finite values of η , i.e., for $y - y_c = O(\epsilon)$. Unfortunately, for many applica-

tions, (including the case of the boundary layer) one boundary point corresponds to finite or large $y - y_c$ while the other corresponds to finite η . Thus neither (8), (9) nor (12) can be adequately used for the solution of a characteristic value problem. This is a central mathematical difficulty in the stability theory.

The second point is that the asymptotic matching is restricted to certain sectors of the complex plane. This leads to the conclusion (later confirmed by more careful mathematical investigations) that the damped and the self-excited solutions are not complex conjugates in the limiting case of infinite Reynolds number even though the inviscid equations appear to lead to that conclusion.

We shall not go into further details of solutions of the type (12) as it will not be used in the following investigations.

b. *Uniformly valid asymptotic solutions of the Orr-Sommerfeld equation.*

To obtain uniformly valid asymptotic solutions of the Orr-Sommerfeld equation, we apply a theory developed for a class of differential equations of that type. In fact, we need only to know the structure of the solution to obtain the desired results. This will now be presented. The general theory will be presented elsewhere.

According to the general theory, asymptotic solutions of the Orr-Sommerfeld equation may be found in the form

$$\phi = K_0 u + K_1 u' + \lambda^{-2}(K_2 u'' + K_3 u'''), \quad (13)$$

In this formula, u is a solution of the equation

$$u^{iv} + \lambda^2(zu'' + \gamma u' + \beta u) = 0 \quad (14)$$

$$\gamma = \lambda^{-2}\gamma^{(1)} + \dots \quad (14a)$$

$$\beta = \beta^{(0)} + \lambda^{-2}\beta^{(1)} + \dots$$

where z is related to y by

$$z = [\tfrac{1}{2} \int_{y_c}^y [i(w - c)]^{\frac{1}{2}} dy]^{\frac{2}{3}}, \quad (15)$$

and the parameters λ^2 and $\beta^{(0)}$ are given by

$$\lambda^2 = \alpha R, \quad \beta^{(0)} = - (w_c''/w_c')(iw_c')^{-\frac{1}{2}} \quad (16)$$

In (13) we also have

$$K_i = K_i^{(0)} + \lambda^{-2}K_i^{(1)} + \dots \quad i = 1, 2, 3, 4 \quad (17)$$

where $K_i^{(j)}(z)$ are regular in z . We note that to each solution $u(z)$, there is a corresponding solution $\phi(z)$.

According to the general theory, the coefficients $K_i^{(j)}(z)$ are essentially determined except for certain arbitrary constants. Furthermore, the initial approximation $[K_1^{(0)}(z), K_2^{(0)}(z), K_3^{(0)}(z), K_4^{(0)}(z)]$ is uniquely determined up to a common numerical factor. The actual calculation of the functions $K_i^{(j)}(z)$ is, however, to be done by solving certain differential equations and is in general very complicated.

A practical method of obtaining the functions $K_i^{(j)}$ can be devised by merely using the structure of the solution (13). If we calculate the formal asymptotic expansions of ϕ , for finite values of $y - y_c$, by using the known expansions for u , we must obtain asymptotic solutions expressible in terms of the solutions (8) and (9). Since this can be done for *four* linearly independent solutions u , we have essentially *four* sets of relations (for each order in λ^{-2}) for the determinations of K_i . In the above argument, the constants of linear combinations are left undetermined. However, the *regularity* condition for the coefficients $K_i^{(j)}(z)$ is found sufficient for the determination of all the constants to the extent specified in the general theory. Further details of the

mathematical theory will be given in the complete publication. Let us now go on to make some remarks on the problem of transition.

5. SOME REMARKS ON TRANSITION

I shall not attempt to give a general survey of the transition problem but wish merely to discuss a few points about the relation between the stability theory and the transition phenomenon.

First, we know that the minimum critical Reynolds number obtained from stability calculations is usually much lower than the transition Reynolds number. Still, there are two useful applications of the stability theory. First, the parameters occurring in the instability theory usually turn out to be the right ones for correlating transition data. For example, the Görtler parameter

$$R_\theta(\theta/r)^{\frac{1}{2}} \quad (18)$$

for the boundary layer over a concave surface. Another example is the form parameter

$$H = \delta^*/\theta \quad (19)$$

which is found to be useful for correlating stability calculations of the boundary layer with suction or pressure gradient or both. A third example is the surface temperature. In all the three cases just cited, the trend of variation of the critical Reynolds number with the parameter in question is the same for stability theory and transition experiment.

Some results obtained by Klebanoff, Schubauer, and Tidstrom [21] give an even more direct correlation between theory and experiment. They found that below the minimum critical Reynolds number of the stability theory, it is practically impossible to induce transition either by roughness or by a spark. The turbulence spot produced by a spark does not spread until it reaches beyond the location defined by the minimum critical Reynolds number. However, if one would consider the situation in a pipe, it is clear that we cannot yet consider such a situation as a general rule for transition from laminar to turbulence. On the other hand, if further experiments can show its validity even only for boundary layers, the practical value of the stability theory would be greatly enhanced.

I wish now to make a few remarks about factors which influence transition but which lie outside the scope of the usual stability theory. Two such important factors are

- a. the level of turbulence in the free stream,
- b. roughness of the surface.

The influence of the level of turbulence is described by the well-known parameter

$$\frac{u'}{U} \left(\frac{x}{L} \right)^{\frac{1}{2}} \quad (20)$$

given by Taylor [1]. I wish now to turn my attention to the effect of roughness.

Dryden [22] analyzed a series of experiments (especially those of Tani and Hama) where the transitions is induced by a single two-dimensional roughness element and arrived at the correlation parameter

$$k/\delta_k^*, \quad (21)$$

where δ_k^* is the displacement thickness of the boundary layer at the roughness elements of height k . In the case of distributed roughness, Klebanoff, Schubauer and Tidstrom [21] found that such a parameter was not good for correlating experimental data (Fig. 2). They instead proposed the parameter

$$u_k k / \nu, \quad (22)$$

where u_k is the velocity of the flow at the edge of the roughness element. The data still show some scatter, but much less (Fig. 3). Now from purely dimensional considerations, it appears that *both* parameters must be used, since the location of the roughness

Figure 2. From Klebanoff, Schubauer and Tidstrom (21), by permission of the Journal of Aeronautical Sciences.

Figure 3. From Klebanoff, Schubauer and Tidstrom (21), by permission of the Journal of Aeronautical Sciences.

element and its height are *two* independent dimensional parameters [cf. Ref. 22]. From them, together with the flow variables, one can indeed construct *both* of the above parameters, which appear to be independent. Thus, if one tries to obtain a one-parameter correlation, there must be an imposed condition (or some other subtle reason) which would reduce a degree of freedom. It therefore seems desirable that experiments should be performed for a wide range of values of *both* parameters in order to clarify the role of roughness.

I wish now to make another suggestion for experiments to decide on the relative significance of two effects on transition. The heating of the solid surface increases the stability of the boundary layer in water by decreasing the viscosity in the immediate

neighborhood of the wall, making the profile fuller. However, if a roughness element is present, the roughness parameter $\rho u_k k / \mu$ would be increased indicating early transition. Such experiments would then clarify the relative importance of the two effects on transition: the velocity profile and the roughness elements. When the roughness effect is dominant, the stability theory might actually give the wrong trend for the effect of heating of the surface.

In conclusion, I wish to summarize my discussion as follows. When the inherent instability of the boundary layer is the primary cause of transition, the process can be described in the five stages outlined in Section 3. The first stages may be called the instability of the boundary layer, while the last stages represent transition proper. Thus, although existing results from the instability theory do throw much light on the transition problem, some further effort is needed to determine quantitatively the behavior of the oscillations of finite amplitudes for the purpose of understanding the transition mechanism. On the other hand, even for practical purposes, the presence of so many factors influencing the transition phenomenon makes it almost necessary to study its basic mechanism. I hope that further joint theoretical and experimental efforts on this fascinating subject will lead us to an even better understanding of the phenomenon and will offer us a better basis for its practical control.

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DISCUSSION

W. Wasow

I wish to add some purely mathematical remarks that may help to clarify the relation between Lin's new asymptotic technique and the method I employed in my paper [1] in the Annals of Mathematics, v.58, pp. 223-252 (1953).

Most work on the asymptotic behavior of the solutions of linear differential equations for large values of a parameter λ is based on the idea of comparing the given differential equation $L_g[y, \lambda] = 0$ with a related differential equation $L_r[y, \lambda] = 0$

whose asymptotic theory is already known and whose solutions are conjectured to have the same asymptotic form as those of the given equation, at least up to terms of some given order $O(\lambda^{-m})$. Since the asymptotic character of the solution of such differential equations is apt to change abruptly from one region of the complex plane to another (so-called "Stokes phenomenon"), it is often difficult to find a related equation whose solutions resemble asymptotically those of the given equation in a domain as large as the physical applications of the theory require. Only very simple types of differential equations can be *directly* analyzed asymptotically. The most common tools for this are integral representations resulting from functional transformation such as the Laplace transformation. Therefore, the asymptotic investigations along the lines described here begin frequently with the study of some *special differential equation* $L_s[y, \lambda] = 0$ for which a direct asymptotic theory can be given.

The complete procedure consists then of the following parts.

1. Choice and asymptotic analysis of a suitable special differential equation $L_s[y, \lambda] = 0$.
2. Construction of a suitable related differential equation $L_r[y, \lambda] = 0$ based on $L_s[y, \lambda] = 0$. The related equation should be such that the asymptotic form of its solutions follows readily from that of $L_s[y, \lambda] = 0$, e.g., by a transformation of variables.
3. Comparison of $L_r[y, \lambda] = 0$ and the given differential equation $L_g[y, \lambda] = 0$, i.e., proof that the solutions of the two equations differ by terms that are $O(\lambda^{-m})$, $m \geq 0$, in the considered domain of the independent variable z . This part frequently required tedious appraisals.

The given differential equation studied by Lin is of the form

$$L_g[y, \lambda] \equiv y^{(4)} + \lambda^2 [P(z, \lambda)y'' + Q(z, \lambda)y' + R(z, \lambda)y] = 0. \quad (1)$$

This is somewhat—but not essentially—different from the differential equation treated in [1]. The Sommerfeld-Orr equation is a special case of (1). In order to obtain the results needed in the hydrodynamic applications, z should be permitted to range over a domain containing one simple zero of $P(z, \infty)$. Let us assume, e.g., that

$$P(0, \infty) = 0, \quad \left(\frac{\partial P}{\partial z} \right)_{\substack{z=0 \\ \lambda=\infty}} \neq 0. \quad (2)$$

It is the inclusion of this "turning point" $z = 0$ into the z -domain that causes most of the difficulties.

The simplest differential equation of the type (1), (2) which still has all its essential features is

$$L_s[y, \lambda] \equiv y^{(4)} + \lambda^2(y'' + \alpha y' + \beta y), \quad \alpha, \beta \text{ constants}. \quad (3)$$

This is the special equation used by Lin and—with $\alpha = 0$ —by myself. Its full asymptotic theory, for $\alpha = 0$, was developed by myself in [1] and in [2], *Annals of Math.*, v. 52, pp. 350-361 (1950), by means of complex Laplace transformation.

In a sufficiently small region *not containing the turning point* asymptotic expansions for the solutions of (1), and hence of (3), have been known for a long time (cf. [3], *Annals of Math.*, v. 53, pp. 852-871 (1948)). Both Lin and myself use these expansions for the construction of a related equation. First, a preliminary transformation of the independent variable has to be performed. In order to explain the meaning of this transformation without entering into technicalities we refer to the fact, proved, e.g. in [3], that the previously mentioned abrupt change of asymptotic character takes place upon the crossing of certain curves ("transition lines"), which, in the case of the equation (3), form three rays issuing at equal angles from $z = 0$. By properly transforming the independent variable the transition lines of (1) can be made to coincide with those of (3). Let us assume that (1) is already written in terms of this new

variable. Now let $u_j(z, \lambda)$, $j = 1, 2, 3, 4$ be a suitable fundamental system of solutions of (3) of known asymptotic form. Then I determine, in [1], four functions $v_j(z, \lambda)$ of the form

$$v_j = k_j(z)u_j, \quad j = 1, \dots, 4, \quad (4)$$

where the $k_j(z)$ are chosen so that the $v_j = v_j(z, \lambda)$ have, in some region away from $z = 0$, the same asymptotically leading term as the solution of (1). The functions $k_j(z)$ may fail to be regular at $z = 0$, which makes the subsequent arguments more involved. The related equation used in [1] is then the easily constructed fourth order linear differential equation that has these v_j , $j = 1, \dots, 4$ as a fundamental system.

By contrast, Lin computes a different set of four functions, which we shall also call v_j , by setting

$$v_j = C_1(z, \lambda)u_j + C_2(z, \lambda)u_j' + C_3(z, \lambda)u_j'' + C_4(z, \lambda)u_j''', \quad j = 1, \dots, 4 \quad (5)$$

and requiring that in a region away from $z = 0$ these v_j have the same asymptotic form as a fundamental system of (1). Since the $C_j(z, \lambda)$ are permitted to depend on λ , Lin has much more freedom in his construction so that he can construct functions v_j that agree with the solutions of (1) up to an arbitrary but finite order $O(\lambda^{-m})$. His coefficients are of the form

$$C_j(z, \lambda) = \sum_{n=1}^m r_{jn}(z)\lambda^{-n}, \quad (6)$$

and it turns out that the $r_{jn}(z)$ can be chosen so as to be regular at $z = 0$. These are important improvements.

The work of part 3 of the general procedure outlined above, in which the related and the given equations have to be compared as to their asymptotic form in a domain *including* the turning point $z = 0$, has not yet been fully carried out for Lin's method. A well developed technique for this task exists, but its application to Lin's differential equations may still involve considerable labor.

T. B. Benjamin

I believe it may be appropriate here to refer to another problem of hydrodynamic stability, one which has received only little attention compared with the stability problems for the boundary layer and pipe flow, which have been in the forefront of theoretical hydrodynamics for about a half a century now.

The case I would like to mention is the flow of a viscous liquid in a thin film, bounded on one side by a plane wall, and on the other by a free surface. Flow of this type is a fairly common experience.

For instance, rain water may sometimes be observed running in a thin sheet on the outside of a window. Again, one may often observe motor oil, or paint, or mayonnaise, flowing in a thin sheet down the sides of some solid object immersed for sampling purposes.

The general mechanics, and in particular, the stability of such flows, is an important question in chemical engineering, for instance, particularly where the film is in contact with a gas stream.

Some work on this problem has been done recently in Cambridge, both in the Chemical Engineering Department and in the hydraulics laboratory, and I would like to outline what the problem was, briefly, and what progress has been made on the theoretical side at least, since this seems the aspect most appropriate to this session of the Symposium.

It is a well-established fact that when a film of water runs down a vertical plate, the flow is laminar and uniform until the Reynolds number, based on film thickness, is raised to a value of the order of unity. Waves with a well defined periodicity then

appear, their velocity downwards being about three times the mean flow velocity, and it is in fact, quite straight forward to measure all features of the transition. It should be noted that this transition is far removed from the development of turbulence, unlike that in the more familiar cases of instability of laminar flow. The film cannot be described as fully turbulent until the Reynolds number is increased to values in the hundreds, at least.

The influence of surface tension must be recognized in any theory which is to be of practical use for water or other common liquids. Surface tension is the only restoring force when the plate is vertical, although, of course, gravity exerts a restoring force when the plate is inclined.

During his talk on Monday, Professor Lighthill remarked that the existing theory of roll waves, as for instance developed by R. F. Dressler, is applicable to flow in thin films. Since this theory has been developed by Dressler for a generalized law of frictional resistance, one may use his results simply by substituting the appropriate resistance of a laminar film. In this case, of course, the resistance is directly proportional to flow speed, and inversely to the film thickness.

In fact, a lengthy treatment having much in common with this approach, was published a number of years ago by the Russian scientist Kapitza. However, a more exact treatment of this problem is possible. The problem may be formulated as a characteristic-value problem in the familiar manner of stability theory for the boundary layer. The only differences are that special boundary conditions are needed to account for the free surface, and that, in the range of interest, the Reynolds number is small. Thus one may resort to asymptotic expansions in ascending powers of Reynolds number, instead of descending powers as in the usual case.

Dr. Yih has already published a description of this method of approach, but restricted his treatment to the case of zero surface tension, and to terms of only the first power of αR . Actually the Reynolds number is about two or three in the range of greatest interest, but α is kept fairly small due to the result of surface tension. αR is considerably less than one, but couldn't be regarded as a small quantity in the usual sense.

In our work at Cambridge, we have considered expansions up to the third power of αR , which has required a power series expansion of the stream function in terms of the coordinate perpendicular to the flow as far as the 15th power. It remains to prove the convergence of this approximation, but the physical aspects are sufficiently reassuring.

The result of a good deal of heavy algebra is a series of curves of neutral stability, with R and α as coordinates in the familiar fashion of stability theory, and with Weber number, representing surface tension, and the slope of the plane as parameters.

A good agreement with experiment has been found, and we have also made some progress with the more difficult problem where the film is stressed by a turbulent gas stream confined with an additional fixed boundary.

In conclusion, I would very much like to comment the film problem for further study by workers in this field. It has a number of important practical aspects. I hope, for instance, that it will attract the great talents of Professor Lin.

