NOTE ON THE SO-CALLED VAHLEN RELATIONS BETWEEN THE MINORS OF A MATRIX.

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(1) Relations between the elements of a rectangular array, though not looked upon as such, make their appearance very early in the history of determinants. In 1748, for example, Fontaine, using a peculiar notation for $|a_1 b_2|$, states in regard to the array

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{array}$$

the identity

$$|a_1 b_2| \cdot |a_3 b_4| - |a_1 b_3| \cdot |a_2 b_4| + |a_1 b_4| \cdot |a_2 b_3| = 0 \quad . \quad . \quad (a)$$

In 1779 along with a series of related results Bezout gives in regard to the array

$$\begin{array}{ccccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{array}$$

the identity

$$|a_1 b_2 c_3| \cdot |d_1 e_2 f_3| - |a_1 b_2 d_3| \cdot |c_1 e_2 f_3| + |a_1 c_2 d_3| \cdot |b_1 e_2 f_3| - |b_1 c_2 d_3| \cdot |a_1 e_2 f_3| = 0 \quad (\beta)$$

And, again, in 1809, Monge, in regard to the 3-by-5 array,

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{array}$$

gives five relations of the form

$$|a_1 b_2 c_3| \cdot |a_1 d_2 e_3| - |a_1 b_2 d_3| \cdot |a_1 c_2 e_3| + |a_1 b_2 e_3| \cdot |a_1 c_2 d_3| = 0 \quad . \quad . \quad (\gamma)$$

each one of the ten minors of the array being involved three times.

(2) Of course, these three identities were soon recognised as belonging to the same family, and were included in one general identity. To this result several writers contributed; but the theorem is spoken of somewhat loosely as Sylvester's, and as giving an expression for the product of two n -line determinants in the form of an aggregate of like products.

Fontaine's is the case of this for $n = 2$; Bezout's is the case for $n = 3$; and Monge's is obtainable from Bezout's by putting the f 's equal to the corresponding a 's. It is important to note, however, that Monge's is with at least equal interest viewable as what is technically called an "extensional" of Fontaine's. Thus, turning to (γ) and dropping the a_1 in it which begins every one of the six minors, we find remaining

$$|b_2 c_3 \cdot d_2 e_3| - |b_2 d_3| \cdot c_2 e_3 + |b_2 e_3 \cdot c_2 d_3| = 0,$$

which is the equivalent of (α) .

(3) It would appear that for long this theorem of Sylvester's formed the usual means of investigating relations between the primary minors of an oblong array. All such investigations therefore dealt with relations that were quadratic; any others, like Monge's cubic relations, may be passed over, because they were directly deducible from those of lower degree.

In the year 1893, however, Vahlen published his paper professedly devoted to the subject, the title being *Ueber die Relationen zwischen den Determinanten einer Matrix* (*Crelle's Journal*, cxii, 306-310*), and he struck out a new course. The theorem which he used was one regarding the multiplication of one n -line determinant by the $(n-1)^{\text{th}}$ power of another, the product being given in the form of a compound determinant. It is most easily established by substituting for the multiplier the adjugate of the second determinant, and noting that every row-multiplication gives an n -line determinant for result. Thus, the two determinants being $|a_1 b_2 c_3|$ and $|x_1 y_2 z_3|$, we have

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^2 &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} |y_2 \ z_3| & |y_3 \ z_1| & |y_1 \ z_2| \\ |x_3 \ z_2| & |x_1 \ z_3| & |x_2 \ z_1| \\ |x_2 \ y_3| & |x_3 \ y_1| & |x_1 \ y_2| \end{vmatrix} \\ &= \begin{vmatrix} |a_1 \ y_2 \ z_3| & |x_1 \ a_2 \ z_3| & |x_1 \ y_2 \ a_3| \\ |b_1 \ y_2 \ z_3| & |x_1 \ b_2 \ z_3| & |x_1 \ y_2 \ b_3| \\ |c_1 \ y_2 \ z_3| & |x_1 \ c_2 \ z_3| & |x_1 \ y_2 \ c_3| \end{vmatrix} \end{aligned}$$

where the element in the place (r, s) of the right-hand member is the result of putting the r^{th} row of $|a_1 b_2 c_3|$ in place of the s^{th} row of $|x_1 y_2 z_3|$. The theorem, or rather an extensional of it, seems to have been first put on record by Bazin in 1851 †; since then, and prior to Vahlen, it has been restated by several writers—for example, by Zehfuss in 1862.

