NOTE ON THE PRODUCT OF ANY DETERMINANT AND ITS BORDERED DERIVATIVE.

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(1) The fundamental result here obtained is the theorem that *the product of two determinants, the second of which is got by bordering the first, is expressible as a bilinear function of which the quasi variables are the cofactors of the bordering elements in the second determinant, and the discriminant is the unbordered determinant.* For example, when the order of the initial determinant is the 3rd we have

$$-\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} = \begin{array}{ccc|c} D_1 & D_2 & D_3 & \\ \hline a_1 & a_2 & a_3 & A_4 \\ b_1 & b_2 & b_3 & B_4 \\ c_1 & c_2 & c_3 & C_4 \end{array} .$$

By way of proof we note that on the right-hand side the cofactor of A_4

$$\begin{aligned}
&= a_1D_1 + a_2D_2 + a_3D_3 \\
&= -a_1\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} + a_2\begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} - a_3\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \\
&= -\begin{vmatrix} a_1 & a_2 & a_3 & \cdot \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 & \cdot \\ \cdot & \cdot & \cdot & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \\
&= -a_4\,|\,a_1\,b_2\,c_3\,| , \\
&= \text{cofactor of } A_4 \text{ on the left;}
\end{aligned}$$

and the outcome is similar when the cofactors of B_4 and C_4 are considered.

(2) As every bordered determinant is already known to be expressible as a bilinear, for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} = \begin{array}{ccc|c} d_1 & d_2 & d_3 & \\ \hline -|b_2c_3| & |b_1c_3| & -|b_1c_2| & a_4 \\ |a_2c_3| & -|a_1c_3| & |a_1c_2| & b_4 \\ -|a_2b_3| & |a_1b_3| & -|a_1b_2| & c_4, \end{array} ,$$

an alternative form for the preceding theorem is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{array}{ccc|c} d_1 & d_2 & d_3 & \\ \hline |b_2c_3| & -|b_1c_3| & |b_1c_2| & a_4 \\ -|a_2c_3| & |a_1c_3| & -|a_1c_2| & b_4 \\ |a_2b_3| & -|a_1b_3| & |a_1b_2| & c_4 \end{array} = \begin{array}{ccc|c} D_1 & D_2 & D_3 & \\ \hline a_1 & a_2 & a_3 & A_4 \\ b_1 & b_2 & b_3 & B_4 \\ c_1 & c_2 & c_3 & C_4 \end{array},$$

where it is curious to note that the two bilinear functions are such that the elements in the square array of the first are the cofactors, in $| a_1b_2c_3 |$, of the elements in the square array of the second, and the elements in the laterals of the second are the cofactors, in the bordered determinant, of the elements in the laterals of the first.

(3) The theorem thus reached recalls another* in which occur two of the same component parts, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{array}{ccc|c} D_1 & D_2 & D_3 & \\ \hline a_1 & a_2 & a_3 & A_4 \\ b_1 & b_2 & b_3 & B_4 \\ c_1 & c_2 & c_3 & C_4 \end{array} = - \begin{vmatrix} |b_2\,c_3| & -|b_1\,c_3| & |b_1\,c_2| & A_4 \\ -|a_2\,c_3| & |a_1\,c_3| & -|a_1\,c_2| & B_4 \\ |a_2\,b_3| & -|a_1\,b_3| & |a_1\,b_2| & C_4 \\ D_1 & D_2 & D_3 & . \end{vmatrix},$$

and, being thus able to combine the two, we deduce the still more interesting equality,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & . \end{vmatrix} = \begin{vmatrix} |b_2\,c_3| & -|b_1\,c_3| & |b_1\,c_2| & A_4 \\ -|a_2\,c_3| & |a_1\,c_3| & -|a_1\,c_2| & B_4 \\ |a_2\,b_3| & -|a_1\,b_3| & |a_1\,b_2| & C_4 \\ D_1 & D_2 & D_3 & . \end{vmatrix},$$

where the two bordered determinants are related in the matter of their elements quite similarly to the two bilinears in §2.

As, however, the first factor on the left-hand side of the theorem which we have quoted is, when general, raised to the power $n-2$, our deduced result in its general form is that *the product of any bordered determinant of the* (n + 1)th *order by the* (n − 1)th *power of the unbordered determinant is expressible as a bordered determinant of the* (n + 1)th *order also.*

(4) We have next to note that the extent of the border in the foregoing need not be restricted to one line: for example,

$$| a_1 b_2 c_3 | \cdot \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & . & . \\ e_1 & e_2 & e_3 & . & . \end{vmatrix} = \begin{array}{ccc|c} |a_3b_4c_5| & -|a_2b_4c_5| & |a_1b_4c_5| & \\ \hline |a_1b_2| & |a_1b_3| & |a_2b_3| & |c_1d_2e_3| \\ |a_1c_2| & |a_1c_3| & |a_2c_3| & -|b_1d_2e_3| \\ |b_1c_2| & |b_1c_3| & |b_2c_3| & |a_1d_2e_3| \end{array}$$

