ON THE DEVELOPMENT OF THE PERTURBATIVE FUNCTION IN THE THEORY OF PLANETARY MOTION.

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(Read April 19, 1916.)

In the *Transactions*, vol. ii, part 3, pp. 301–317, 1911, I have a paper upon the Newcomb operators used in the algebraical development of the elliptic perturbative function. The present paper deals with a further extension of the uses of these Newcomb operators. At best the development of the perturbative function is a very cumbersome business, and reference must be made to Newcomb's original paper in vol. v. of the *Astronomical Papers of the American Ephemeris* for a full statement. On p. 15 Newcomb quotes his remarkable theorem by which the operators, acting on the development in powers of e , combine with the operators acting on the development in powers of e_1 , e and e_1 being the eccentricities of the inner and outer planets respectively. The operators connected with e he writes as Π_j^n , those with e_1 as $\Pi_{0,j'}^{0,n'}$, and he proves that

$$\Pi_{j,j'}^{n,n'} = \Pi_j^n \times \Pi_{0,j'}$$

These operators consist of a string of differential coefficients of the Laplacian functions b_i or $a'A_i$.

Thus we have—

$$8\Pi_{0,0}^{1,2} = [8i^3 + (4i^2 - 2i)D + (-2i - 1)D^2 - D^3]b_i,$$

in which—

$$D = a \frac{d}{da}.$$

Operators which are factored by powers of $\sigma^2 = \text{sine squared half mutual inclination of orbits}$ are indicated by the affix k thus : $\Pi_{j,j'}^{k,n,n'}$.

Unfortunately Newcomb only shows how the operators can be combined symbolically, and therefore in numerical work it is necessary to consider the eccentricities of both orbits simultaneously.

It appears, however, that an extension can be made. It will be proved that if we possess a development of the disturbing function in which one of the eccentricities is supposed to be vanishingly small (in fact zero, but with a place of perihelion), we can include the effect of the second eccentricity by a series of simple operations. Thus, starting with a development with $e = 0$ and $e_1 = e_1$ we can arrive at the complete development for e and e_1 .

Newcomb writes any term of the development as follows :

$$e^j e_1^{j'} \sigma^{2k} P_{j,j'}^k(i) C(i'g' - ig + \theta),^*$$

in which—

$$P_{j,j'}^k = \Pi_j^k + e^2 \Pi_{j,j'}^{2,0} + e_1^2 \Pi_{j,j'}^{0,2} + \text{terms factored by } e^4, e^2 e_1^2, e_1^4, \text{ etc.}$$

In applying the new method it is a matter of choice which eccentricity is assumed to be zero. If each in turn is taken as zero and the two partial developments completed, then an independent check is furnished. Generally as $\Pi_{0,j'}$ is larger than Π_j it will be better to assume that e , the eccentricity of the inner planet, is zero.

Implicitly the method not only computes the development, but also its derivatives; but as these are actually wanted, it is so much to the good.

If we suppose that e_1 is zero, we will indicate the fact by replacing the upright P used by Newcomb by a slanting P ; if e is zero, we must add a dash so as to distinguish between—

$$P_{0,0} = \Pi_{0,0} + e^2 \Pi_{0,0}^{2,0} + e^4 \Pi_{0,0}^{4,0} + \text{etc.},$$

and—

$$\acute{P}_{0,0} = \Pi_{0,0} + e_1^2 \Pi_{0,0}^{0,2} + e_1^4 \Pi_{0,0}^{0,4} + \text{etc.}$$

Assuming now that e_1 is zero, any term in the development takes this form—

$$\begin{aligned} & \overset{0}{P}_j \quad (i) \quad C[ig' + (-i+j)g + i(w'-w) + 0(w'+w)] \\ & + \overset{1}{P}_{j-2} \quad (i-1) \quad C[ig' + (-i+j)g + (i-1)(w'-w) + 1(w'+w)] \\ & + \overset{1}{P}_{-2} (-i-1) C[ig' + (-i+j)g + (i+1)(w'-w) - 1(w'+w)] \\ & + \overset{2}{P}_{j-4} \quad (i-2) \quad C[ig' + (-i+j)g + (i-2)(w'-w) + 2(w'+w)] \\ & + \overset{2}{P}_{-j-4} (-i-2) C[ig' + (-i+j)g + (i+2)(w'-w) - 2(w'+w)] \\ & + \text{etc., etc., etc.} \end{aligned}$$

We cannot write down the general expression for $\overset{k}{P}_j$, but we know it contains a series of terms—

$$\Pi_j^k + e^2 \Pi_{j,0}^{2,0} + e^4 \Pi_{j,0}^{4,0} + \text{etc.}$$

* Leaded C indicates "cosine."