(4) Vahlen's mode of using it in connection with the subject under

* Pascal in his "Determinanti" gives a full account of it. (See pp. 148-152; or in Leitzmann's translation, pp. 115-117.)

† BAZIN, H., "Sur une question relative aux déterminants," *Journ. (de Liouville) de Math.*, xvi, pp. 145-160.

discussion is to take every primary minor of the given array in succession for the multiplicand, and the adjugate of the first of them for the constant multiplier and make application of the theorem. If the array consist of k rows and m columns, there are thus $(m)_k$ multiplications to be performed, and in regard to these he reasons as follows. First of all the multiplicand may have *all* its columns in common with the first minor, in which case the outcome is manifestly nugatory. Or it may have $k-1$ columns in common, when there is a like outcome, as the right-hand member reduces at once to exactly the same form as the left-hand member. The number of such cases is $k(m-k)$, there being one for every way in which we can replace a column of the multiplicand by an outside column. All the other cases,

$$(m)_k - 1 - k(m-k)$$

in number, are real relations between the primary minors of the array. Not only so, but it is affirmed that all these latter relations are mutually independent, the reason being that each one involves a minor, the multiplicand, which necessarily does not appear in any of the others.

(5) All this needs to be received with some caution and scrutiny. As a matter of fact, in the case of every one of the

$$(m)_k - 1 - k(m-k)$$

multiplications viewed by Vahlen as effective, the $(n-2)^{th}$ power of the first of the primary minors can be removed from both sides, leaving on the left-hand the product of two minors, and on the right-hand an aggregate of products of pairs of other minors—leaving, that is to say, what Sylvester's theorem would have given at once.

(6) As the matter is of some moment, let us illustrate by the case of a 4-by-9 array, say the array

$$\begin{array}{cccccccccc} a_1 & b_1 & c_1 & d_1 & . & . & . & . & h_1 & i_1 \\ a_2 & b_2 & c_2 & d_2 & . & . & . & . & h_2 & i_2 \\ a_3 & b_3 & c_3 & d_3 & . & . & . & . & h_3 & i_3 \\ a_4 & b_4 & c_4 & d_4 & . & . & . & . & h_4 & i_4 \end{array}$$

Here the number of minors is 126, and of the 126 multiplications 21 are properly called nugatory; for example, the columnwise multiplication of

$$|a_1 b_2 c_3 h_4| \text{ by } |A_1 B_2 C_3 D_4|$$

gives

$$\begin{vmatrix} |a_1 b_2 c_3 d_4| & . & . & . \\ . & |a_1 b_2 c_3 d_4| & . & . \\ . & . & |a_1 b_2 c_3 d_4| & . \\ |h_1 b_2 c_3 d_4| & |a_1 h_2 c_3 d_4| & |a_1 b_2 h_3 d_4| & |a_1 b_2 c_3 h_4| \end{vmatrix}$$

i. e. $|a_1 b_2 c_3 d_4|^3 \cdot |a_1 b_2 c_3 h_4|,$

so that we end exactly as we started. Of the remaining 105 considered to be effective and independent we may select that in which the multiplicand is $|a_1 f_2 g_3 h_4|$, there being in this only one column that occurs also in $|a_1 b_2 c_3 d_4|$. The result of multiplying by $|A_1 B_2 C_3 D_4|$ now comes out in the form

$$\begin{vmatrix} |a_1 b_2 c_3 d_4| & . & . & . \\ |f_1 b_2 c_3 d_4| & |a_1 f_2 c_3 d_4| & |a_1 b_2 f_3 d_4| & |a_1 b_2 c_3 f_4| \\ |g_1 b_2 c_3 d_4| & |a_1 g_2 c_3 d_4| & |a_1 b_2 g_3 d_4| & |a_1 b_2 c_3 g_4| \\ |h_1 b_2 c_3 d_4| & |a_1 h_2 c_3 d_4| & |a_1 b_2 h_3 d_4| & |a_1 b_2 c_3 h_4| \end{vmatrix}$$

whence, of course, it is concluded that

$$|a_1 f_2 g_3 h_4| \cdot |a_1 b_2 c_3 d_4|^2 = \begin{vmatrix} |a_1 f_2 c_3 d_4| & |a_1 b_2 f_3 d_4| & |a_1 b_2 c_3 f_4| \\ |a_1 g_2 c_3 d_4| & |a_1 b_2 g_3 d_4| & |a_1 b_2 c_3 g_4| \\ |a_1 h_2 c_3 d_4| & |a_1 b_2 h_3 d_4| & |a_1 b_2 c_3 h_4| \end{vmatrix},$$