P. S. Klebanoff

The comments I have to make consist of some of the recent results obtained from a continuing experimental investigation of boundary layer transition which is being conducted at the National Bureau of Standards under the sponsorship of the National Advisory Committee for Aeronautics. They bear on Dr. Lin's paper in that they deal with the region of finite amplitude with which we are all concerned. One objective of the investigation is to observe by hot-wire techniques the growth and evolution of a wave from its source to transition. It was possible to carry out this objective by using the vibrating ribbon technique used so successfully by Schubauer and Skramstad in their experimental confirmation of the Tollmien-Schlichting stability

Figure 1.

theory. The study is being conducted on a flat-plate boundary layer with zero pressure gradient and wind speed of 50 feet per second.

Figure 1 represents the intensity of the longitudinal fluctuation and shows its distribution in the z -direction, which is the direction parallel to the plate and normal to the flow. The measurements were made one foot downstream from the vibrating ribbon, at which point transition began, and at a distance from the surface equal to half the boundary layer thickness. The intensity is plotted in terms of the root-mean-square value and is referred to the intensity u_0' , which is the value at a fixed point 1.2 inches above the center line of the plate, and was on the order of 1 percent. Although the vibrating ribbon excites a wave which is 2-dimensional and uniform along its crest for a distance of about one foot there has developed a strong 3-dimensionality as shown by the large variation of intensity. This variation had its beginning well upstream from transition where the wave is still quite pure. The breaking pattern associated with transition is also shown. At the peak the breaking pattern is different from that in the hollow. The breaking pattern at the peak consists of bursts of high frequency fluctuations occurring each cycle on the low-velocity side of the primary

wave. In the hollow the fluctuations are of much smaller intensity and occur on the high velocity side of the wave. The frequency of these fluctuations is about six times that of the primary wave which had a frequency of 145 cycles per second. Frequencies this high would indicate that these high intensity fluctuations are probably not a consequence of non-linear effects but are more in the nature of a secondary instability. However, more study is called for in this connection. Preliminary measurements have indicated that these high frequency fluctuations at the peak and in the hollow occur at the same time, indicating that the smaller intensity fluctuation may be just an excitation caused by the very intense pattern at the peak, and that the initial breakdown of laminar flow has a very short extent in the z-direction which might be described loosely as point-like. These localized regions are spaced along the wave front giving the impression of streets along which a succession of bursts were occurring.

In Figure 2 there is shown the breakdown of laminar flow from the beginning of transition to where it becomes fully turbulent. It was too inconvenient to move the hot wire through the transition region; instead the oscillograph traces shown in the figure were obtained by moving the transition region over the hot-wire by continuously increasing the ribbon amplitude. The hot-wire was at a position corresponding to a peak in the intensity distribution shown in Figure 1. At first we see the rather pure wave somewhat distorted with a flat-top appearance on the high velocity side, then the appearance of high-frequency fluctuations which consume more and more of the primary wave, and then the onset of fully turbulent flow. This process, i.e. from the beginning of transition to where we have fully developed turbulent flow, takes place within a distance equivalent to 50 boundary layer thicknesses.

An attempt was made using the china clay technique to determine whether the 3-dimensionality associated with the intensity of the wave in the region where the wave was still quite pure could possibly be due to Görtler-type vortices. The drying pattern on the china-clay surface which was initially sprayed with oil of wintergreen is shown in Figure 3. The dark line at the left of the figure is the vibrating ribbon. The limitations of the method prevented any definite conclusions as to the existence of Görtler-type vortices in the region before transition. As far as could be determined the street-

Figure 2.

Figure 3.

like array manifested by the transition process as shown began only after the laminar flow was disrupted and turbulent bursts were present.

An important aspect especially in the region of finite amplitude is the amplification of the wave. Figure 4 shows preliminary measurements of the amplification of the wave from the ribbon to transition at a frequency of 160 c.p.s. for different input levels, and was measured along a line corresponding to a peak in the intensity distribution. The intensity u_0' is the level at x_0 which is a reference position two inches downstream from the ribbon. The distance from the surface at which the measurements were made was at about 0.2 the thickness of the boundary layer, which is where the maximum in the intensity distribution across the boundary layer occurs in the Tollmien-Schlichting theory. At input levels of 0.2 and 0.3 percent transition did occur past the neutral point, but here it occurred in an intermittent fashion with the wave sometimes damping and at other times increasing and causing transition. As the input level was raised still further, transition occurred steadily and at positions successively farther upstream. The data shown ends at about the beginning of the transition region. The departure from the small disturbance rate is an exponential-like growth. The departure for the four highest input levels takes place at an intensity level of about 1.5 percent. As pointed out by Dr. Lin on a visit to our laboratory the other day, a level of this magnitude would correspond to a fluctuating shear stress which is on the order of one-tenth the viscous shear and would be a reasonable assumption for the beginning of non-linear effects. However, it should be mentioned that there is no discernible effect on the mean-velocity distribution. In fact there is no significant distortion of the mean-velocity profile until the actual breakdown of laminar flow occurs. At any rate it is felt that the significance of non-linearity needs further investigation and that the effect of curvature as proposed by Görtler should not be overlooked.

Figure 4.

This briefly summarizes some of the experimental data. The 3-dimensionality that I have been talking about was eventually traced to small span-wise variations in the mean flow in the boundary layer, which when coupled with the circumstance that the frequency of the primary wave was near branch II of the neutral curve, gave rise to local variations in the amplification rate. In fact by choosing a frequency near branch I of the neutral curve the peaks and hollows could be reversed. The 3-dimensional character is of practical significance inasmuch as the frequency selected for amplification in natural transition lies near branch II and would be especially susceptible to irregularities in the mean flow. Also a random input would give rise to distorted waves. Although the results obtained to date indicate the presence of 3-dimensionality prior to transition the question as to whether it is an inherent part of the transition process has not as yet been resolved. In this connection it should be mentioned that Hama at the University of Maryland, conducted an investigation of transition on a flat plate in water, using dye and a trip wire and concluded that distortion in the wave always occurs prior to transition. The difficulty has been in getting a flow free of irregularities. Span-wise variations in mean velocity on the order of a few percent appear to be significant. The results obtained do not exclude the possibility of breakdown of a 2-dimensional wave, nor exclude the usefulness of a 2-dimensional finite amplitude theory.

C. C. Lin

I wish to thank Dr. Wasow for this able comparison between his work and our present approach. It may be well to emphasize that the connection coefficients $C_\nu(z, \lambda)$ are independent of the particular solutions u_j while $k_j(z)$ depends on the index j . The appropriateness and desirability of using the form

$$v = \sum_{\nu=0}^3 C_\nu(z, \lambda) u^{(\nu)} \quad (29)$$

become especially obvious, when the original equation is regarded as a system of four differential equations of the first order.

The third step mentioned by Dr. Wasow is essentially an existence proof. Whereas this is usually omitted in applied mathematical research—since the solution constructed carries every conviction of being a uniformly valid solution—Mr. Rabenstein and I are carrying out this proof in view of the fundamental nature of the problem.

Note added in proof (Nov. 7, 1957). The proof referred to in the above discussions has been completed in the meantime by Rabenstein and myself. A paper is in preparation to give a comprehensive description of the theory.

XV

SOME CURRENT PROBLEMS IN TURBULENT SHEAR FLOWS

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INTRODUCTION

This paper is intended to be a heuristic account of several problems in the dynamics of turbulent shear flow. More to formulate questions than to answer them, it is in no sense a survey of the current research status in the field; the admirable new monograph of Townsend [1] will fill this role for some time to come. Unfortunately, it has not been possible to incorporate here the relevant contributions of his book, but I have attempted to make some references to it.

The turbulent transfer of a passive scalar contaminant in a mean gradient has become partly clarified through the use of Lagrangian dispersion analysis (in the ideal case, just by following indelibly tagged fluid) [2] instead of the Eulerian representation more convenient for turbulence dynamics. In approaching the problem of turbulent momentum transfer, it is therefore unavoidable that we ask the naive question of whether an idealized problem analogous to the scalar case can be sensibly formulated. Can we treat kinematically the turbulent dispersion of momentum from a "point source", at least in some kind of "small perturbation" limit?

An immediate difficulty is the conceptual one of visualizing a dispersing fluid (or solid) particle indelibly tagged with a mean momentum increment. In the very act of being convected by turbulent fluid the particle partakes of the general flow dynamics. On the other hand, if this action can be described by a linear differential equation, and if it does not affect the basic turbulence, there remains some hope for the utility of this approach. If the interaction is essentially non-linear, not only will the point-source dispersion be complex, but also the entire notion of synthesizing a mean gradient by superposition of sources will be invalid.

In a flowing isotropic turbulence [mean velocity $\bar{U}$, turbulent velocity $u_i \ll \bar{U}$], the possibility of an equation linear in v_i , the perturbation velocity of a particle, depends upon the negligibility of effects quadratic in v_i .

Figure 1. Turbulent wake of bullet. [courtesy of Ballistic Research Laboratory, Aberdeen Proving Grounds]

Inequalities like

$$v_j \frac{\partial v_i}{\partial x_j} \ll u_k \frac{\partial v_i}{\partial x_k} \quad (1)$$

can be gotten in principal by setting

$$v_k' \ll u_k', \quad (2)$$

but the static pressure contribution of v_i to the Navier-Stokes equation will probably behave like

$$p_v \sim \rho \bar{U} v_j \quad (3)$$

so that the inequality (2) may be insufficient to permit neglecting static pressure effects.* Nevertheless, if we can devise a convenient quantitative representation for static pressure perturbation, there is some hope of exploiting a point source method.

The foregoing matter is carried no farther here. This paper deals instead with some traditional shear flows such as boundary layer, channel, wake and jet—especially in terms of shear stress, strain-rate and their fluctuations.

THE PROXIMITY OF BOUNDARIES IN TURBULENT SHEAR FLOWS

In theoretical analysis of turbulence, restriction to homogeneity brings considerable simplification, and such an approach is being made to shear flow by Reis and Lin [3] and by Burgers and Mitchner [4]. Homogeneity, the invariance to translation of all statistical properties of the turbulence, requires that the boundaries be at infinity, or at least far apart compared with the maximum correlation distance of the fluctuations. To evaluate the possible role of a theory of homogeneous shear flow, it is instructive to recall some pertinent experimental results.

Figure 1 is a shadowgraph of the wake of a supersonic bullet, several hundred diameters downstream. In the ballistic range the camera is fixed with respect to the undisturbed air, and at this station all velocities are far below sonic. The wake is visible because of residual temperature fluctuations in the turbulence. Of interest here

* Townsend appears to disagree: [1], p. 57.

Extent of Turbulent Correlation Across Shear Flow

Fig. 2

is the fact that even though the shadowgraph technique responds to the second derivative of the index of refraction field, thus giving heavy emphasis on the fine structure, there is still detectable structure of a size comparable with the wake diameter.

For a quantitative display, we look at the comparison between mean velocity profiles and transverse correlation functions of turbulent velocities in representative shear flows. Figures 2 and 3 show that in channel, boundary layer, wake and jet [5, 6, 7, 1, 8, 9], there is measurable correlation over distances comparable with the shear

Extent of Turbulent Correlation Across Shear Flow

Fig. 3

Fig. 4

zone width. Since the shear stress is carried principally by the large eddies, and these may be influenced by the boundaries, there exists a question of whether a turbulent shear theory postulating homogeneity can be expected to give quantitative results for momentum transfer.

Figure 4 demonstrates the momentum transfer burden carried by the large structure, and shows that the average wave number for transfer $_{uv}(k_1)_{ave}$, corresponds to a length greater than the displacement thickness of the boundary layer. The result would look even more emphatic in terms of wavelength $\frac{2\pi}{_{uv}(k_1)_{ave}}$.

It seems likely, however, that homogeneous shear flow theories could give semi-quantitative explanation of the momentum transfer, and should certainly be quantitatively applicable to the large wave number region, including that in which the shear correlation coefficient spectrum $_n R_{uv}$ is small but not zero.

MEAN STRESS AND STRAIN-RATE

The early theories of turbulent shear flow focussed attention upon the mean momentum equations (the Reynolds equations), seeking to render them determinate by various postulates expressing the turbulent shear stress $(-\rho uv)$ in terms of the local

mean strain rate $\left(\sim \frac{\partial \bar{U}}{\partial y} \right)$ or its derivatives. Reserving criticism of the local feature

for later, it is instructive to look at the magnitudes and directions of the mean principal stresses and strain-rates in some turbulent flows.

Fig. 5

Figures 5, 6 and 7 give this information for plane channel, boundary layer and wake:

$$\begin{aligned}
 E_{1,2} &\approx \pm \frac{1}{2} \frac{\partial U}{\partial y}, \\
 -\frac{1}{\rho} \sigma_{1,2} &= \frac{\overline{u^2} + \overline{v^2}}{2} \pm \sqrt{\left(\frac{\overline{u^2} - \overline{v^2}}{2}\right)^2 + (\overline{uw})^2}, \\
 \alpha_{E_1} &\approx \frac{1}{2} \tan^{-1} \left\{ \frac{\frac{1}{2} \frac{\partial \overline{U}}{\partial y}}{\frac{\partial \overline{U}}{\partial x}} \right\} \approx 45^\circ, \\
 \alpha_{\sigma_1} &= \frac{1}{2} \tan^{-1} \left\{ \frac{2\overline{uw}}{\overline{u^2} - \overline{v^2}} \right\}.
 \end{aligned} \tag{4}$$

It seems noteworthy that the magnitudes of principal stress and strain-rate tend to vary together, loosely speaking. However, the directions of the corresponding principal axes are quite different in the two bounded flows, and this should recall the well known fact that assumptions of simple gradient transport of momentum in these flows have been rather unsuccessful.

In contrast, the wake shows a wide zone of rough coincidence of principal directions, a result consistent with the fact that its velocity profile is close to that obtained with the assumption of simple gradient transport with constant exchange coefficient. Some consequences of this directional coincidence are explored by Townsend ([1], pp. 117 et seq.).

LOCAL ISOTROPY

Recent measurements on the anisotropy produced by irrotational, homogeneous strain-rate on the (roughly) isotropic turbulence downstream of grids [10, 11] (Figure 8) suggest re-examination of Kolmogoroff's postulate of local isotropy in shear flow [12]. The ubiquitous strain-rate of shear flow acts upon "eddies" of all sizes, tending to make the turbulence anisotropic at all wave numbers.

In shear flow the situation is complicated by the fact that energy is received by the turbulence from the mean flow in a highly anisotropic condition, as can be seen from component energy equations. In plane turbulent channel flow, for example,

$$\rho \overline{uw} \frac{d\overline{U}}{dy} + \frac{\rho}{2} \frac{d}{dy} (\overline{vu^2}) + u \frac{\partial \overline{p}}{\partial x} + (\text{viscous terms}) = 0, \tag{5}$$

$$\frac{\rho}{2} \frac{d}{dy} (\overline{v^3}) + v \frac{\partial \overline{p}}{\partial y} + (\text{viscous terms}) = 0, \tag{6}$$

$$\frac{\rho}{2} \frac{d}{dy} (\overline{vw^2}) + w \frac{\partial \overline{p}}{\partial z} + (\text{viscous terms}) = 0, \tag{7}$$

which shows that *all* turbulent energy appears first as $\overline{u^2}$. This is very nearly true for all flows in which the boundary layer approximation is valid.

Fig. 6

Fig. 7

1 Effect of contraction on homogeneous turbulence

2 Subsequent approach to isotropy

Figure 8. Curves from Uberoi (11), by permission of the Journal of Aeronautical Sciences.

There must also be a general inertial tendency toward isotropy through the pressure-velocity correlations at all wave numbers.

A necessary condition for local isotropy is that there exist a wave number beyond which the spectral times characterizing inertial and viscous mechanisms favoring isotropy [$\vartheta_i(k)$ and $\vartheta_v(k)$] are much smaller than the time characterizing the gross straining process, $\frac{1}{E_1}$:

$$\vartheta_i(k) \quad \text{and} \quad \vartheta_v(k) \ll \frac{2}{\left| \frac{\partial U}{\partial y} \right|}. \quad (8)$$

Of course, in a purely inertial range only the ϑ_i inequality is necessary, while in the purely viscous range (if a thing exists in stationary turbulence), only the ϑ_v inequality is necessary.

With $E(k)$ as turbulent energy spectrum, we may follow Onsager's definition [13] of inertial time,

$$\vartheta_i(k) \equiv \frac{1}{\sqrt{k^3 \epsilon(k)}}. \quad (9)$$

Similar reasoning leads to

$$\vartheta_v(k) \equiv \frac{1}{\nu k^2}. \quad (10)$$

Figure 9

As a basis for rough numerical estimates, it is convenient to replace $E(k)$ by the solution to Heisenberg's transfer equation [13]. Then, in terms of the Kolmogoroff wave number, which locates the viscous region of the spectrum,

$$k_c \equiv \left(\frac{\phi}{\nu^3} \right)^{\frac{1}{4}}, \quad (11)$$

(where ϕ is viscous dissipation rate and ν is kinematic viscosity), and

$$\vartheta_i \left(\frac{k}{k_c} \right) \approx \left(\frac{\nu}{\phi} \right)^{\frac{1}{2}} \left[1 + \frac{27}{2} \left(\frac{k}{k_c} \right)^4 \right]^{\frac{2}{3}} \left(\frac{k}{k_c} \right)^{-\frac{2}{3}}. \quad (12)$$

In this estimate, Heisenberg's empirical constant has been chosen as $\frac{4}{9}$.