The question is if we operate on $\overset{0}{P}_j$ with say the operator $\Pi_{0,1} = \frac{1}{2}(2i + 1 + D)$, which reduces $\Pi_{j,0}$ to $\Pi_{j,1}$, etc., so as to obtain $\overset{0}{P}_{j,1}$, can we include $\overset{k}{P}_{\pm j - 2k}$ in this operation? We can see that this is so for j positive because the operator $\overset{k}{\Pi}_j(i)$ is equal to the operator $\overset{0}{\Pi}_j(i)$ if i is replaced by $i - k$. This will also include the negative suffices, because we pass from $\overset{k}{\Pi}_j$ to $\overset{k}{\Pi}_{-j}$ by replacing i by $-(i - 2k)$.

Hence—

$$\overset{k}{P}_{-j - 2n}(-i - k) = \overset{k}{P}_{j + 2n}(i - k).$$

Similarly for the external planet the operator $\Pi_{0,j'}(i)$ is equal to $\overset{0}{\Pi}(i + 2k)$, and—

$$\overset{k}{P}_{0,-j' - 2n}(-i - k) = \overset{k}{P}_{0,j' + 2n}(i - k).$$

Hence when we have reduced each term to the form—

$$\begin{aligned} &\overset{0}{P}C\theta C(i'g' + ig) - \overset{0}{P}S\theta S(i'g' + ig)^* \\ &\overset{1}{P}C\theta' C(i'g' + ig) - \overset{1}{P}S\theta' S(i'g' + ig) \\ &\overset{2}{P}C\theta'' C(i'g' + ig) - \overset{2}{P}S\theta'' S(i'g' + ig) \end{aligned}$$

we can operate on the sum so as to include the effect of the eccentricity of the outer planet—and *vice versa*.

As already mentioned, an absolute check is obtained by considering firstly one eccentricity nil and then operating, and then considering the other nil in its turn and operating. The resulting sums for each term must be identical, and they will be so if all sensible terms are included and the calculation is correct.

Each operation introducing e removes the term from the argument $i'g' + ig$ to the next argument $i'g' + (i + 1)g$, and the introduction of e^n removes it n times.

As an example we give a portion of the process as applied to some of the terms which go to make up the great inequality of Jupiter and Saturn.

Suppose, firstly, that Jupiter's orbit is treated as being without eccentricity,

We have then—

$$\begin{aligned} &e_1^3 P_{0,3} \quad (2)C[5g' - 2g + 2(w' - w) + 0(w' + w)] \\ &+ e_1 \sigma^2 \overset{1}{P}_{0,1} \quad (3)C[5g' - 2g + 3(w' - w) + 1(w' + w)] \\ &+ e_1 \sigma^4 \overset{2}{P}_{0,-1} (4)C[5g' - 2g + 4(w' - w) + 2(w' + w)] \\ &+ \text{terms of the 9th order.} \end{aligned}$$

* Leaded S indicates "sine."

Into these coefficients, and until they are insensible, we introduce the powers of e by operating firstly with $\Pi_0^2 = \frac{1}{4}(-16 + D + D^2)$, then with Π_0^4 , and so on.

If these results are added together, the effect is to change P into P .

The next coefficient will arise from the term—

$$\begin{aligned} & e_1^2 P_{0,2}(3) C[5g' - 3g + 3(w' - w) + 0(w' + w)] \\ & + e_1^2 P_{0,0}(4) C[5g' - 3g + 4(w' - w) + 1(w' + w)] \\ & + \text{etc.} \end{aligned}$$

In this case the succession of operators is $e\Pi_1^0 + e^3\Pi_1^3 + e^5\Pi_1^5 + \text{etc.}$, and so on.