and there Vahlen allows the matter to rest. If, however, we expand the compound determinant here in terms of the elements of the first row and their complementary minors, we find by what we have called the extensional of Fontaine's identity that the said minors all have $|a_1 b_2 c_3 d_4|$ for a factor, being equal to

$$|a_1 b_2 g_3 h_4| \cdot |a_1 b_2 c_3 d_4|, \quad -|a_1 b_2 c_3 d_4| \cdot |a_1 c_2 g_3 h_4|, \quad |a_1 b_2 c_3 d_4| \cdot |a_1 d_2 g_3 h_4|;$$

so that there results the simple equality

$$|a_1 f_2 g_3 h_4| \cdot |a_1 b_2 c_3 d_4| = |a_1 f_2 c_3 d_4| \cdot |a_1 b_2 g_3 h_4| - |a_1 b_2 f_3 d_4| \cdot |a_1 c_2 g_3 h_4| + |a_1 b_2 c_3 f_4| \cdot |a_1 d_2 g_3 h_4|$$

in agreement with Sylvester's theorem.

(7) The first to note and to establish the reducibility of Bazin's theorem, to Sylvester's was Rubini in 1878.* The theorem, in fact, found in Rubini another discoverer, one too who was not content merely to state and prove his theorem, but to note special instances of it, draw deductions from it, view it from more than one standpoint, and indicate its affinities. The connected question, however, of the relations between the minors of a matrix he did not specifically refer to. This was left for Pascal, who in 1896 drew pointed attention to it in a paper whose title could leave no doubt as to its object.† In the first section of this he establishes what he calls his "fundamental identity A"—not recognised, strange to say, as Sylvester's—then shows how the so-called Vahlen relations are dependent on it, and ends with the words "in general all the possible relations between the

* RUBINI, R., "Formole di trasformazioni nella teorica dei determinanti." *Giornale di Mat.*, xvi, pp. 198–208.

† PASCAL, E., "Sulle varie forme che possono darsi alle relazioni fra i determinanti di una matrice rettangolare. *Annali di Mat.*, xxiv, pp. 241–253.

determinants of a matrix are but opportune transformations of the formula A."

(8) With this pronouncement most mathematicians would agree. Unfortunately it does not appear to have led in twenty years to any better understanding of the subject: and one is thus induced to inquire on what grounds the application of Bazin's theorem to the matter can be viewed as an "opportune transformation." What advantage, for example, is there to be got by multiplying each of the $(m)_k$ minors of the given array by the $(n-1)^{th}$ power of the first of them, as Vahlen directs, rather than by the first power of the said minor in accordance with Sylvester's theorem? For one thing the sifting of the results into

$$1+k(m-k) \text{ nugatory}$$

$$\text{and} \quad (m)_k - 1 - k(m-k) \text{ effective}$$

cannot be the reason, because the same sifting takes place in both cases. Thus, the array being

$$\begin{array}{cccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{array}$$

we multiply, according to Vahlen, each of the 20 minors by $|a_1 b_2 c_3|^2$, finding 10 cases nugatory, 9 cases in which the result has to be simplified by the removal of the first power of $|a_1 b_2 c_3|$, and only 1 case, namely,

$$|d_1 e_2 f_3| \cdot |a_1 b_2 c_3|^2 = \begin{vmatrix} |d_1 b_2 c_3| & |a_1 d_2 c_3| & |a_1 b_2 d_3| \\ |e_1 b_2 c_3| & |a_1 e_2 c_3| & |a_1 b_2 e_3| \\ |f_1 b_2 c_3| & |a_1 f_2 c_3| & |a_1 b_2 f_3| \end{vmatrix},$$

which is considered to be useful as found; in other words, if we arrange the 20 minors in dictionary order and denote them by their order numbers, Vahlen's result is

$$\begin{array}{ll} 8 \cdot 1 = 6 \cdot 2 - 5 \cdot 3 & 14 \cdot 1 = 12 \cdot 2 - 11 \cdot 3 \\ 9 \cdot 1 = 7 \cdot 2 - 5 \cdot 4 & 15 \cdot 1 = 13 \cdot 2 - 11 \cdot 4 \\ 10 \cdot 1 = 7 \cdot 3 - 6 \cdot 4 & 16 \cdot 1 = 13 \cdot 3 - 12 \cdot 4 \\ & 17 \cdot 1 = 12 \cdot 5 - 11 \cdot 6 \\ & 18 \cdot 1 = 13 \cdot 5 - 11 \cdot 7 \\ & 19 \cdot 1 = 13 \cdot 6 - 12 \cdot 7 \\ & 20 \cdot 1^2 = \begin{vmatrix} 11 & 5 & 2 \\ 12 & 6 & 3 \\ 13 & 7 & 4 \end{vmatrix}, \end{array}$$

the nugatory cases being those whose left-hand members are—

$$1 \cdot 1^2, 2 \cdot 1^2, \dots, 7 \cdot 1^2, 11 \cdot 1^2, 12 \cdot 1^2, 13 \cdot 1^2.$$