Both (10) and

$$\vartheta_v \left(\frac{k}{k_c} \right) = \left(\frac{\nu}{\phi} \right)^{\frac{1}{2}} \left(\frac{k}{k_c} \right)^{-2} \quad (13)$$

are plotted in Figure 9 [14].

The minimum value of ϑ_i turns out to be

$$\vartheta_i(0.40) = 1.8 \left(\frac{\nu}{\phi} \right)^{\frac{1}{2}}, \quad (14)$$

so that one necessary condition for local isotropy is

$$\left(\frac{\phi}{\nu} \right)^{\frac{1}{2}} \gg \frac{\partial \bar{U}}{\partial y}. \quad (15)$$

Under still more simplifying assumptions (14), this can be translated into a Reynolds number inequality.

Both inertial and viscous forces may be appreciable in the range $0.2 < \frac{k}{k_c} < 5$.

Hence, rather severe conditions for local isotropy may exist since

$$\vartheta_v(0.2) \approx 30 \left(\frac{\nu}{\phi} \right)^{\frac{1}{2}}, \quad \text{and} \quad \vartheta_i(5.0) \approx 100 \left(\frac{\nu}{\phi} \right)^{\frac{1}{2}}. \quad (16)$$

Turning to actual numbers, the homogeneous strain rate of Figure 8 has a maximum of 11 per second, and Uberoi's highest [11] (a 16-1 contraction) reached 63 per second. Townsend's homogeneous plane strain-rate was 9.4 per second. These numbers may be compared with principal strain rates varying from zero to 10^4 per second in Laufer's plane channel [5] at a Reynolds number of 61,000. A general order of

magnitude in this example is $\frac{1}{2} \frac{\bar{U}_{\max}}{d} \approx 120$ per sec. Klebanoff's boundary layer [7]

has strain-rates of the same order, while Townsend's wake strain-rates are appreciably smaller [15], e.g. roughly 2.0 per second at 800 diameters behind a 0.159-inch cylinder at Reynolds number 1360. Except near solid boundaries, these shear flow strain-rates are comparable with the homogeneous ones imposed by Townsend and Uberoi, so it is possible that the data of the latter may be employed semi-quantitatively to explain some features of shear turbulence. Townsend [1] has apparently done considerable work along this line.

In a discussion of local isotropy in shear flow, it is pertinent to note the strong departures from local isotropy in the homogeneous strain experiments, at least out as far as the spectral region contributing to mean square first derivatives. In Figure 10, for example, we see Uberoi's evidence at a strain-rate of about 30 per second. From

the decay rate just before the contraction, $\left(\frac{\phi}{\nu} \right)^{\frac{1}{2}}$ may be estimated as 16 per second, evidently not an order of magnitude larger than 30. This is consistent with the lack of local isotropy.

Taking Klebanoff's boundary layer data [7] as typical and considering a point $\frac{y}{\delta} = 0.2$, in the fully turbulent region, we may check inequality (15) by estimating

$$\frac{\left(\frac{\phi}{\nu} \right)^{\frac{1}{2}}}{\frac{\partial \bar{U}}{\partial y}} \approx 10. \quad (17)$$

This implies that local isotropy is more likely here than in the homogeneous cases reported.

Figure 10. From Uberoi (11), by permission of the Journal of Aeronautical Sciences.

Superficially distinct from the characteristic time criteria is the obvious requirement that the locally isotropic spectral region be at wave numbers much larger than those at which energy is fed into the turbulence. With a mixing-length type of behavior qualitatively in mind, we may estimate the latter as

$$k_p \approx \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}, \quad (18)$$

so that we cannot expect local isotropy at wave numbers which do not satisfy the inequality

$$k \gg \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}. \quad (19)$$

Hence, for dissipative local isotropy, a necessary condition is that

$$k_c = \left(\frac{\phi}{\nu^3} \right)^{\frac{1}{2}} \gg \frac{1}{v'} \frac{\partial \bar{U}}{\partial y}. \quad (20)$$

At the same boundary layer position,

$$\frac{\left(\frac{\phi}{\nu^3} \right)^{\frac{1}{2}} \cdot v'}{\frac{\partial \bar{U}}{\partial y}} \approx 45, \quad (21)$$

which is enough greater than unity to be consistent with locally isotropic dissipation.

Klebanoff reports roughly isotropic dissipation for $.05 < \frac{y}{\delta} < 1.0$, which includes this position.

SHEAR SPECTRUM

Since elementary dimensional reasoning has had a bit of success in predicting the character of local regions in the energy spectrum, it seems natural to attempt the simplest kind of dimensional argument on the shear spectrum, $\tau_1(k_1)$ defined by

$$-\overline{uv} = \int_0^\infty \tau_1(k_1) dk_1. \quad (22)$$

We are interested in predicting from gross turbulence data the wave number above which the coefficient of shear correlation ${}_k R_{uv}$ is much less than unity. If $F_1(k_1)$ and $F_2(k_1)$ are the one-dimensional spectra of $\overline{u^2}$ and $\overline{v^2}$,

$${}_k R_{uv}(k_1) \equiv \frac{\tau_1}{\sqrt{F_1 \cdot F_2}}, \quad (23)$$

and this has been measured in several flows [5, 7, 16, 17].

For large Reynolds number we naively assume that at the "small eddy" end of the shear spectrum, $\tau_1(k_1)$ depends primarily upon the local mean velocity gradient $\frac{\partial \overline{U}}{\partial y}$. Then, dimensionally,

$$\tau_1(k_1) \sim \left(\frac{\partial \overline{U}}{\partial y} \right)^2 k_1^{-3}, \quad (24)$$

SPECTRUM OF SHEAR CORRELATION COEFFICIENT
[straight line slope = $-4/3$]

Fig. 11

which could be compared with experiment. However, the actual comparison here has been made with $k_1 R_{uv}$ under the additional assumption that $F_1 \sim F_2 \sim k_1^{-5/3}$, since the high wave number end borders on the region of local isotropy. Then

$$k_1 R_{uv} \sim k_1^{-4/3} \quad (25)$$

which does not look particularly successful in Figure 11. The abscissa there is frequency rather than wave number.

By a considerably more sophisticated approach, Tchen [18] arrived essentially at

$$\tau_1 \sim k_1^{-7/3} \quad (26)$$

for "small $\frac{\partial \bar{U}}{\partial y}$ ".

THE GRADIENT TRANSPORT POSTULATE

It is a familiar fact that the validity of any simple gradient transport description requires the transfer mechanism to have a characteristic length much smaller than the gross dimensions of the field. With a random "free path" type of mechanism, for example, the transfer rate Q of some property $\psi(x)$ may often be expanded in a power series in the "mean free path" l :

$$Q(x_1) \sim \frac{l}{2} \left(\frac{d\psi}{dx} \right)_{x_1} + \frac{\left(\frac{l}{2} \right)^3}{3!} \left(\frac{d^3\psi}{dx^3} \right)_{x_1} + \dots \quad (27)^*$$

Provided ψ' and ψ''' are properly behaved, (27) reduces to simple gradient transport when

$$l \ll \sqrt{6 \frac{\psi'}{\psi'''}} \quad (28)$$

a characteristic length of the gross field.

In Figures 1, 2, 3, 4 we were reminded of the well-known fact that shear turbulence has structure comparable with the shear zone width. Furthermore, it turns out that if we apply (27) to heat diffusion measurements in isotropic turbulence, where the Lagrangian integral scale serves in place of a free path, the second term is about 30% of the first.

Of course, (27) indicates only constant coefficients, and it seems reasonable to see whether

$$Q(x) \approx A\psi'(x) + B\psi'''(x) \quad (29)$$

can be represented as gradient transport with coefficient dependent on x^{**} , i.e. we define $D(x)$ such that

$$Q(x) = D(x)\psi'(x) = A\psi'(x) + B\psi'''(x). \quad (30)$$

* The even powers vanish if we choose x_1 at the center of the mean free path, and consider particles traveling only between $(x_1 - \frac{l}{2})$ and $(x_1 + \frac{l}{2})$.

**This mathematical trick is not intended to include inhomogeneous flows, in which the transport mechanism actually does vary from place to place.

Then

$$D(x) = A + B \frac{\psi'''(x)}{\psi'(x)}, \quad (31)$$

which displays an unfortunate singularity wherever the gross field happens to have an extremum.

The surprising fact is that some turbulent shear flows, notably the so-called "free shear flows" without solid boundaries (wakes and jets), seem to have mean velocity distributions rather close to those given by assuming that turbulent shear stress is Newtonian in character. Clearly this cannot be correct in principle.

Although boundary layers and channels cannot be approximated even roughly as quasi-Newtonian, it turns out that two-fluid models having a wall layer $0 \leq \frac{yU_*}{\nu} \leq 12$ with the true viscosity and an outer fluid with a "turbulent viscosity" can be instructive. I have used such a pedagogical device for channel flow; it was independently introduced into turbulent boundary layer analysis by F. H. Clauser [20].

Here y is distance from the wall, $U_* \equiv \sqrt{\frac{\bar{T}_0}{\rho}}$, the "friction velocity", $\bar{T}_0 \equiv \mu \left(\frac{\partial \bar{U}}{\partial y} \right)_{y=0}$, the mean skin friction stress.

The superficial success of a "turbulent viscosity" in describing the mean velocity pattern of jets and wakes suggests evaluation of the corresponding "effective Reynolds number" $R_e \equiv \frac{U_1 L}{\nu_T}$, in order to see how it varies with the true Reynolds number,

$R \equiv \frac{U_1 L}{\nu}$. U_1 and L are characteristic velocity and shear zone width.

*For jets and wakes, it turns out that R_e is independent of R .** For the round jet entering fluid at rest,

$$R_e = \frac{\bar{U}_m \cdot \Delta}{\nu_T} \approx 15, \quad (32)$$

where $\bar{U}_m$ is the maximum velocity at a cross section and Δ is the "momentum diameter," defined by

$$\frac{\pi \Delta^2}{4} \bar{U}_m^2 \equiv 2\pi \int_0^\infty U^2(r) r dr, \quad (33)$$

for isopycnic flow. For the plane wake,

$$R_e \equiv \frac{(\bar{U}_\infty - \bar{U}_{\min})h}{\nu_T} \approx 12, \quad (34)$$

where $\bar{U}_\infty$ is free stream velocity, $\bar{U}_{\min}$ is the velocity on the axis at a section and h is the lateral coordinate at which $(\bar{U}_\infty - \bar{U}) = \frac{1}{2} (\bar{U}_\infty - \bar{U}_{\min})$.

It is noteworthy that these values of R_e are both of the same order as the "lower critical Reynolds numbers" for the stability of laminar free shear flows.

When a two fluid model is used for turbulent flow through a round tube, for example, the "effective Reynolds number" defined with the ν_T of the turbulent core

* This observation has also been made by Townsend [1].

flow, the maximum velocity $\bar{U}_m$ and the tube radius a , turns out to vary with the actual Reynolds number:

$$R_e \equiv \frac{\bar{U}_m \cdot a}{\nu_T} \approx \frac{4}{c_f} - \frac{24\sqrt{2}}{\sqrt{c_f}}, \quad (35)$$

where $c_f \equiv \frac{\bar{T}_0}{\frac{1}{2}\rho\bar{U}_m^2}$ is a slowly varying function of $R = \frac{\bar{U}_m a}{\nu}$. (36)

As numerical example, we may take Laufer's two cases:

$$\begin{aligned} R_e &\approx 525 \quad \text{for } R = 25,000, \\ R_e &\approx 900 \quad \text{for } R = 250,000. \end{aligned} \quad (37)$$

It is possible that there exists a more suitable definition of R_e for tube flow, i.e. a definition which does not display this slow variation with R . However, in contrasting the actual Reynolds number dependence of turbulent jet and tube flows we recall that the mean velocity profile of the former is virtually invariant, while that of the latter is appreciably sensitive to Reynolds number.

In terms of the turbulence parameters, we may regard the "turbulent diffusivity" as behaving like

$$\nu_T \sim v' \mathcal{L}, \quad (38)$$

where v' is the r.m.s. velocity fluctuation in the transport direction, and $\mathcal{L}$ is a corresponding Lagrangian integral length scale. It is found experimentally that $\frac{v'}{\bar{U}_m}$ is

relatively insensitive to R in jets, but decreases with R in channels and tubes. The R dependence of $\mathcal{L}$ is not established, but as a property of the large eddies it is likely to be more or less proportional to shear zone width. The foregoing assumptions give $\nu_T \sim \bar{U}_m \mathcal{L}$ for jets [$\therefore R_e = \text{Const}$] and ν_T slowly decreasing with increasing R for tubes.

The low empirical values of R_e for jet and wake, and the superficial agreement of the corresponding constant exchange coefficient postulate, suggest that an engineering approximation to such flows may be gotten via a quasi-Oseen approximation to the Reynolds equations, incorporating a constant ν_T as well.*

SHEAR STRESS FLUCTUATIONS

For some applications it may be interesting to know the shear stress fluctuation level in a turbulent flow. In a fully turbulent region the mean (Reynolds) turbulent stress $-\rho\bar{u_i u_k}$ dominates the viscous momentum transport, and we may begin by a look at the fluctuation in turbulent stress. For simplicity, we restrict the discussion to a boundary layer type flow, in which only one kind of mean shear component, $\bar{T}_T \equiv -\rho\bar{uv}$, is important.

Since the so-called turbulent shear stress is actually the average *y-convection of x-momentum* attributable to the turbulence, it seems plausible to identify

$$\tau_T \equiv -\rho[\bar{U}v + \bar{V}u + (uv - \bar{u}\bar{v})] \quad (39)$$

as the instantaneous fluctuation in turbulent shear stress. Then, for $\bar{V} \ll \bar{U}$ and turbulence levels not too high,

$$\frac{\tau'_T}{\bar{T}_T} \approx \frac{\bar{U}v'}{\bar{u}\bar{v}}, \quad (40)$$

a number considerably greater than unity. Figure 12 shows how τ'_T varies across the turbulent boundary layer. $\bar{T}_0$ is the mean shear stress at the wall. It should be

* This has been applied to the wake by I. Imai (lecture at Fluid Mechanics Colloq., The Johns Hopkins University, November, 1955).

noted that in the fully turbulent zone, $\bar{T}_T$ decreases monotonically with increasing y/δ , so that $\frac{\tau'_T}{\bar{T}_T}$ goes to even larger values than $\frac{\tau'_T}{\bar{T}_0}$.

If (39) is indeed a proper identification, then the large values of $\frac{\tau'_T}{\bar{T}_T}$ describes a process in which instantaneous transfer against the mean gradient is only a little less frequent than instantaneous transfer down the mean gradient.

Parenthetically we may look at the fluctuations in the molecular transport of laminar flow. In the gas kinetic formulation of momentum transport at low Mach numbers, the expression analogous to (39) is dominated by the $(uv - \bar{u}\bar{v})$ term since molecular velocities are much higher than "drift" velocities. Hence the gas kinetic quantity corresponding to our $\frac{\tau'_T}{\bar{T}_T}$ is

$$\frac{\tau'}{\bar{T}} = \sqrt{\frac{\overline{u^2 v^2}}{\bar{u} \bar{v}}} - 1, \quad (41)$$

where u and v are, for this discussion only, molecular velocity components. If we assume u and v to be jointly normal,

$$\overline{u^2 v^2} = \overline{u^2} \overline{v^2} + 2\bar{u}\bar{v}, \text{ and}$$

$$\frac{\tau'}{\bar{T}} = \sqrt{\frac{\overline{u^2} \overline{v^2}}{\bar{u} \bar{v}}} + 1 \approx \frac{\bar{u}' \bar{v}'}{\bar{u} \bar{v}}, \quad (42)$$

which is much larger than unity if the inverse of the mean strain rate is much larger than the mean free time of the molecules.

Toward a solid boundary, the mean and turbulent velocities approach zero, so $\tau'_{\mathcal{T}}$ decreases rapidly with decreasing y/δ . Figure 13 displays this behavior in terms of dimensionless length coordinate

$$y_* \equiv \frac{yU_*}{\nu}, \quad (43)$$

where

$$U_* \equiv \sqrt{\frac{\overline{T}_0}{\rho}}.$$

On this same figure is a very rough estimate of the laminar shear stress fluctuation level,

$$\frac{\tau'_L}{\overline{T}_0} = \frac{\mu \sqrt{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}}{\mu \left(\frac{\partial \bar{U}}{\partial y}\right)_{y=0}}. \quad (44)$$

This dashed curve was sketched by its estimated value at $y = 0$, the skin friction fluctuation, and the measured mean square derivatives at large values of y_* . The indicated monotonic decrease of τ'_L with increasing y is an assumption based upon its apparent flatness at the wall: both u' and $\bar{U}$ are linear in y as $y \rightarrow 0$, and $\frac{\partial U}{\partial y} \approx \frac{U}{y}$ in this vicinity,

so $\tau_0 \approx \tau_{L0} \approx \mu \frac{u}{y}$ and $\tau'_0 \approx \mu \left(\frac{u'}{y}\right)_{y \rightarrow 0}$. It is, of course, conceivable that τ'_L could have a maximum away from the wall.