Similarly the term starting with—

$$P_{0,4}(1) C[5g' - g + (w' - w)],$$

when operated on with $e\Pi_{-1}^1 + e^3\Pi_{-1}^3 + \text{etc.}$,

will give the coefficient $P_{-1,4}(1)$ etc., etc.

In practice each term $PC[5g' - ig + \theta]$ will be brought to the form—

$$P_{0,n} C\theta C[5g' - ig] - P_{0,n} S\theta S[5g' - ig]$$

and all the coefficients $PC\theta$, $PC\theta'$, etc., summed so that we have finally—

$$\Sigma P_{0,n} C\theta C[5g' - ig] - \Sigma P_{0,n} S\theta S[5g' - ig].$$

Then the operator which starts with $e^n \Pi_{n,0}^{n,0} + e^{n+2} \Pi_{n,0}^{n+2,0} + \text{etc.}$, acting on this sum will yield us—

$$\Sigma P_{n,n} C\theta C[5g' - (i - n)g] - \Sigma P_{n,n} S\theta S[5g' - (i - n)g],$$

which is complete in P_{nn} .

In thus making a complete expansion of the perturbing function and its derivatives, the saving of labour will be great. In Newcomb's and Le Verrier's expansions each coefficient requires a separate and sometimes very lengthy computation, whereas, as just demonstrated, blocks of coefficients are computed in one process. The machinery by which the other eccentricity is introduced is uniform throughout and simple. Moreover, it is more inclusive. In algebraical expansions of the perturbing function, the computer must decide at what limit he will stop. In Le Verrier's theory of Jupiter and Saturn, he adopted roughly the 4th and 5th powers of the eccentricities. In other words, he was satisfied with such an expansion as—

$$e_1 \Pi_{0,1}^{0,1} + e_1^3 \Pi_{0,1}^{0,3} + e_1^2 e_1 \Pi_{0,1}^{2,1} + e_1^5 \Pi_{0,1}^{0,5} + e_1^4 e_1 \Pi_{0,1}^{4,1}.$$

The new method starting with—

$$e_1 \Pi_{0,1}^{0,1} + e_1^3 \Pi_{0,1}^{0,3} + e_1^5 \Pi_{0,1}^{0,5},$$

which is the same order of approximation, will at the first operation introduce—

$$e^2 e_1 \Pi_{0,1}^{2,1} + e^2 e_1^3 \Pi_{0,1}^{2,3} + e^2 e_1^5 \Pi_{0,1}^{2,5}.$$

The next operation indicated by $\Pi_{0,0}^{4,0}$ introduces—

$$e^4 e_1 \Pi_{0,1}^{4,1} + e^4 e_1^3 \Pi_{0,1}^{4,3} + e^4 e_1^5 \Pi_{0,1}^{4,5},$$

and so on, and of these Le Verrier only included the first of each. The point is, that with exactly the same labour as Le Verrier used for the lower terms, we introduce in addition a crowd of higher terms, some of which may be sensible.

Newcomb, on pp. 27–32, gives the expressions for the operators Π_j^n and $\Pi_{0,j'}^{0,n'}$ so far as regards development to the 8th powers of the eccentricities—these or the modification of them already given in *Transactions*, vols. ii, part 3, and iii, part 2, 1911 and 1913, must still be used; but the tables for $\Pi_{j,j'}^{n,n'}$, which Newcomb gives on pp. 32 to 42 are not required. Similarly for the Π_k . The number of pages involved shows pretty closely the ratio of the saving—viz., as $4\frac{1}{3}$ pp. are to 10 pp.

JOHANNESBURG, March 27, 1916.

P.S.—The process outlined above can be generalised. If we have any development involving the distances of two planets from the sun, and suppose both eccentricities to be zero, then the effects of the eccentricities can be introduced bit by bit so far as they are sensible. Thus, if with eccentricities zero, we have a term

$$f(\alpha) \mathbf{C}(i'g' - ig + \theta)$$

then the Newcomb operator $P_{i,j'}$ will introduce the effect of $e_j e_{1j'}$ and change the angle from—

$$i'g' - ig + \theta$$

to—

$$(i' + j') g' - (i - j) g + \theta.$$

A numerical example of this new view of the use of the Newcomb operators will be found in the *Astronomical Journal* for June 30, 1916.

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