Instead of all this the other procedure only requires that we write down the simply identity

$$|a\beta\gamma| \cdot |123| = |a\beta 1| \cdot |\gamma 23| + |a\beta 2| \cdot |1\gamma 3| + |a\beta 3| \cdot |12\gamma|,$$

where the determinants are denoted by their column-numbers, and put the triad a, β, γ equal to each of the 20 triads

$$1, 2, 3; 1, 2, 4; \dots; 4, 5, 6,$$

in succession. Line for line the outcome is the same until we come to the last, when we obtain at once

$$|456| \cdot |123| = |145| \cdot |236| - |245| \cdot |136| + |345| \cdot |126|$$

i.e.,

$$20 \cdot 1 = 8 \cdot 13 - 14 \cdot 7 + 17 \cdot 4,$$

which is what Vahlen's last reduces to when we express the compound determinant in it in terms of the elements of the last row and their complementary minors.

(9) It is interesting to note in passing that the compound determinant just referred to has every one of its nine primary minors exactly divisible by $|a_1 b_2 c_3|$; indeed Vahlen's first nine relations are statements to that effect. The expression for $20 \cdot 1$ which the first row and its complementary minors have just furnished can thus be matched by using any one of the other rows or any one of the columns, the six equivalents obtained being

$$\begin{aligned} & 11 \cdot 10 - 5 \cdot 16 + 2 \cdot 19, \\ & - 12 \cdot 9 + 6 \cdot 15 - 3 \cdot 18, \\ & 13 \cdot 8 - 7 \cdot 14 + 4 \cdot 17, \\ & 11 \cdot 10 + 12 \cdot 9 + 13 \cdot 8, \\ & - 5 \cdot 16 + 6 \cdot 15 - 7 \cdot 14, \\ & 2 \cdot 19 - 3 \cdot 18 + 4 \cdot 17. \end{aligned}$$

On this point may be consulted a paper in the *Proceed. R. Soc. Edinburgh*, xxv (1904), pp. 366-371, on "The three-line determinants of a six-by-three array."

(10) The property utilised in the preceding paragraph holds for all orders, and deserves to be noted quite apart from its connection with the present subject. Stated in its simplest form it is that *every minor of a Bazin compound determinant is a determinant of the same kind.*

(11) The additional term for the expression $20 \cdot 1$ is, of course, due to the fact that the determinants 1 and 20 have no column in common. As, however, there are nine of the determinants of the array that have one column in common with 20, there are necessarily nine expressions of the shorter type which all involve 20, and of which one could be used to make the set of ten relations more uniform; for example,

$$20 \cdot 2 = 8 \cdot 15 - 9 \cdot 14.$$

In this connection it is important to note that there is considerable advantage in writing the vanishing trinomials of the Monge type as Pfaffians: thus

$$\begin{vmatrix} 2 & 8 & 9 \\ & 14 & 15 \\ & & 20 \end{vmatrix} = 0.$$

They can then be grouped in interesting sets of five, namely, the five primary minors of a quasi-Pfaffian. Written in this way the complete set of thirty is:

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & 9 \\ & & & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ & 11 & 12 & 13 \\ & & 14 & 15 \\ & & & 16 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 6 & 7 \\ & 11 & 12 & 13 \\ & & 17 & 18 \\ & & & 19 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 5 & 8 & 9 \\ & 11 & 14 & 15 \\ & & 17 & 18 \\ & & & 20 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 8 & 10 \\ & 12 & 14 & 16 \\ & & 17 & 19 \\ & & & 20 \end{vmatrix} = \begin{vmatrix} 4 & 7 & 9 & 10 \\ & 13 & 15 & 16 \\ & & 18 & 19 \\ & & & 20 \end{vmatrix}, \end{aligned}$$

and from the thirty the chosen set of ten is got at once by taking the nine which involve 1 and any one of the nine which involve 20.

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