We note from figure 13 that $\tau'_L < \tau'_T$ except for $y_* < 3$. This is a bit different than the case of many other physical properties in this region; it is found that laminar and turbulent contributions to various transport and dissipation phenomena are equal at $y_* \approx 12$, a common choice of boundary between a "wall layer" and the "outer flow." It seems preferable to restrict the term "laminar sublayer" as designation for the zone in which turbulent transport is actually negligible.

WALL LAYER AND LAMINAR SUBLAYER

The physical picture of a wall neighborhood in which viscous effects are not negligible has remained virtually unchanged since its description by Taylor in 1916 [22]. He described it essentially as a Couette flow violently disturbed by the turbulence in the main body of the shear zone, suggesting that it should therefore have a Reynolds

number $R_\varepsilon \equiv \frac{\bar{U}(\varepsilon)\varepsilon}{\nu}$ roughly equal to the theoretical lower critical value for stability of such a Couette flow, 150 to 200.

This assumption permits engineering estimates of turbulent boundary layer and channel flow (via the Kármán integral relation, for example). As is well known, it has since turned out to be quantitatively correct if we identify Taylor's layer as extending from the solid wall out to where laminar and turbulent shear are exactly equal. As mentioned in the previous section, this occurs at $y_* \approx 12$ [23]. This is the "wall layer."

Because of the no-slip condition, there must be a non-inertial layer at the wall, presumably characterized by a length made up of the characteristic ("friction") velocity

$U_* \equiv \sqrt{\frac{\bar{T}_0}{\rho}}$ and the kinematic viscosity:

$$l_* = \frac{\nu}{U_*}. \quad (45)$$

In this "laminar sublayer" [$y_* < 1$], the direct turbulent transport is negligible.

Recent hot-wire studies, like [7] and [23], have given information on r.m.s. velocity fluctuation levels well inside the wall layer, and Klebanoff [7] has presented a u' spectrum at $y_* = 3$, virtually in the laminar sublayer. Still it seems likely that more information will be necessary before a suitable theoretical picture can be developed.

By passing to the $y = 0$ limits of the equations of motion and their derivatives, it is possible to deduce some restricted relations. For example, it is well known that the Reynolds equations reduce to Stokes flow, with only one appreciable component:

$$\left(\frac{\partial^2 \bar{U}}{\partial y^2} \right)_0 = \frac{1}{\mu} \left(\frac{\partial \bar{P}}{\partial x} \right)_0. \quad (46)$$

The fluctuation momentum equations reduce to a simplified two-dimensional Stokes flow:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}, \quad (47)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial z}, \quad (48)$$

since continuity requires that $\bar{V}(y)$ and $v(y)$ start out parabolically from $y = 0$. Further, since

$$u(y) \sim y + \text{-----}, \tag{49}$$

$$w(y) \sim y + \text{-----}, \tag{50}$$

$$v(y) \sim y^2 + \text{-----}, \tag{51}$$

we see that the turbulent shear stress can rise no faster than a cubic:

$$-\overline{uv} \sim y^3 + \text{-----}. \tag{52}$$

In fact, there is some evidence that $\left|\frac{\overline{uv}}{u'v'}\right|$ sometimes decreases toward the wall [24], in which case $-\overline{uv}$ may build up even more slowly.

Some of these conditions have been exploited, for example, in a report by Phillips [25].

A different but qualitative look at the wall region can be gotten by dye injection methods. The “leaching” of colored water from a glass tube by flowing clear water through it in a turbulent state leaves dye only near the wall after a short time. In figure 14 is such a situation as photographed by Beatty, Ferrell and Richardson.* The Reynolds number is about 7000. The significant property seems to be the strong orientation into stream-wise filaments of the residual dye. Presumably this indicates a predominance of axial vorticity near the wall, “sweeping” the (dyed) wall fluid into these long narrow strips. Of course the very high mean velocity gradient in this zone is expected to rotate and strain all fluid lines axially, but the parabolic decrease of v'

Figure 14. Flushing of dye by a turbulent pip flow, R 7000. [Courtesy of Chemical Engineering Department, North Carolina State College]

toward the wall would seem to prevent the dominance of this vorticity component in the laminar sublayer itself.

Figure 15 is a somewhat analogous picture taken by Hama, with dye injected from a flush cross-stream slit into the wall layer of a turbulent boundary layer. In this case the interpretation is made more difficult by the apparent non-uniformity of dye

* I should like to thank Drs. Beatty, Ferrell and Richardson of the Chemical Engineering Dept. at North Carolina State College for the loan of their manuscript, and also Dr. Hama of the University of Maryland for supplying figure 15.

Figure 15. Dye streaks near wall in a turbulent boundary layer. [Courtesy of Dr. F. R. Hama, University of Maryland]

Figure 16. Oscillograms of u -fluctuations near wall. Velocity increases upward.

ejection along the slit. Since this differed in different photographs it is probably due to the local fluctuations in skin friction and static pressure forces.

A rather intermittent structure of the wall layer has been shown clearly by an oscillogram of Ruetnik [24], which is here reproduced as figure 16. As is to be expected, the "bursts" containing higher frequencies are always accompanied by an increase in velocity; this is "turbulent" fluid coming in to $y_* = 5$ from the outer part of the wall layer.

Independent observations on the intermittent character of this region have encouraged Einstein and Li [26] to propose a periodically growing and collapsing laminar "Rayleigh model." Although proper adjustment of constants enables this model to give encouraging agreement with measurements of some flow properties, it appears to suffer from the basic ailment of an infinite r.m.s. skin friction fluctuation.

It seems likely that a "Stokes model" of the sublayer, with sinusoidal rather than sawtooth periodicity, would overcome this difficulty. However, a model of the entire wall layer must doubtless be three-dimensional, as we see from the photographs.

FLUCTUATIONS IN SKIN FRICTION AND SURFACE STATIC PRESSURE

Since $u \sim y$ near $y = 0$, it may be possible to interpret Klebanoff's u' -spectrum at $y_* = 3$ [7] as a spectrum of skin friction fluctuations. This possibility is encouraged by the properties of a Couette flow with fixed wall at $y = 0$ and moving wall (at $y = h$), having mean velocity U_0 plus sinusoidal oscillation $u_0 \sin \omega t$ in its own plane. The solution of this problem gives

$$\frac{\left(\frac{u'}{\bar{U}}\right)}{\left(\frac{u'}{\bar{U}}\right)_h} = \frac{h}{y} \sqrt{\frac{\cosh^2 \beta y - \cos^2 \beta y}{\cosh^2 \beta h - \cos^2 \beta h}}, \quad (53)$$

$$\text{where} \quad \left(\frac{u'}{\bar{U}}\right)_h = \frac{1}{\sqrt{2}} \frac{u_0}{U_0} \quad \text{and} \quad \beta \equiv \sqrt{\frac{\omega}{2\nu}}. \quad (54)$$

This gives at the fixed wall

$$\frac{\left(\frac{u'}{\bar{U}}\right)_0}{\left(\frac{u'}{\bar{U}}\right)_h} = \frac{\sqrt{2} \alpha h}{\sqrt{\cosh^2 \alpha h - \cos^2 \alpha h}}. \quad (55)$$

If ω is identified as an average frequency of the "shear-carrying eddies," i.e.

$$\omega = u_v(k_1)_{av} \cdot \bar{U}_{y_*=3} \approx 100 \text{ sec}^{-1}, \quad (56)$$

and if h is chosen such that $\frac{hU_*}{\nu} = 3$, i.e. $h \approx .008 \text{ cm}$, we find $\alpha h \approx 0.15$. Hence

$\left(\frac{u'}{\bar{U}}\right)_0 \approx \left(\frac{u'}{\bar{U}}\right)_h$, and we conclude that at $y_* = 3$ the relative longitudinal velocity fluctuations have reached their boundary value.

Klebanoff's power spectrum, though plotted in terms of wave number, was actually measured in time, and in figure 17 is the Fourier transform, essentially the

time autocorrelation of the skin friction fluctuations. The time interval is made dimensionless by the characteristic sublayer time,

$$\frac{U_*^2}{\nu} = \left(\frac{\partial \bar{U}}{\partial y} \right)_0^{-1}. \quad (57)$$

Einstein and Li [26] have measured the autocorrelation of surface static pressure fluctuations on the bottom of a turbulent open channel flow of oil. Their curve, somewhat smoothed, is plotted on the same sheet as the skin friction correlation. Considering the differences between the two flow systems, the general consistency is remarkable. The situations are doubtless too diverse to permit drawing detailed conclusions, but it does appear that the two kinds of surface force fluctuations have about the same duration of autocorrelation.

BOUNDARY LAYER THICKNESS FLUCTUATIONS

Recent attempts at prediction of sound radiation by a turbulent boundary layer have taken two apparently diverse paths. Working from the viewpoint of Lighthill's formulation, Phillips [25] has related sound radiation to skin friction fluctuations. Working from a more explicit picture of how the free stream "sees" a boundary layer, Liepmann [27] has focussed attention upon the fluctuations in boundary layer displacement thickness, presumably acting as a random distribution of pistons.

Of course there must be some analytical connection between skin friction fluctuations and boundary layer thickness fluctuations, although we may expect to find the connection with momentum thickness rather than displacement thickness.

Taking a very simplified *two-dimensional* viewpoint, we may define local displacement and momentum thickness by

$$\Delta^* \equiv \int_0^\infty \left(1 - \frac{U}{U_\infty} \right) dy, \quad (58)$$

$$\Theta \equiv \int_0^\infty \frac{U}{U_\infty} \left(1 - \frac{U}{U_\infty} \right) dy. \quad (59)$$

Introducing $U = \bar{U} + u$, we get mean thicknesses

$$\bar{\Delta}^* = \int_0^\infty \left(1 - \frac{\bar{U}}{U_\infty} \right) dy, \quad (60)$$

$$\bar{\Theta} = \int_0^\infty \frac{\bar{U}}{U_\infty} \left(1 - \frac{\bar{U}}{U_\infty} \right) dy - \int_0^\infty \frac{\bar{u}^2}{U_\infty^2} dy, \quad (61)$$

and mean square fluctuations

$$\overline{\delta^{*2}} = \frac{1}{U_\infty^2} \int_0^\infty \int_0^\infty \overline{u(y_1)u(y_2)} dy_2 dy_1, \quad (62)$$

and

$$\begin{aligned} \overline{\vartheta^2} = \frac{1}{U_\infty^2} \int_0^\infty \int_0^\infty \left\{ \left[1 - \frac{\bar{U}}{U_\infty} - \frac{2\overline{u(y_1)}}{U_\infty} \right] \left[1 - \frac{2\bar{U}(y_2)}{U_\infty} \right] \right. \\ \left. \cdot \overline{u(y_1)u(y_2)} + O(\bar{u}^3) \right\} dy_2 dy_1, \end{aligned} \quad (63)$$

where terms of order $\bar{u}^3$ and higher may be neglected.

Using very rough approximations for mean velocity profile and lateral correlation function from reference 6, we estimate the following values for $\delta^{*'}$ and ϑ' :

$$\frac{\delta^{*'}}{\delta} \approx .027; \quad \frac{\delta^{*'}}{\bar{\Delta}^*} \approx 0.23 \quad (64)$$

$$\frac{\vartheta'}{\delta} \approx .014; \quad \frac{\vartheta'}{\bar{\Theta}} \approx 0.15 \quad (65)$$

δ is a "geometrical boundary layer thickness," the distance at which $\bar{U} \approx U_\infty$

It would, of course, be most desirable to be able to predict these values from a rough estimate of skin friction fluctuations. The proper approach must involve a two-dimensional random field of skin friction and the proper three-dimensional generalization of the von Kármán integral relation.

A rough estimate was attempted, however, by using an over-simplified form of two-dimensional Kármán integral relation:

$$\frac{d\Theta}{dx} = \frac{T_0}{\rho U_\infty^2}, \quad (66)$$

so that

$$\frac{d\vartheta}{dx} = \frac{\tau_0}{\rho U_\infty^2}, \quad (67)$$

or

$$\vartheta(x) = \frac{1}{\rho U_\infty^2} \int_0^x \tau_0(x_1) dx_1. \quad (68)$$

Hence

$$\overline{\vartheta^2}(x) = \frac{1}{\rho^2 U_\infty^4} \int_0^x \int_0^x \overline{\tau_0(x_1) \tau_0(x_2)} dx_2 dx_1. \quad (69)$$

This integral was evaluated by assuming the skin friction spatial autocorrelation to be equivalent to the time correlation by $(\Delta x) \approx \overline{U}(\Delta t)$, a very poor assumption near the wall as shown experimentally by Klebanoff [7]. With the further assumptions of similarity and $\delta \sim x^{4/5}$ mean growth rate of the boundary layer, the momentum thickness fluctuation was estimated from (69) as

$$\frac{\vartheta'}{\delta} \approx .001, \quad (70)$$

at the boundary layer position where the computed value is .014. The failure of this estimate may be due in part to the erroneous shear stress correlation and in part to the general omission of z -variations both in the differential equation and the surface stress.

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SYMBOLS

A, B	constants
c_f	skin friction coefficient
d	pipe diameter
D	diffusivity
$E_{1,2}$	principal strain rates
ε	three-dimensional spectrum
$F_{1,2}$	one-dimensional spectra
k	wave number
k_1	wave number in x_1 direction
k_c	"cutoff" wave number
k_p	wave number for turbulent energy "production"
l	characteristic diffusive length
l_*	thickness of "laminar sublayer"
L	characteristic length
$\mathcal{L}$	Lagrangian integral length scale
P	static pressure
$\bar{P}$	mean static pressure
p	static pressure fluctuation [$= P - \bar{P}$]

p_v	perturbation static pressure fluctuation due to v_i
Q	transport rate
${}_k R_{uv}$	turbulent shear stress spectrum $\left(\text{dimensionless form} = \sqrt{\frac{\tau_1}{F_1 \cdot F_2}} \right)$
R	Reynolds number
R_e	Reynolds number based on "turbulent kinematic viscosity,"
T	total shear stress
$\overline{T}_0$	mean skin friction stress
$\overline{T}_T$	mean turbulent shear stress
$\overline{T}_L$	mean viscous shear stress
U, V, W	cartesian velocities in x, y, z directions respectively
$\overline{U}, \overline{V}, \overline{W}$	mean velocities
u, v, w	velocity fluctuations $[= (U - \overline{U}), (V - \overline{V}), (W - \overline{W})]$
u_j	velocity fluctuation in x_j direction
U_*	"friction velocity" $\left[\equiv \sqrt{\frac{\overline{T}_0}{\rho}} \right]$
v_i	perturbation velocity fluctuation in x_i direction
x, y, z	cartesian coordinates
x_i	cartesian coordinates ($i = 1, 2, 3$)
$y_* =$	$\frac{y U_*}{\nu}$
α_{E1, σ_1}	principal directions
β	wave number
δ	geometrical boundary layer thickness
Δ	momentum diameter of round jet
Δ^*	displacement thickness of boundary layer
δ^*	displacement thickness fluctuation
ε	thickness of wall layer in turbulent shear flow
$\vartheta_{i, \nu}$	characteristic spectral times
$\textcircled{H}$	momentum thickness of boundary layer
ϑ	momentum thickness fluctuation
ν	kinematic viscosity
ν_T	"turbulent kinematic viscosity"
ρ	density
$\sigma_{1,2}$	principal stresses
$\tau_{0,T,L}$	shear stress fluctuations
ϕ	rate of dissipation of kinetic energy
ψ	a scalar property

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DISCUSSION

K. Wieghardt

The following remark refers to a complication which might occur in the boundary layer along a three-dimensional body without incidence. A study of the flow around the run of double-models of simplified ships in a wind tunnel has shown a longitudinal vortex-pair near the keel plane produced by the flow around the sharp bilges (see Fig. 1).

Figure 1.

The strength of the vortices was so that they induced a downward velocity of almost half the main speed in the center line of the wake. This might be of practical interest for double screw ships. In most cases outward turning propellers are best, i.e. screws running in the opposite sense to these vortices. This would seem plausible since in this case the residual vorticity is as small as possible. Of course, on a real ship, the influence of bossings and the interference with the wave drag will also be of great importance. Further on, this effect will certainly be much smaller with bilges well rounded off. E.g. with a second extreme model, a circular cylinder and a run with elliptical sections leading to the same vertical edge, no such vortices were observed. However, since some ships are more similar to the first model such a vortex-pair might play a certain role with real ships.

L. Landweber

First I would like to ask Professor Corrsin's opinion of the relative importance in the boundary-layer momentum equation of the term

$$\frac{d}{dx} \int_0^\delta (\bar{u}^2 - \bar{v}^2) dy. \quad (71)$$

It was shown in several papers, in about 1950, that this term could become large in adverse pressure gradients, but recently it has been indicated that the term is negligible. This apparent contradiction should be clarified.

Next let us consider the validity of the so-called universal logarithmic laws of turbulent boundary layers which is assumed in even the most recent work of Clauser, Coles and Townsend. These laws are based upon two similarity laws, the law of the wall and the velocity-defect law. Of these, the former may be derived by dimensional reasoning and has been well-confirmed by experiment; the latter can be derived theoretically only for pipe and channel flows, and its adoption for the case of flat-plate boundary layers is based upon its correlation of velocity profiles over the narrow range of Reynolds numbers for which data are available.

It has recently been shown by Townsend, on the basis of the concept of approximate self-preservation of the structure of the turbulence in the outer part of a boundary layer on a flat plate, that the velocity defect law may be retained, provided the previous history of the flow is taken into account by replacing the local velocity scale by that corresponding to some upstream position. Townsend's subsequent analysis is approximate and retains the logarithmic relations by assumption. A more precise analysis of Townsend's theory yields a family of possible universal laws of which the logarithmic law is only one member, and among which that one which best fits the boundary-layer data is to be selected.

The theory assumes the validity of the law of the wall

$$\frac{u}{u_\tau} = f(y^*), \quad y^* = \frac{yu_\tau}{\nu} \quad (72)$$

where $u_\tau = \sqrt{\tau/\rho}$, τ is the shear stress at the wall, and ρ and ν are the density and kinematic viscosity of the fluid; and in the outer part of the boundary layer, the validity of a modified velocity-defect law

$$\frac{U_0 - u}{u_\tau} = \frac{F(\zeta)}{g(\sigma)}; \quad \zeta = \frac{y}{L}, \quad \sigma = \frac{U_0}{u_\tau} \quad (73)$$

where U_0 is the free stream velocity, L is a length scale proportional to the boundary-layer thickness, and $g(\sigma)$ is an unspecified function. Townsend assumes a specific form for $g(\sigma)$, but if it is further assumed that there is a range of values of y in which the inner and outer similarity laws overlap, it is found that there is no freedom of choice concerning either the form of $g(\sigma)$ or the similarity laws.

By an exact functional analysis of the foregoing assumptions, one obtains the result

$$y^* \frac{f''(y^*)}{f'(y^*)} = n - 1 \quad (74)$$

If $n = 0$, this yields the well-known logarithmic relations for part of the velocity distribution and the variation of the coefficient of shear stress with Reynolds number. If $n \neq 0$, an entirely new set of relations may be derived. The results for the two cases are compared in the following table:

	$n = 0$	$n \neq 0$
$f(y^*)$	$c \log y^* + d$	$ay_n^* + b$
$F(\zeta)$	$-c \log \zeta + d'$	$b'(1 - a\zeta^n)$
$g(\sigma)$	1	$b' / (\sigma - b)$
c_τ	$2/\sigma^2$	$2/\sigma^2$
R_δ	$c\sigma e^{\sigma/k}$	$\zeta_0 \sigma (\sigma - b)^{1/n}$
R_{δ_1}	$c\beta e^{\sigma/k} + \lambda$	$\alpha_0 + \alpha_1(\sigma - b)^{1+1/n}$
R_{δ_2}	$ce^{\sigma/k} \left(\beta - \frac{\gamma}{\sigma} \right) - \lambda - \frac{\alpha}{\sigma}$	$\frac{1}{\sigma} [\beta_0 + \beta_1(\sigma - b) + \beta_2(\sigma - b)^{1+1/n} + \beta_3(\sigma - b)^{2+1/n}]$
R_x	$ce^{\sigma/k} [\beta \sigma^2 - (\gamma + 2\beta k)\sigma + 2k(\gamma + \beta k)] + \alpha \sigma + c$	$\gamma_0 + \gamma_1(\sigma - b) + \gamma_2(\sigma - b)^{1+1/n} + \gamma_3(\sigma - b)^{2+1/n} + \gamma_4(\sigma - b)^{3+1/n}$

F. R. Hama

I am not quite prepared to discuss Dr. Corrsin's interesting and stimulating lecture, and I am rather anxious to read his manuscript carefully.

However, in connection with his Fig. 15, which was taken in a fully-developed turbulent boundary layer at Dr. Corrsin's request, I would like to show one other picture (Fig. 1) taken in a laminar boundary layer disturbed by vortices due to a trip wire.

In the absence of the wire, the dye comes out uniformly and no filamentation of the dye occurs. Once the wire is applied, the flow of the dye shows a concentration as shown in Fig. 1, indicating the existence of secondary spiral motion near the boundary surface, although the boundary layer still remains laminar or, more properly, is in the stage toward transition.

There is naturally a distinctive difference between the two cases. Whereas the attached eddies are smaller and randomly distributed in turbulent flow, the wisps are larger and regular in the other flow. Nevertheless, there seems to be a resemblance for the two flow conditions, i.e., secondary spiral motion.

I don't know, first, how we can explain this secondary flow formation which takes place when a laminar boundary layer is disturbed by strong-two-dimensional vortices. I don't know, second, whether there is any relation between the predominant secondary flow in turbulent boundary layers and that in laminar layers in the transition region.

R. W. Stewart

We seem to get very close to the boundary. There is one phenomenon that must be connected with this, which I thought I would mention. We have examined very closely the bottom of some ships in dry dock. Whenever these ships were painted in Hongkong, apparently they used very soft paint, and along the bottom of the hull, along the direction of the mean flow—this has no relation whatsoever to the way the paint brush strokes went, or the direction of the plates on the ships—but in the direction of the mean flow one finds streaks of raised paint. Sometimes a single streak will be

Figure 1. (F. R. Hama)

as many as several yards long. These streaks are about an 8th of an inch or a millimeter apart.

The effect must be somewhat related to this type of phenomena.

The thing interesting about it, since these have actually deformed a fairly solid body like this paint, is that there must be some self-stabilizing feature, so that once a little seed is developed, this process continues from then on, in essentially the same place.

Also another interesting thing about this, of course, is that the boundary layer

on the ship varies a great deal in thickness from the bow to the stern, close to the propellers, but the streaks are all about the same distance apart, whether at the front or the back.

It must be closely associated with something close to the boundary, because, for example, there is a wake behind each rivet head. I think this streaky phenomenon is something that occurs very characteristically close to the boundary even in fully developed boundary flows.

E. Silberman

The remarks just made (by the previous discussor) bring to mind similar results obtained on the walls of a curved duct. The duct cross section was 6 in. square and the walls were of Lucite. The inner surfaces were coated with a thin paste of aluminum powder in machine oil. The fine streaks mentioned appeared in the oil and apparently took the directions of the streamlines at the walls. (Directions given by the streaks coincided with the limiting directions of fine yarns moved from the interior to the boundary of the fluid.) Observations of the erosion process producing the streaks gave the impression that the paste was first dislodged in a direction other than the streak direction, but almost immediately turned in the streak direction. Reynolds number based on duct hydraulic diameter was about 5×10^4 for the experiments.

G. K. Batchelor

I don't think these remarks have any bearing on the explanation of the observed streaks of dye. The same phenomenon of longitudinal streaks or ridges on thin films has been observed on aircraft wings smeared with oil, and in fact has been used to make visible the shape of the streamlines of the air flow. I don't think it has been analyzed, but I feel pretty sure it is related to the characteristics of the film itself, and is the result of some kind of instability tending to make the oil move sideways and gather itself into longitudinal ridges and troughs. I think one can see this as fairly natural if one thinks of the way in which the oil ought to arrange itself in order to have the least amount of viscous dissipation as a result of the shearing action produced by the airflow outside the film, and the fixed wall underneath it.

In other words, this seems to me to be a film phenomenon, whereas the problem we are talking about is concerned with the flow in the boundary layer itself. Surely the streaks of dye are related to the idea of longitudinal diffusion, which Taylor talked about a few years ago in connection with the dispersion of salt solution in a pipe. He pointed out that if you produce a small cloud of salt solution in turbulent flow in pipe, the variation of mean velocity across the pipe tends to draw out the cloud longitudinally. Wouldn't this also happen here, when Professor Hama's dye is ejected across the flow? Will not the part of the dye that lies farther from the wall also be a part which moves forward more rapidly, thus producing a streak of dye?

F. R. Hama

The distinct difference between the streak formation and the non-streaky flow is that, if you don't have the trip, the dye comes out uniformly and no filamentation occurs, and, only when the trip is applied, the wisp formation takes place. Therefore, the wisp formation must have resulted from the secondary spiral flow.

M. P. Tulin

Just before Dr. Batchelor got up I was thinking of experiments we (Mr. Ralph Cooper and myself) had made at the Model Basin several years ago, investigating the effects of studs on transition. These measurements were made on a bi-convex, thin plate which was towed in a model basin. Transition was determined by coating the

Inside of Bend

Bottom of Bend

Streaks on interior surfaces of 6-in. square duct at a 90° bend. Walls are Lucite. Photographed from outside. Coating is paste of aluminum powder in SAE 30 oil.

Figure 1. (E. Silberman)

plate with a solid substance which would wash away in the turbulent region and would remain on the plate in the laminar region.

In regions of transition, and just preceding transition, we noticed a regular and very fine pattern of streamwise striations, perhaps of the order of a millimeter distance apart, these distances considerably less than the distances between the studs, and apparently not much depending on the stud distance.

This experience suggests that in order to determine whether the boundary layer streaks are really connected with film behavior it might be instructive to perform further observations in water using the aforementioned technique.

XVI

WAVE SCATTERING DUE TO TURBULENCE

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SUMMARY

When a sound wave is incident upon a part of the medium which contains small irregularities, scattered waves are set up. The irregularities, which may be either variations of velocity or of physical properties such as temperature, are here assumed to be random as a result of the action of turbulence in the medium. The basic equations that describe the propagation of sound in both air and water containing various kinds of non-uniformity are derived. After a general exposition of the theory of scattering in continuous media, the scattering cross-section for each of these cases is obtained, and is found to depend on the spectrum of the spatial distribution of the relevant property of the medium.

It happens occasionally in science that a problem suddenly becomes important and interesting in several different contexts simultaneously. This is exactly what has happened in the case of propagation of waves through a medium whose physical properties are not quite uniform and vary from point to point in a random fashion as a result of the presence of turbulence. This problem has sprung into prominence in the post-war years in no fewer than four different physical settings.

One setting concerns the propagation of radio waves of about one metre wavelength through the atmosphere, and has led to the spectacular development of a new method of long-range communication, which employs scattering of the radio waves through a small angle by small random variations of electron density in the ionosphere. Another concerns the old problem of the scintillation of starlight; that the twinkling of stars is due to variations of refractive index in the atmosphere was known to Newton, but only recently have attempts been made to analyse the phenomenon quantitatively and to relate it to the turbulence in the atmosphere. In these two contexts the waves are electro-magnetic, whereas in the other two I have in mind sound waves are involved.

Most people here will already know of the current interest in various aspects of the propagation of sound in the ocean. If the ocean were perfectly uniform, ordinary acoustic theory would tell us all that we need to know in order to design and operate equipment for sound-ranging and sound-detection. But the ocean is not uniform, and contains currents which convect the sound and variations of temperature and of salt concentration which refract the sound. These non-uniformities are made random by the ever-present turbulence, and are broken down in scale so that there are always some Fourier components with wave-lengths comparable with that of the incident sound. As a consequence, sound signals are subject to large fluctuations in amplitude, to scattering about their original direction, and to rapid attenuation. This is not by any means a new problem, but the needs of naval warfare gave a great stimulus to its investigation, and interest in it has remained active during the post-war years.

Finally, there is the allied problem of propagation of sound in the atmosphere.

The effect of non-uniformities in the atmosphere came under notice during the first world war in connection with devices for the detection of enemy aircraft, but these devices were not successful and the problem was not examined fully. Interest in it has more recently been revived, as a part of a comprehensive investigation of the noise produced by jet engines. Instead of wanting to pick up the noise of an aircraft, we now wish to suppress or avoid it, but in either case we require a knowledge of the effect of random variations of the properties of the air on the sound waves passing through it.

Thus radio engineers, astronomers, oceanographers and aerodynamicists all have good reasons for wanting to know more about the same basic problem. So far as I can tell from a limited acquaintance with the separate literatures, they seem on the whole to have pursued the problem in their own ways, without making much use of advances made in other physical contexts. There is a need, I think, for an investigation which is framed in as general a manner as possible, and which brings together all the available results. The general aspects of the subject are especially suitable as material for a survey at the present time, and, although this lecture does not provide the time that would be needed for a proper account, I hope at least to arouse your interest and to give you the right orientation.

In order to keep fairly close to the theme of this symposium, I propose to confine my attention to sound waves, thereby excluding the applications of the theory to radio communication and stellar scintillation. (The analysis required in these latter applications is essentially similar, such differences as there are being mostly those arising from the need to specify electromagnetic waves by a vector, instead of a scalar, quantity.)

When reading the published papers on the propagation of sound waves in a non-uniform medium, I have noticed that many of the authors assume intuitively the validity of a certain equation describing the propagation of sound, and then proceed to investigate some special feature of this equation, such as the nature of the scattering. This procedure has its dangers, since mistakes in the form of the governing equations cannot be detected by the usual checks on consistency of the analysis. Some authors have assumed, for instance, that any non-uniformity of the physical properties of the medium is equivalent to a variation of the phase velocity, that is, of refractive index; others have assumed that the effect of turbulent convection with velocity $\mathbf{u}$ in the medium can be represented by replacing $\partial/\partial t$ in the conventional acoustic wave equation by $\partial/\partial t + \mathbf{u} \cdot \nabla$. Both of these ideas are plausible, but actually they are too simple, and lead to correct results only in special circumstances. In view of the evident differences of opinion about the equations on which the whole problem is based, I shall spend a large part of my time on a careful derivation of the governing equations appropriate to each kind of non-uniformity of the medium. I regret that there will not be time to get to grips with the observational data about scattering of sound.

There are three essentially different aspects of a theoretical treatment of the problem of propagation of sound waves through a non-uniform medium in turbulent motion:

- a. the derivation of the equations describing the propagation of sound waves in these circumstances;
- b. the analysis of the scattering and of other consequences of the non-uniformity;
- c. the determination, from turbulence theory or experiment, of those statistical properties of the medium that the analysis in b. has shown to be relevant.

Consideration of this last aspect would take us into the complexities of turbulence theory, and must be omitted for lack of time. I shall begin with an analysis of scattering in general form, since talk about the form of governing equations is not interesting until one knows the kind of use to which they are to be put.

The scattered wave: general theory

We shall see later that the propagation of sound waves can be described, in all the situations under consideration, by a linear modified wave-equation of the form

$$\nabla^2 \psi - \frac{1}{c_0^2} \frac{\partial^2 \psi}{\partial t^2} = \sum_n P_n(\xi) D_n(\psi). \quad (1)$$

(It is desirable to use a complete wave theory, rather than the methods of geometrical optics, in a discussion of scattering, because the relative scattered acoustic power is greatest under conditions for which a treatment by geometrical optics would not be valid.) ψ is here a scalar wave-function, such as the fluctuating pressure or density, the exact interpretation being left unspecified for the moment. c_0 is a constant, D_n is a linear differential or integral operator with respect to space or time, and $P_n(\xi)$ is a function of the group of variables ξ that specify the (departures from the mean of the) relevant local properties of the medium. The summation is over all terms of the type shown, the usual number being not more than two. The variables ξ , of which temperature is a typical example, are themselves random functions of position and time as a result of the action of turbulent motion in the medium. The physical problems that we have in mind are such that the medium is only slightly non-uniform, and P_n is small compared with unity when all quantities are made dimensionless with the aid of parameters characteristic of the sound wave.

We wish to find a solution of this equation representing the passage of a sound wave, generated by given external means, through a region in which the random variables ξ are known in the statistical sense. It is important for the theoretician that he be allowed to assume that the incident sound wave has a frequency which is high by comparison with the fractional rates of change of the variables ξ , and fortunately this is nearly always true in practice. The consequence of this assumption is that ψ is effectively the only quantity in the above equation that depends on t , and, since ψ occurs linearly in all terms, the dependence on t plays only a minor role in the problem.

Since the right-hand side of the equation is small, whereas terms on the left-hand side are not, it is appropriate to seek a solution by a perturbation procedure. The first approximation to ψ , represented by ψ_0 , is given by

$$\nabla^2 \psi_0 - \frac{1}{c_0^2} \frac{\partial^2 \psi_0}{\partial t^2} = 0. \quad (2)$$

We need consider only one Fourier component (with respect to t) of the incident wave, and we shall suppose the incident wave that is generated externally to be harmonic, with frequency ω , and to be approximately plane; thus

$$\psi_0 = A e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (3)$$

where

$$|\mathbf{k}| = \kappa = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}. \quad (4)$$

The second approximation to ψ is then $\psi_0 + \psi_1$, where

$$\nabla^2 \psi_1 - \frac{1}{c_0^2} \frac{\partial^2 \psi_1}{\partial t^2} = \sum_n P_n(\xi) D_n(\psi_0). \quad (5)$$

The linear operations represented by D_n can be carried out explicitly, thus allowing the equation for ψ_1 to be written as

$$\nabla^2 \psi_1 - \frac{1}{c_0^2} \frac{\partial^2 \psi_1}{\partial t^2} = A e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} Q(\mathbf{x}) \quad (6)$$

say, where $Q(\mathbf{x})$ is independent of t . The formal solution of this equation as a "retarded potential" is

$$\psi_1(\mathbf{x}, t) = -\frac{A}{4\pi} \int \frac{e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t) + ik|\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} Q(\mathbf{y}) d\mathbf{y}. \quad (7)$$

The sound wave represented by ψ_1 is the first approximation to the effect of the non-uniformity of the medium on the incident wave ψ_0 . ψ_1 has the same form as the wave-function appropriate to a volume distribution of acoustic sources in a uniform medium, each source being of frequency ω and the density of source strength at position $\mathbf{y}$ being proportional to $AQ(\mathbf{y})$. It is as if the incident wave "polarized" each volume element of the medium with a strength proportional to $|\psi_0|$ and to a quantity representing the local departure of the medium from uniformity, the radiation from these "poles" then being, to this same order of approximation, the same as if the medium were uniform. The usual physical interpretation is that ψ_1 represents the collection of singly-scattered waves, that is, those waves that would be set up in a uniform medium by the incidence of the wave ψ_0 on single isolated scattering elements.

It is not often possible to go to the higher approximation $\psi_0 + \psi_1 + \psi_2$, and most of the available theoretical work concerns single scattering. It is therefore important that we should know when $\psi_0 + \psi_1$ is a valid approximation to the wave-function. In the case of a plane wave passing through a medium such that the velocity of sound has different, but uniform, values inside and outside a region of volume V , it has been established that $\psi_0 + \psi_1$ is a good approximation to the wave-function provided $\varepsilon V^{1/3}/\lambda \ll 1$, where ε is a measure of the magnitude of the fractional variation in the velocity of sound. When the properties of the medium are random, having values above the ambient level in some parts of V , and below it in others, the requirement will be weaker and seems likely to be of the form $\varepsilon(V^{1/3}L)^{1/2}/\lambda \ll 1$, where L is a length characterizing the scale of variation of the properties of the medium. A numerical example will help to convey the significance of this criterion. Suppose sound waves of wave-length 10 cm pass through either water or air in which the temperature varies, with an r.m.s. fluctuation of 1°C and a length scale of a metre. Single scattering will then be a reasonable approximation provided the region containing the non-uniformities has linear dimensions small compared with a kilometre. The restriction on the size of the volume V can be thought of as ensuring that the chances of a scattered wave being itself scattered before leaving V are slight.

The wave-function that describes the waves associated with a point singularity of some kind takes a simpler form at large distances from the singularity. The wave motion is purely radiative at these large distances, with an intensity that is a measure of the rate of output of energy from the source. There is thus both analytical convenience and physical significance in a consideration of the radiation fields of the scattered waves emanating from the various elements of the volume V in which the medium is non-uniform. If we wish to describe the fluctuations in sound intensity at some point within the scattering volume V , we must of course take into account the "near" fields of the scattered waves from neighbouring elements. Investigations of this kind have been made for certain choices of the governing equation (Ellison 1951, Mintzer 1953 & 1954). In order to keep this lecture within bounds, only the algebraically simpler quantities depending on the radiation fields will be considered.

A point whose distance from V is large compared with the linear dimensions of V will be in the radiation field of all the scattering elements, and at such a point

$$\begin{aligned}
 \psi_1(x, t) &\sim -\frac{A}{4\pi x} \int_V e^{i(\mathbf{\kappa} \cdot \mathbf{y} - \omega t) + i\kappa x(1 - \mathbf{x} \cdot \mathbf{y}/x^2)} Q(\mathbf{y}) d\mathbf{y} \\
 &= -\frac{A}{4\pi x} e^{i(\mathbf{\kappa} x - \omega t)} \int_V e^{i\mathbf{k} \cdot \mathbf{y}} Q(\mathbf{y}) d\mathbf{y} \\
 &= -\frac{A}{4\pi x} e^{i(\mathbf{\kappa} x - \omega t)} \Gamma(\mathbf{k})
 \end{aligned} \tag{8}$$

where Γ is the Fourier transform of the function Q that describes the distribution of the relevant property of the medium, and $\mathbf{k} = \mathbf{\kappa} - \kappa \mathbf{l}$ (where $\mathbf{l} = \mathbf{x}/x$) is the so-called scattering vector that bisects the direction of the incident wave and that of $\mathbf{x}$ reversed (see figure). The scattered wave at a considerable distance from V thus consists of a spherical wave of frequency ω propagating radially outwards with an amplitude proportional to that of the incident wave and to the function $\Gamma(\mathbf{k})$. The appearance of $\Gamma(\mathbf{k})$ in the analysis is to be expected from considerations of the mutual interference of the scattered waves from the different volume elements. The Fourier component of $Q(\mathbf{y})$ with vector wave-number $\mathbf{k} = \mathbf{\kappa} - \kappa \mathbf{l}$ is the only one for which the phases of the scattered waves from the various volume elements combine in such a way as to produce a sinusoidal variation of ψ_1 along a line drawn from V in the direction $\mathbf{l}$; all other Fourier components of the distribution of the relevant property of the medium produce scattered waves which annul each other at distance points in the direction $\mathbf{l}$.

Now the properties of the medium vary randomly as a result of the action of turbulence, and the function $\Gamma(\mathbf{k})$, and consequently $\psi_1(\mathbf{x}, t)$, take different values from one realization to another. Only the statistical properties of $\Gamma(\mathbf{k})$ and $\psi_1(\mathbf{x}, t)$

Figure 1

are relevant, and we therefore calculate the mean square of ψ_1 , this being the most significant of these properties. The mean intensity of the scattered wave at position $\mathbf{x}$, expressed as a fraction of the intensity of the incident wave, is

$$\frac{\overline{\psi_1 \psi_1^*}}{A^2} = \frac{\overline{\Gamma(\mathbf{k}) \Gamma^*(\mathbf{k})}}{16\pi^2 x^2}$$

or

$$= \frac{1}{16\pi^2 x^2} \int_V \int_V \overline{Q(\mathbf{y}) Q(\mathbf{z})} e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{z})} d\mathbf{y} d\mathbf{z}. \quad (9)$$

When the physical properties of the medium are approximately stationary random functions of position, and the volume V is large enough to contain many fluctuations of Q , this relation can be simplified to

$$\begin{aligned} \frac{\overline{\psi_1 \psi_1^*}}{A^2} &\approx \frac{V}{16\pi^2 x^2} \int_V \overline{Q(\mathbf{y}) Q(\mathbf{y} + \boldsymbol{\xi})} e^{-i\mathbf{k} \cdot \boldsymbol{\xi}} d\boldsymbol{\xi} \\ &\approx \frac{\pi V}{2x^2} \Phi(\mathbf{k}), \end{aligned} \quad (10)$$

where $\Phi(\mathbf{k})$ is the function usually referred to as the spectrum of the quantity Q .

The form of this expression for the mean intensity of the scattered wave at position $\mathbf{x}$ is worthy of note, not only for its simplicity. It reveals the prominent role played by the spectrum of the relevant property of the medium, showing that this is one of the functions that must be supplied from turbulence theory. A good many authors have analysed the scattering of sound and radio waves in terms of distributions of temperature, etc., in physical space. This makes the mathematical work cumbersome, and conceals the dependence of the intensity of the scattered wave on a very particular aspect of the turbulence. If, for instance, we wish to calculate the scattering of incident waves of high frequency, in directions not too close to that of the incident wave, we see from the above formula that a knowledge of the spectrum function $\Phi(\mathbf{k})$ at large wave-numbers is required. Quite large variations in the form of $\Phi(\mathbf{k})$ at large k correspond to relatively slight changes in the form of the correlation function $\overline{Q(\mathbf{y}) Q(\mathbf{y} + \boldsymbol{\xi})}$ near $\boldsymbol{\xi} = 0$, so that the latter function would need to be known very accurately indeed if it were to be used in a calculation of the scattering.

The flux of energy in the scattered wave in direction $\mathbf{l}$ is sometimes expressed in terms of a quantity $\sigma(\mathbf{l})$ called the "scattering cross-section." The interpretation of our wave-function ψ has been left unspecified, but it will bear the same relation to the energy flux for both the incident and scattered waves.

Thus $\sigma(\mathbf{l}) = (\text{mean flux of energy in scattered wave per unit solid angle in direction } \mathbf{l} \text{ per unit scattering volume}) \div (\text{energy flux in incident wave, at position of scattering volume, per unit area of wave-front})$

$$\begin{aligned} &= \frac{x^2}{VA^2} \overline{\psi_1(\mathbf{x}, t) \psi_1^*(\mathbf{x}, t)} \\ &= \frac{\pi}{2} \Phi(\mathbf{k}). \end{aligned} \quad (11)$$

The fraction of the energy of the incident wave that is lost by scattering, per unit

length traversed by the incident wave-front, that is, the logarithmic decrement of the attenuation by scattering, is then

$$\begin{aligned}\gamma &= \int \sigma(\mathbf{l}) d\mathbf{l} \\ &= \frac{\pi}{2} \int \Phi(\mathbf{\kappa} - \kappa \mathbf{l}) d\mathbf{l}\end{aligned}\quad (12)$$

where the integration is over all directions of the unit vector $\mathbf{l}$.

The special case of sound of very high frequency is worth noting, in view of its common use in under-water instruments and devices. If $\mathbf{\kappa}$ is considerably larger than the reciprocal of the length scale of the inhomogeneities of the medium, $\Phi(\mathbf{\kappa} - \kappa \mathbf{l})$ will be small for all directions of $\mathbf{l}$ except those close to the direction of $\mathbf{\kappa}$. Most of the energy of the scattered wave is thus thrown forwards, at small values of the scattering angle θ . To the first order in the angle of the cone containing most of the scattered energy,

$$\gamma = \frac{\pi}{2\kappa^2} \int \Phi(\mathbf{m}) d\mathbf{m} \quad (13)$$

where $\mathbf{m}$ is a vector in the plane perpendicular to $\mathbf{\kappa}$ and the (two-dimensional) integration is over all values of $\mathbf{m}$. If we choose the x_1 -axis to be parallel to $\mathbf{\kappa}$, this can be written as

$$\begin{aligned}\gamma &= \frac{\pi}{2\kappa^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(o, m_2, m_3) dm_2 dm_3 \\ &= \frac{1}{2\kappa^2} \int_0^\infty \overline{Q(y_1 + \xi_1, y_2, y_3) Q(y_1, y_2, y_3)} d\xi_1,\end{aligned}\quad (14)$$

showing the dependence of γ , in this case, on a kind of integral scale of the quantity Q .

To proceed further we must know more about the right-hand side of the equation that was adopted at the beginning of this discussion, and we therefore turn now to a consideration of the equations that describe the propagation of sound waves. There are two important cases, first a medium of variable physical properties which is stationary except for the disturbance caused by the incident wave, and second a medium of uniform physical properties in turbulent motion. These two cases often occur together in practice, but separate consideration is desirable so that we can see which is likely to be the more important effect.

Stationary medium of variable properties

Provided we can neglect the effects of molecular viscosity, which usually plays the minor role of a small damping agent in these problems, the exact equations describing any kind of fluid motion are

$$\begin{aligned}\frac{D\mathbf{v}}{Dt} &= -\frac{1}{\rho} \nabla p, \\ \frac{D\rho}{Dt} &= -\rho \nabla \cdot \mathbf{v},\end{aligned}\quad (15)$$

where $\mathbf{v}$, ρ and p are the local velocity, density and pressure. Since the changes in the

state of a material element of fluid are adiabatic (in the absence of molecular transfer effects), p and ρ are related by

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} = -\rho c^2 \nabla \cdot \mathbf{v} \quad (16)$$

where c is a parameter describing one of the physical properties of the material element under consideration and is equal to the phase velocity that sound waves would have in a uniform medium having the same physical properties as that element. Thus we have two equations for $\mathbf{v}$ and p , the form of the equations showing that the only two physical properties of the medium that are relevant are the local inertia per unit volume ρ , and the local phase velocity of sound waves c .

Our interest is in sound waves of small amplitude, and we therefore linearize in the departures from the undisturbed state in which $\mathbf{v}$ and ∇p are zero, obtaining

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho} \nabla p \\ \frac{\partial p}{\partial t} &= -\rho c^2 \nabla \cdot \mathbf{v} \end{aligned} \quad (17)$$

where ρ and c are now symbols denoting the local density and acoustic phase velocity of the medium in the undisturbed state, and are independent of t according to the assumption explained earlier. The equation for p alone is

$$\frac{\partial^2 p}{\partial t^2} - \rho c^2 \nabla \cdot \left(\frac{\nabla p}{\rho} \right) = 0, \quad (18)$$

while that for $\mathbf{v}$ is

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot \mathbf{v}) = 0. \quad (19)$$

The velocity field $\mathbf{v}$ is not irrotational when ρ varies with position, but it is clearly possible to define a quasi-potential such that

$$\mathbf{v} = -\frac{1}{\rho} \nabla \varphi, \quad (20)$$

the equation for φ then being the same as that for p . If the length scale on which ρ varies is large compared with the wave-length of the incident sound wave, the equation for p may, for some purposes, be reduced approximately to

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0, \quad (21)$$

which is the ordinary wave-equation with variable phase velocity; however, the Fourier components of the distribution of ρ that are most effective in scattering energy through a large angle are just those with length-scale comparable with the wave-length, so that we shall not make use of this approximation.

If the variations of ρ and c in the medium are relatively small, we can write

$$\rho = \rho_0 + \rho_1, \quad c = c_0 + c_1 \quad (22)$$

where ρ_1 , c_1 are local departures from the uniform mean values ρ_0 , c_0 , and proceed to linearize in ρ_1 and c_1 . The equation for p then becomes

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = (\nabla p) \cdot \left(\nabla \frac{\rho_1}{\rho_0} \right) - \frac{2c_1}{c_0} \nabla^2 p, \quad (23)$$

which is now in the standard form adopted earlier in the lecture, p taking the part of the wave-function ψ .

The relative importance of the two terms on the right-hand side of this equation depends on the nature of the medium, and some discussion of the properties of the two common cases of air and water is needed. In the case of air, variations of ρ and c are likely to be attributable to variations of either temperature or water-vapor concentration, or both. It happens that the ratio of the specific heats of moist air is about the same as that of dry air, so that, for stationary air at uniform pressure, variations of c and ρ are related by

$$c^2 \propto \rho^{-1}, \quad i.e. \quad \frac{2c_1}{c_0} = - \frac{\rho_1}{\rho_0}, \quad (24)$$

irrespective of whether the variations of ρ and c are a consequence of variations of temperature or of water-vapor concentration. Of the two possible causes of variations in ρ , variations in temperature seem likely to be more important in the atmosphere, since the difference in density between completely dry air and saturated air at 10°C is no more than that produced by a variation of temperature of about 1.5°C.

For air, then, the governing equation becomes

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = (\nabla p) \cdot \left(\nabla \frac{\rho_1}{\rho_0} \right) + \frac{\rho_1}{\rho_0} \nabla^2 p, \quad (25)$$

and from this point onwards the analysis of the scattering follows the path already laid down. The function $Q(\mathbf{x})$ of the general analysis is here given by

$$Q(\mathbf{x}) = i\kappa \cdot \nabla \frac{\rho_1}{\rho_0} - \kappa^2 \frac{\rho_1}{\rho_0}, \quad (26)$$

and its Fourier transform is

$$\begin{aligned} \Gamma(\mathbf{k}) &= (\kappa \cdot \mathbf{k} - \kappa^2) \Lambda(\mathbf{k}) \\ &= -\kappa^2 \cos \theta \Lambda(\mathbf{k}), \end{aligned} \quad (27)$$

where $\Lambda(\mathbf{k})$ is the Fourier transform of $\rho_1(\mathbf{x})/\rho_0$. The function $\sigma(\mathbf{l})$ that describes the directional distribution of intensity of the scattered wave is

$$\sigma(\mathbf{l}) = \frac{\pi}{2} \kappa^4 \cos^2 \theta \Psi(\mathbf{k}), \quad (28)$$

where $\Psi(\mathbf{k})$ is the spectrum of the relative density fluctuations $\rho_1(\mathbf{x})/\rho_0$ in the sense in which $\Phi(\mathbf{k})$ was defined to be the spectrum of $Q(\mathbf{x})$.

The method of obtaining more specific results for the case of high frequency sound has already been described; it will be seen to show that the total scattered energy flux is then proportional to κ^2 . Another particular case of some interest is that in which the distribution of density is statistically isotropic (as is always likely to be a valid

approximation, in the atmosphere, for the small-scale components, according to the universal equilibrium theory). $\Psi(\mathbf{k})$ then depends only on the vector magnitude

$$k = 2\kappa \sin \frac{\theta}{2}, \quad (29)$$

and the decrement of the incident wave becomes

$$\begin{aligned} \int \sigma(l) dl &= \pi^2 \kappa^4 \int_0^\pi \cos^2 \theta \Psi \left(2\kappa \sin \frac{\theta}{2} \right) \sin \theta d\theta \\ &= 2\pi^2 \kappa^2 \int_0^{2\kappa} s \left(1 - \frac{s^2}{2\kappa^2} \right)^2 \Psi(s) ds. \end{aligned} \quad (30)$$

A similar set of results can be obtained for water. Variations of ρ and c will arise here from variations of temperature and of salt concentration (and possibly from the presence of air bubbles, in some circumstances). From data given in the textbook by A. B. Wood (1932), it can be calculated that, for a fluctuation δT in absolute temperature under normal conditions,

$$\frac{\delta \rho}{\rho} \approx -0.06 \frac{\delta T}{T}, \quad \frac{\delta c}{c} \approx 0.80 \frac{\delta T}{T}, \quad (31)$$

and, for a fluctuation δC about the normal level of salt concentration (roughly 0.035 gm salt/cm³ water),

$$\frac{\delta \rho}{\rho} \approx 0.025 \frac{\delta C}{C}, \quad \frac{\delta c}{c} \approx 0.025 \frac{\delta C}{C}. \quad (32)$$

The relative importance of all these separate influences on the scattering will probably depend on the prevailing conditions, but it seems likely that temperature fluctuations in the sea will often be more effective in changing ρ and c than fluctuations in salt concentration, and that temperature fluctuations will be relevant to sound scattering primarily through their influence on c . At any rate, I shall assume so here for illustrative purposes.

With this assumption, we can put $\rho_1/\rho_0 \approx 0$ and $c_1/c_0 \approx 0.8 \frac{T_1}{T_0}$, and the equation for the sound pressure becomes

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = -1.6 \frac{T_1}{T_0} \nabla^2 p, \quad (33)$$

which is simply the ordinary wave equation with non-uniform phase velocity. We can now proceed to apply the general analysis of scattering once again. The function $Q(\mathbf{x})$ is here given by

$$Q(\mathbf{x}) = 1.6\kappa^2 \frac{T_1(\mathbf{x})}{T_0}, \quad (34)$$

and the associated scattering cross-section is

$$\sigma(l) = 1.28\pi\kappa^4 \Omega(\mathbf{k}), \quad (35)$$

where $\Omega(\mathbf{k})$ is the spectrum of the relative fluctuations in absolute temperature $T_1(\mathbf{x})/T_0$ in the sense in which $\Phi(\mathbf{k})$ and $\Psi(\mathbf{k})$ are the spectra of $Q(\mathbf{x})$ and $\rho_1(\mathbf{x})/\rho_0$.

The only difference between the scattering cross-sections for stationary air and sea-water, apart from the difference in the numerical factors, lies in the appearance of the $\cos^2\theta$ factor in the former but not in the latter. In the case of sea-water, the scattered energy flux is fairly evenly distributed over all directions, but in the case of air

there is a minimum at the lateral plane $\theta = \frac{\pi}{2}$. The expressions for the total energy

flux $\int \sigma(l) dl$ in the cases of air and water should not differ by more than a factor of order unity as a result of the presence or absence of the $\cos^2\theta$. Much more substantial differences can occur as a result of differences between the forms, and the magnitudes, of the spectra of density fluctuations in the one case and temperature fluctuations in the other. However, these differences cannot be described in a general way since they depend on particular considerations such as the value of the sound frequency.

Uniform medium in turbulent motion

A discussion of the effect of turbulent fluctuations of velocity on the propagation of sound waves encounters many difficulties, and a good deal of care is necessary in the use of approximations. Many papers have been written on the problem, and three people have independently derived the cross-section for single scattering correctly. The first was Blokhintzev in 1945, who made use of the equation for the propagation of sound in turbulent flow derived by Obukhoff two years earlier, and the others were Lighthill and Kraichnan in 1953. These people used completely different kinds of argument, and it takes a good deal of work to show that in fact they used the same assumptions and arrived at the same results. None of these pieces of work seems yet to have been assimilated into the literature of the subject. By standing on the shoulders of these pioneers I have devised my own derivation, which I believe to be simpler and freer from logical imperfections. The result is important enough to warrant continued attempts to improve the argument.

One or two numbers will help us first to see what is perturbing what, in this combination of sound and turbulence. The r.m.s. particle velocity in a plane sound wave of N decibels sound level in air is about $5 \times 10^{-6} \times 10^{N/20}$ cm/sec, which comes to 0.5 cm/sec for a very loud noise of 100 decibels. On the other hand, the r.m.s. of the turbulent velocity fluctuations in the lower levels of the atmosphere is often in the neighbourhood of 50 cm/sec. Thus the fluid velocity in the turbulence field alone is very much bigger than that in the sound field alone. However, this is not necessarily true also of the corresponding fluctuations in density and pressure, since density variations in a sound wave increase as the first power of the accompanying particle velocity whereas in a turbulent motion they increase as the second power. Use of the above figures for the atmosphere shows that the density fluctuations in the sound and turbulence fields alone are comparable in magnitude. Similar calculations for underwater sound waves, having intensities like those commonly used in connection with ranging equipment, show that here again the sound waves are likely to be accompanied by particle velocities much smaller than, and density fluctuations comparable to, those produced by turbulence in the sea.

Despite this approximate parity of the separate density variations, the incident sound wave cannot have much effect on the turbulent velocity field. The shearing motions of the turbulence are determined primarily by the boundary conditions on velocity, and are not affected appreciably by superimposed relative density variations of order M_t^2 (where M_t is the Mach number of the turbulence alone)—as indeed can be seen from the fact that the velocity distribution in the pure turbulence is known to be approximately independent of M_t . Thus the velocity distribution in the combined incident sound and turbulence field is approximately the same as that in the turbulence alone.

We shall find it convenient to write the density, pressure and velocity in the combined incident sound and turbulence field in the following form:

$$\rho = \rho_0 + \rho_t + \rho_s, \quad p = p_0 + p_t + p_s, \quad \mathbf{U} = \mathbf{u} + \mathbf{v}, \quad (36)$$

where ρ_0 and p_0 are the uniform density and pressure in the completely undisturbed medium, and $\rho_0 + \rho_t$, $p_0 + p_t$, $\mathbf{u}$ are *exactly* the values of the density, pressure and velocity in the turbulent field in the absence of the incident sound wave. From what has already been said, $|\mathbf{v}| \ll |\mathbf{u}|$ although $|\rho_s|$ and $|p_s|$ may be comparable with $|\rho_t|$ and $|p_t|$. Moreover, since the Mach numbers of the incident sound wave alone (M_s) and of the turbulence alone (M_t) are both small, we can safely assume that

$$|\rho_t + \rho_s| \ll \rho_0, \quad |p_t + p_s| \ll p_0. \quad (37)$$

We shall ignore again the effects of molecular conductivity, so that the motions are adiabatic and

$$\begin{aligned} \frac{p_s}{p_0} &= \left(1 + \frac{\rho_t + \rho_s}{\rho_0} \right)^\gamma - \left(1 + \frac{\rho_t}{\rho_0} \right)^\gamma \\ &\approx \frac{\gamma \rho_s}{\rho_0} + \frac{1}{2} \gamma (\gamma - 1) \left[\left(\frac{\rho_t + \rho_s}{\rho_0} \right)^2 - \left(\frac{\rho_t}{\rho_0} \right)^2 \right]. \end{aligned} \quad (38)$$

These are the approximations that we can now use in the equations of motion.

The equations for the quantities ρ , p , $\mathbf{U}$ relating to the combined field can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U_i)}{\partial x_i} = 0, \quad (39)$$

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \quad (40)$$

without approximation, apart from the neglect of the effects of viscosity. A combined form of these equations is

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p - \frac{\partial^2(\rho U_i U_j)}{\partial x_i \partial x_j} = 0. \quad (41)$$

This equation is satisfied when ρ , p , $\mathbf{U}$ are given by (36), and also when

$$\rho = \rho_0 + \rho_t, \quad p = p_0 + p_t, \quad \mathbf{U} = \mathbf{u}, \quad (42)$$

so that

$$\frac{\partial^2 \rho_s}{\partial t^2} - \nabla^2 p_s - \frac{\partial^2}{\partial x_i \partial x_j} [\rho_s(u_i + v_i)(u_j + v_j) + (\rho_0 + \rho_t)(2u_i v_j + v_i v_j)] = 0, \quad (43)$$

again without approximation. ρ_s and p_s are the increments in density and pressure in the combined field due to the incident sound wave, and they are unlikely to be smaller than the corresponding increments in the absence of the turbulence. Thus ρ_s/ρ_0 and p_s/p_0 are at least of order M_s . Examination of the magnitudes of the different terms of this equation in the light of the above permissible approximations, together with a little (uncontentious) guesswork about the magnitudes of the spatial derivatives of $\mathbf{u}$, then shows that to a first approximation the equation reduces to

$$\nabla^2 \frac{\rho_s}{\rho_0} - \frac{1}{c_0^2} \frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} = 0. \quad (44)$$

The boundary conditions on ρ_s/ρ_0 are those appropriate to the circumstances of generation of the incident sound wave. Thus to a first approximation ρ_s , p_s and $\mathbf{v}$ have the same values as for the incident sound wave in a stationary medium.

It appears, then, that the interaction between the sound and turbulence fields is weak, and that they each perturb the other. This lack of coupling is essentially a consequence of the big difference in the time scales on which changes take place in the two fields.

The equation that describes the perturbation of the incident sound wave due to the presence of the turbulence is obtained by going to the second approximation of the above exact equation for ρ_s ; thus

$$\begin{aligned}\nabla^2(\rho_s/\rho_0) - \frac{1}{c_0^2} \frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} &= - \frac{2}{c_0^2} \frac{\partial^2(u_i v_j)}{\partial x_i \partial x_j}, \\ &= - \frac{2}{c_0^2} \frac{\partial}{\partial x_j} \left(u_i \frac{\partial v_j}{\partial x_i} \right)\end{aligned}\quad (45)$$

to a consistent approximation. Terms on the left-hand side of this equation are of order M_s/λ^2 , that on the right-hand side is of order $M_s M_t/\lambda^2$ (assuming, for the sake of definiteness, that the spatial derivatives of $\mathbf{u}$ have a magnitude of an order equal to that of the r.m.s. of $\mathbf{u}$ divided by λ), and the largest term of those neglected is of order M_s^2/λ or $M_s M_t^2/\lambda^2$, whichever happens to be the larger.

It should be noted that this equation describing the changes in the incident sound wave contains approximations additional to the conventional assumptions of linearized sound theory, unlike the corresponding equations obtained earlier for the various cases of a non-uniform stationary medium. These additional approximations may have consequences for the perturbation procedure on which the scattering analysis is based. While the equation certainly allows the investigation of single scattering, an investigation of multiple scattering from the same equation would need further justification.

For the investigation of single scattering, $\mathbf{v}$ on the right-hand side can be expressed in terms of ρ_s/ρ_0 by the ordinary linear acoustic equation

$$\partial \mathbf{v} / \partial t = - c_0^2 \nabla(\rho_s/\rho_0),$$

and so, for an incident harmonic plane wave of the usual form, the equation becomes

$$\nabla^2(\rho_s/\rho_0) - \frac{1}{c_0^2} \frac{\partial^2(\rho_s/\rho_0)}{\partial t^2} = \left(\frac{2u_i}{c_0} \kappa \kappa_i - \frac{2i}{c_0} \frac{\partial u_i}{\partial x_j} \frac{\kappa_i \kappa_j}{\kappa} \right) \frac{\rho_s}{\rho_0}. \quad (46)$$

This equation is in the standard form used earlier for the general analysis of scattering. The function $Q(\mathbf{x})$ introduced there is here given by

$$Q(\mathbf{x}) = \frac{2u_i}{c_0} \kappa \kappa_i - \frac{2i}{c_0} \frac{\partial u_i}{\partial x_j} \frac{\kappa_i \kappa_j}{\kappa}, \quad (47)$$

and the scattering cross-section is

$$\sigma(l) = 2\pi \kappa^4 \cos^2 \theta \frac{\kappa_i \kappa_j F_{ij}(\mathbf{k})}{\kappa^2 c_0^2}, \quad (48)$$

where $F_{ij}(\mathbf{k})$ is the spectrum of the turbulent velocity $u_i(\mathbf{x})$ in the sense in which $\Phi(\mathbf{k})$ was defined to be the spectrum of $Q(\mathbf{x})$. $\frac{\kappa_i \kappa_j}{\kappa^2} F_{ij}(\mathbf{k})$ is simply the spectrum of

the component of turbulent velocity in the direction of propagation of the incident sound wave, showing that only one component of the turbulent velocity is relevant to single-scattering of a plane wave. This expression for $\sigma(\mathbf{l})$ is effectively the same as the results obtained by Blokhintzev, by Lighthill, and by Kraichnan.

It will be noticed that the single-scattering due to a turbulent velocity field $\mathbf{u}(\mathbf{x})$ in a medium of uniform physical properties is exactly the same as that due to a density variation $\rho_1(\mathbf{x})$ in stationary air provided

$$\frac{\kappa \cdot \mathbf{u}(\mathbf{x})}{\kappa c_0} = -\frac{1}{2} \frac{\rho_1(\mathbf{x})}{\rho_0} \quad (49)$$

This is not an entirely unexpected result, since the two sides of this relation represent the local fractional increases in apparent phase velocity of the incident wave as a result of the non-uniformity of the medium, the increase occurring in one case by bodily convection and in the other by a change in the physical properties of the medium. However, lest this result should begin to seem obvious, may I remind you that it holds only for single-scattering and only for the case of air, for which variations in inertia and elasticity are connected in a particular way. It should be noted, moreover, that $\kappa \cdot \mathbf{u}/\kappa c_0$ and $-\rho_1/2\rho_0$ would not have identical distributions (even statistically) in two real cases, since one quantity is subject to the kinematical requirements of the continuity condition and the other is not. A comparison of the relative importance of the scattering effects of turbulent velocities and of variations of density or temperature, as they occur in reality, would require a consideration of turbulent spectra, which lies outside the scope of this lecture. So far as I know, an analytical comparison of this kind has not yet been made.

Particular cases can now be discussed as before. When the turbulent motion is isotropic, we have

$$F_{ij}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{E(k)}{4\pi k^2}, \quad (50)$$

where $E(k)$ is the usual energy spectrum function, and hence

$$\sigma(\mathbf{l}) = \frac{\kappa^2 \cos^2 \theta}{8 \tan^2 \theta/2} \frac{E(2\kappa \sin \theta/2)}{c_0^2}. \quad (51)$$

When k is very small, $E(k)$ is proportional to k^4 and so σ falls to zero at the straight-ahead direction $\theta = 0$ (as is also true when the turbulence is not isotropic). For an incident sound wave of very high frequency, and turbulence which is not necessarily isotropic, the total scattered power is given approximately by

$$\int \sigma(\mathbf{l}) d\mathbf{l} = 2\pi \kappa_i \kappa_j \int \frac{F_{ij}(\mathbf{m})}{c_0^2} d\mathbf{m} \quad (52)$$

where $\mathbf{m}$ is a vector in the plane at right angles to κ . This expression can be written as

$$\int \sigma(\mathbf{l}) d\mathbf{l} = 2\kappa^2 \frac{\overline{u_\kappa^2}}{c_0^2} L_\kappa, \quad (53)$$

where $\overline{u_\kappa^2}$ stands for the mean-square of the turbulent velocity component parallel to κ , and L_κ is the integral scale in this same direction, defined in the usual manner.

This brings me to the end of my account of scattering of sound waves due to turbulence. It is not a particularly dramatic end, and no startling conclusions or new

ideas have been evolved. What I have tried to do is to present reasonably reliable analysis of the process of single-scattering of sound waves in some important cases. It took me a long time to make sense out of the large and confused literature on the problem, and I hope that the account I have given will save other people from having to spend the same amount of time.

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DISCUSSION

D. Mintzer

I would like to make a few comments about a possible new experimental technique which may be useful in analyzing turbulence. As shown in Dr. Batchelor's paper, and in previous published works, we now have a good understanding of the scattering of sound waves by turbulence. This leads me to believe that we should start thinking of sound scattering as a technique for determining correlation functions in media which are randomly inhomogeneous. It has been shown from scattering experiments that one can determine values for some of the variables of interest, such as the scale of turbulence (and characteristic times, in time-varying problems). This may turn out to be a very useful technique, especially in large-scale turbulence, such as in the ocean and the atmosphere.

As a simple example, you can show that by taking the differential scattering cross-sections in two directions close to each other, you can obtain information about the derivatives of the energy spectrum of the turbulence.

R. H. Kraichnan

There are two subjects I should like to discuss briefly in connection with Dr. Batchelor's interesting talk: the domain of validity of sound-turbulence scattering theory based on the first Born approximation, and the effect on the scattering process of the time-dependence of the turbulence field.

As Dr. Batchelor pointed out, the errors implied in the first Born (single-scattering) approximation are not wholly separable from the errors involved in arriving at the usual isentropic inhomogeneous wave-equation for the scattering of sound by turbulence. The equations of conservation of mass and momentum do not alone provide a complete description of the dynamics of a compressible fluid; they must be supplemented by an expression for the internal energy function of the fluid and a law of entropy flow and production. A general consequence is a multi-valued relationship between density and pressure in the fluid. If one makes the usual adiabatic approxi-

mation, a single-valued relationship is established between pressure and density, and there results a complete set of equations for the fluid motion in which the entropy does not appear explicitly.

The magnitude of the errors introduced both by an isentropic approximation and by later use of a single-scattering approximation to solve the associated wave-equation, may be estimated from the *exact* inhomogeneous wave equation for the pressure field, the right side of which will involve density, pressure and velocity. One may transform this equation to the corresponding integral equation and determine the contributions of successive orders of an appropriate iteration procedure, using the other equations of motion for relations between pressure, density, and velocity.

For examination of a scattering process, the appropriate starting point for the iteration could be zeroth order density, pressure, and velocity fields which are the sums of the fields corresponding to an infinitesimal sound wave and very low Mach number turbulence, together with the entropy field associated with these infinitesimal disturbances. It is to be noted that the infinitesimal sound wave represents a solution of the *homogeneous* wave-equation; because the integral equation for the pressure field provides for explicit introduction of this homogeneous part of the solution (automatically maintaining proper boundary conditions at infinity during iteration), it is a more suitable instrument for investigating orders of magnitudes of correction terms than the differential equation, for which boundary conditions at infinity must be taken account of at every step.

Now, if the amplitudes of sound wave and turbulence are *sufficiently small*, and if the volume in which scattering takes place is *sufficiently small*, the first Born approximation—the solution resulting from a single iteration starting with the zeroth order fields described—may be justified (R. H. Kraichnan, J. Acoust. Soc. Am. 25, 1096 (1953)). This implies that departures from isentropic conditions, and what is loosely termed “multiple scattering” will both have small effect on the scattered wave, for both these phenomena here show up in higher steps of the iteration procedure.

The expression “sufficiently small” which I have used requires discussion. It is notoriously difficult in physics to give compact and generally applicable sufficient conditions for the validity of the first Born approximation. An obvious (but sometimes ignored) necessary condition can be stated however: The difference between the exact field and the zeroth order field must be small compared to both throughout the region of scattering. When an applicable criterion of this sort is not satisfied, and the first Born approximation breaks down, it is frequently meaningful to ask to what extent the breakdown is due to nonisentropic conditions or to inappropriate procedure in solving the isentropic equations. The two effects (each represented by terms in the iteration expansion of the source function for the pressure field) are often fairly distinct physically.

For a simple example, consider a plane sound wave propagating through a very large region of homogeneous turbulence. If the turbulence is very weak, the error in the first Born approximation may be ascribable principally to a non-isentropic effect—the attenuation of the sound wave by viscosity. If the turbulence is stronger, however, the attenuation of the plane sound wave may be due principally to energy scattered out of the wave by the turbulence, and an accurate description of this could be obtained by applying better scattering approximations to the isentropic equations. In either of these simple cases, provided there is small attenuation within the correlation length of the turbulence, a better description may be obtained by using a “doctored” Born approximation in which the zeroth order sound field is exponentially damped. (This may be considered the first Born approximation to an inhomogeneous wave equation in which the homogeneous part contains a damping term.)

On the other hand, in the interaction of intense sound waves with strong turbulence, as in a high-speed turbulent jet, it is not very meaningful to speak separately of non-isentropic effects and multiple scattering effects. In this case, in fact, the distinction

between sound and turbulence is unclear, although one may distinguish a transverse velocity and a longitudinal velocity. What is most important to keep in mind is that generalizations about the validity of a particular scattering approximation are risky, and the self-consistency of a given approximation should be investigated for the application at hand. For example, in certain applications the scattering due to turbulent density fluctuations may be negligible, in others not.

Dr. Batchelor noted some similarities between scattering by random scalar inhomogeneities (scattering of radio and light waves by turbulent refractive index fluctuations) and the scattering of sound by turbulence. I should like to stress that there are essential differences in the range of validity of the Born approximation in the two cases. In the scalar case, given a size of the scattering volume, the first Born approximation can remain valid at indefinitely high frequency if the scalar index fluctuations are small. In the sound-scattering case, the interaction between sound and turbulence involves quadratically the *gradients* of the velocities, thereby leading to a strong divergence of the first Born approximation cross-section toward short sound wave-lengths. By the criterion mentioned previously, this approximation is therefore not valid for indefinitely short wave-lengths; for example, it cannot properly be applied to the scattering of energy out of a shock-wave traversing turbulence. In some cases, the sound-scattering at very short wavelengths is satisfactorily described by a "physical optics" approximation in which the corrugation of coherent wavefronts is computed. Here again, *safe* criteria of validity are not easy to state.

Now I should like to turn briefly to the effects of time-variation of the turbulence. Although in most cases where electromagnetic or sound waves are scattered by turbulence the characteristic frequencies of the turbulence are much smaller than the wave frequency, it does not follow that the effect of the time variation is unimportant. Slow time dependence will not alter the time-average scattering cross-sections, but it will alter the time-structure and spectrum of the scattered radiation. In general, the scattered radiation will be smeared over a frequency bandwidth range the order of the characteristic frequencies of the turbulence, and it will exhibit fading times the order of the characteristic period of the turbulence. In the scattering of sound by turbulence the spreading in frequency of the first Born approximation scattered wave has an interesting angular dependence; the energy scattered in nearly forward directions is very little spread in frequency, compared to radiation at larger scattering angles.

Further effects of the time-structure of the turbulence include phenomena of energy exchange between wave field and turbulence. For electromagnetic waves, these effects are not of much interest in nonionized media, but they may be significant for the propagation of sound through turbulence, especially in large regions. The energy exchange has two aspects: radiation of broad-band acoustic noise by the turbulence, and transformation of energy from the incident sound field into turbulent kinetic energy (that is to say, the scattering is not exactly "elastic," or conservative). The relative magnitude of the two effects depends on the ratio of turbulence and sound particle velocity Mach numbers, and on the spectral structure of the fields.

Both of these effects tend to degrade an initially monochromatic sound field by increasing the ratio of energy in diffuse spectrum to that in the line spectrum. For strong enough turbulence, the noise radiation may actually contain much more energy than the scattered sound wave, although the latter may still be distinguishable because of its sharper frequency structure. When Mach numbers of both sound and turbulence are high, the distinction between scattered and radiated energy is not very meaningful.

Finally I should like to comment on Dr. Mintzer's proposal for investigation of turbulence structure by scattering experiments. It is of interest in this connection that different information would be obtainable in principle by the scattering of electromagnetic or acoustic waves in the same wavelength range, because the coupling is scalar in one case and tensor in the other.

I hadn't actually prepared anything, but I had a feeling Dr. Batchelor would say some interesting things, and I was right.

There is this question about any differences between the approach he has described and the approaches that Dr. Kraichnan and myself derived in 1953. I found it a little difficult to see just what the differences were, in the mathematical details of what he showed us and the thing that I did.

But I would like just to mention something which isn't connected with mathematical details, but rather with a physical interpretation which I found helped me, though it won't necessarily help everyone. That was this.

When Lord Rayleigh worked out the explanation of the blue colour of the sky, he showed that this was due to variations of refractive index. Later, the theory of refractive index was worked out, and it was seen that this was due to polarization of dipole molecules in the direction of the incident beam. So we can say that this scattering of light (with some preference for the shorter waves) is due to polarization of molecules in a random sort of way, with some sort of spatial correlation.

The point of view from which I found it helpful to look at the problem we have just heard about was this. We have the following equation which Dr. Batchelor used—I deliberately leave out any question of variation of properties and just leave in the terms

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j} (\rho u_i u_j). \quad (1)$$

We have this equation, and this equation also came out in the theory of aerodynamic noise, and one said that the sound produced by turbulence was effectively that due to the distribution of quadrupoles indicated on the right of (1), namely quadrupoles of strength

$$\rho u_i u_j \quad (2)$$

per unit volume.

Now, Dr. Batchelor pointed out that in the presence of an incident wave the term inside the second derivative in (1) would become

$$\rho(u_i + v_i)(u_j + v_j), \quad (3)$$

where v_i is the velocity field of the incident sound wave. We can write (3), the new quadrupole strength per unit volume, as

$$\rho(u_i u_j + u_i v_j + v_i u_j + v_i v_j). \quad (4)$$

Now, here you have got four lots of quadrupoles, and we can interpret them physically as follows. The first term (see (2)) represents the sound that is being generated by the turbulence itself, which we are not concerned with. The middle two terms represent the scattered sound due to the interaction of the turbulence and the sound beam. And the last term $\rho v_i v_j$ represents the effect of the sound wave on itself, which we know is a steepening, tending to produce shock waves, and so on, but will be very small in the case of ordinary weak waves.

The middle two terms

$$\rho(u_i v_j + v_i u_j) \quad (5)$$

are the part we are interested in, and it struck me that a way of looking at this part would be to say that here you had quadrupoles that were polarized in the direction of the incident sound beam v_i , you see. Well, only one axis of each quadrupole was polarized—in the first term of (5) it is the v_j , in the second the v_i . But you have an analogy in this to the case of light scattering.

I may, perhaps, conclude with some remarks concerning multiple scattering. If β is the proportion of sound energy scattered per unit distance, then at a distance l from the source the scattered sound will be a substantial proportion of the whole if βl is not small. In this case one must, and may, consider multiple scattering by successive use of results from single-scattering theory.

If βl is actually large, then most of the sound that is present will have been scattered many times, and may be thought of as having described a crooked path like the "random walks" treated by statisticians.

The particular kind of random walk which is involved has been insufficiently studied. However, to one simple question about it there is a rather simple answer.

If we ask, what change of direction on the average will a ray of sound undergo (due to multiple scattering) in travelling over a *total* distance (no longer the distance from the source "as the crow flies", but somewhat greater), then we have first that a proportion

$$e^{-\beta l} \frac{(\beta l)^n}{n!} \quad (\text{Poisson's distribution}) \quad (6)$$

of the energy carried by such rays will have been scattered n times. During this process their *direction* may be regarded as performing a random walk (on the unit sphere). If $\frac{1}{2}p(n, \theta) \sin \theta d\theta$ is the probability that after n scatterings the direction will make an angle between θ and $\theta + d\theta$ with the initial direction (where the factor $\frac{1}{2}\sin\theta$ has been inserted to ensure that for a uniform directional distribution $p = 1$), then p satisfies approximately the partial differential equation (of diffusion)

$$\frac{\partial p}{\partial n} = \frac{1}{4\overline{\omega^2}} \frac{\partial}{\sin \theta} \left(\sin \theta \frac{\partial p}{\partial \theta} \right) \quad (7)$$

where $\overline{\omega^2}$ is the mean square deviation in direction at each scattering. The solution of (7) such that the direction is initially $\theta = 0$ is

$$p = \sum_{m=0}^{\infty} (2m+1) P_m(\cos \theta) e^{-\frac{1}{2}\overline{\omega^2} m(m+1)n} \quad (8)$$

Summing the product of (6) and (8) for $n = 0$ to ∞ , we obtain the distribution of direction after propagation through a distance l , as

$$\begin{aligned} p &= \sum_{m=0}^{\infty} (2m+1) P_m(\cos \theta) e^{-\beta l} \left(1 - e^{-\frac{1}{2}\overline{\omega^2} m(m+1)} \right) \\ &= 1 + 3 \cos \theta e^{-\beta l(1-\frac{1}{2}\overline{\omega^2})} + \dots \end{aligned} \quad (9)$$

Equation (9) shows that to make the sound directionally uniform ($p = 1$) needs a distance

$$l \gg \frac{1}{\beta(1 - e^{-\frac{1}{2}\overline{\omega^2}})} \doteq \frac{2}{\beta\overline{\omega^2}}. \quad (10)$$

For homogeneous isotropic turbulence this condition may be written approximately as

$$l \gg \frac{2a^2}{\pi \int_0^{\kappa} k E(k) dk} \quad (11)$$

where a is the velocity of sound, κ the radian wave-number of the incident wave, k that of the turbulence, and $E(k)$ its three-dimensional energy spectrum.

G. K. Batchelor

I think that Professor Lighthill's remarks about double and multiple scattering take it for granted that an investigation by the usual iterative procedure is permissible. In the particular case of sound waves scattered by turbulent motion of the medium, it is not yet proven that this procedure is valid beyond the stage of single scattering. To say again in one sentence what I said in the lecture, if one takes the quantities appropriate to the scattered wave, and inserts them into the right-hand side of the basic equation, one may obtain terms that are of the same order of magnitude as those that have already been neglected in deriving the equation; in this way, an inconsistency may arise.

R. Kraichnan

One of the problems may be that it is a very broad problem, trying to decide which higher order effects are important where. Take the specific problem Lighthill brought up: a region which receives no single scattering at all. Then it is quite possibly safe to use the double scattering approximation to find out what happens there. But if you consider a region which can receive both multiple and single scattering, and where the scattered energy accumulates from large volumes, you must take account of slight deviations from single scattering results, and this requires taking into account neglected terms in the differential equation. Then, there are a lot of high order terms that come in.

It is extremely complicated to separate all of these high order effects.

G. K. Batchelor

May I add a word concerning Professor Lighthill's remark earlier that he wasn't quite clear why I preferred my derivation to the ones that he and Dr. Kraichnan have given? I am sure this needs more informal argument over a table, rather than discussion here, but I can perhaps give a qualitative description of the reason.

There are really two reasons. The more obvious one concerns Professor Lighthill's presentation, and arises from the history of his work on scattering. His work was done when he was in full pursuit of his theory of the generation of aerodynamic noise, which, as everybody now recognizes, has made a fundamental advance in our understanding of that problem. An essential and important part of that theory is the recognition that the generation of noise by turbulence is equivalent to the action of quadrupole acoustic sources. Then, when he came to write his paper on the scattering of sound by turbulence, he worked out the analysis within the framework of the theory of aerodynamic noise, and interpreted the scattered sound in terms of the action of quadrupole sources.

I avoided that presentation because I think it makes for unnecessary complication. In this problem of scattering, it is not really necessary to know that the turbulent fluctuations in velocity are equivalent in their action to quadrupole sources of sound. The multi-pole character of the equivalent sources does not play an important part in scattering theory, essentially because the wavelength of the sound is here comparable with the length-scale of the turbulence, whereas the sound generated by turbulence has a much larger wave-length.

I think that probably Professor Lighthill will agree with that.

M. J. Lighthill

Yes.

G. K. Batchelor

The second reason applies mostly to Dr. Kraichnan's work. I felt that he was

not entirely rigorous about his decomposition of the motion into turbulence on the one hand and sound waves on the other.

The basic exact equation for a motion in which the total pressure, density and velocity are p , ρ and $\mathbf{U}$ is

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 p = \frac{\partial^2 (\rho U_i U_j)}{\partial x_i \partial x_j}.$$

We wish to obtain from this equation an estimate of the effect of a given field of turbulence on an incident sound wave, and a perturbation procedure corresponding to the different orders of scattering. One cannot simply assume, as Dr. Kraichnan seems to have done, that the density and pressure fluctuations are due mainly to the incident sound wave, and that the term on the right-hand side of this equation is small compared with either of the other two terms when values of p , ρ , $\mathbf{U}$ appropriate to the combined turbulence-sound field are substituted in the equation. This is not a correct procedure because, as I showed with the help of some typical values, the fluctuations in p and ρ due to the turbulence alone may be as large as those due to the incident sound wave alone. For the combined turbulence-sound field, the term on the right-hand side may be as large as the other two terms and one cannot begin the perturbation procedure by neglecting the right-hand side.

An equation whose solution has the incident sound wave as its first approximation can be obtained only by inserting in the above equation first the quantities appropriate to the combined turbulence-sound field and second those exactly appropriate to the turbulence alone, and by subtracting the two resulting equations. Only in this way can the unwanted density and pressure variations arising from the turbulence alone be removed from the problem. Of course, one is now left with dependent variables which cannot immediately be given a physical interpretation, since they are simply the increments in pressure, density and velocity that result from the incidence of the sound wave on the existing turbulent motion; however, as I showed in my lecture, the first approximation to this new equation does indeed describe the incident sound wave, and the way is open for an investigation of sound scattering due to the turbulence by the usual perturbation procedure (at least as far as the stage of single scattering). In this way, it seems to me, procedure is made a little more rigorous, and a little clearer, than in the two existing presentations.

R. Kraichnan

I think the whole idea of the decomposition into "turbulence" and "sound" is intrinsically non-rigorous, and at least in the formal structure of the theory, I tried to avoid that by making a rigorous decomposition into transverse and longitudinal velocity fields. If you do that, and work not with the differential equation where the fact is perhaps obscured that the Laplacian operator and time operator largely cancel each other—but with the integral formulation of that equation and consider explicitly the contribution of the homogeneous solution, you can rigorously establish a scale of order of magnitude of the different terms involved. This may have been done sketchily in my paper, but it can be done quite rigorously, and I think that is the fundamental way to attack the problem of rigor.

From The Floor

I would like to throw one more question at Dr. Batchelor or any other gentleman, and that is how much formal analysis they might have done about the time dependent case.

If you want to get the broadening of a single line, you have to get into the space-time correlation of the fluctuations in order to get the line broadening in the scattered energy.

Have you done anything about that?

G. K. Batchelor

I think Dr. Kraichnan is the only one to have given thought to that.

R. Kraichnan

I did this formally, but as to what exactly the time dependence is, we don't know too much about that. Guesses about that are probably less secure than about the space-correlation.

From the Floor

You have the formula with the space-time correlation?

R. Kraichnan

Yes, a four dimensional treatment.

From the Floor

I am not going to attempt to discuss Dr. Batchelor's valuable contribution to the study of wave scattering due to turbulence, but there is a practical observation I should like to make, and I don't think it is entirely irrelevant.

It concerns the interpretation of the twinkling of sound transmitted in the ocean. In the sea, of course, we have a moving boundary, the surface of the sea, and a fixed surface, the bottom, and it may well be that these have a greater effect on the twinkling under certain conditions than does the volume scattering.

D. Mintzer

It is true that at long ranges, where you may have reflections from the surface, this will play an important role; however, there are many cases, including several reported in the literature, in which the sound wave goes completely through the volume without touching the surface; in this case the simple formulation is correct.

I will make another comment, concerning time correlation problems, which someone previously mentioned. The scattering of sound in a time-varying randomly-inhomogeneous medium has been done for the case of scalar inhomogeneities.

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