* 'Trans. R. Soc. Edinburgh.,' xxxii (1885), § 35.

and a quite similar mode of proof suffices. Thus, taking the aggregate of the first three of the nine terms on the right, namely,

$$\left\{ |a_3b_4c_5|\ |a_1b_2| - |a_2b_4c_5|\ |a_1b_3| + |a_1b_4c_5|\ |a_2b_3| \right\} \cdot |c_1d_2e_3|$$

we see that it

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ a_1 & a_2 & a_3 & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot \end{vmatrix} \cdot |c_1d_2e_3|$$

$$= \begin{vmatrix} \cdot & \cdot & \cdot & a_4 & a_5 \\ \cdot & \cdot & \cdot & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ a_1 & a_2 & a_3 & \cdot & \cdot \\ b_1 & b_2 & b_3 & \cdot & \cdot \end{vmatrix} \cdot |c_1d_2e_3|$$

$$= |\,a_4b_5\,| \cdot |\,a_1b_2c_3\,| \cdot |\,c_1d_2e_3\,| ,$$

which is also one of the three terms got on the left by using Laplace's expansion on the five-line determinant.

The order of the square array of the bilinear depends on the order of the initiating determinant and the breadth of the border; for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & \cdot & \cdot \\ f_1 & f_2 & f_3 & f_4 & \cdot & \cdot \end{vmatrix}$$

is equal to the bilinear whose square array is

$$\begin{matrix} |\,a_1b_2\,| & |\,a_1b_3\,| & \ldots\ldots & |\,a_3b_4\,| \\ |\,a_1c_2\,| & |\,a_1c_3\,| & \ldots\ldots & |\,a_3c_4\,| \\ \cdots & \cdots & \cdots & \cdots \\ |\,c_1d_2\,| & |\,c_1d_3\,| & \ldots\ldots & |\,c_3d_4\,| \end{matrix}$$

and whose laterals are

$$|a_3b_4c_5d_6|, -|a_2b_4c_5d_6|,\ |a_2b_3c_5d_6|,\ |a_1b_4c_5d_6|, -|a_1b_3c_5d_6|,\ |a_1b_2c_5d_6|,$$

and

$$|c_1d_2e_3f_4|, -|b_1d_2e_3f_4|,\ |b_1c_2e_3f_4|,\ |a_1d_2e_3f_4|, -|a_1c_2e_3f_4|,\ |a_1b_2e_3f_4|.$$

The general theorem may with useful enough fulness be enunciated as follows: *The product of an* n*-line determinant by the determinant got from it by bordering with* r *rows and* r *columns is expressible as a bilinear function whose quasi variables are the* n*-line minors of the multiplier that contain the bordering rows and the* n*-line minors that contain the bordering* *columns, and whose square array is that of the* r^{th} *compound of the original determinant.*

(5) When we pass on to the case where the initiating determinant is axisymmetric and is axisymmetrically bordered, another previously known theorem can be brought into service with advantage, namely, the theorem regarding the determinant whose matrix is the sum of an axisymmetric and a zero-axial skew matrix. The 3rd order being taken this theorem is *

$$\begin{vmatrix} a & f+\nu & e-\mu \\ f-\nu & b & d+\lambda \\ e+\mu & d-\lambda & c \end{vmatrix} = \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} + \begin{array}{ccc|c} \lambda & \mu & \nu & \\ \hline a & f & e & \lambda \\ f & b & d & \mu \\ e & d & c & \nu \end{array} .$$

Returning then to our first result above, and taking g, h, k for the border of the basic determinant here, we have

$$- \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} \cdot \begin{vmatrix} a & f & e & k \\ f & b & d & h \\ e & d & c & g \\ k & h & g & . \end{vmatrix} = \begin{array}{ccc|c} \mathrm{K} & \mathrm{H} & \mathrm{G} & \\ \hline a & f & e & \mathrm{K} \\ f & b & d & \mathrm{H} \\ e & d & c & \mathrm{G} \end{array} ,$$

and consequently, by the theorem just recalled,

$$\begin{vmatrix} a & f+\mathrm{G} & e-\mathrm{H} \\ f-\mathrm{G} & b & d+\mathrm{K} \\ e+\mathrm{H} & d-\mathrm{K} & c \end{vmatrix} = \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} - \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} \begin{vmatrix} a & f & e & k \\ f & b & d & h \\ e & d & c & g \\ k & h & g & . \end{vmatrix} ,$$

an unexpected result in pure determinants.

(6) When the determinant with the peculiar matrix referred to at the beginning of §5 is of the 4th order, there is in the expansion an additional term of a quite different type, namely, we have—

$$\begin{vmatrix} a & g+w & f-v & l+u \\ g-w & b & e+z & k-y \\ f+v & e-z & c & h+x \\ l-u & k+y & h-x & d \end{vmatrix} = \begin{vmatrix} a & g & f & l \\ g & b & e & k \\ f & e & c & h \\ l & k & h & d \end{vmatrix} + \mathrm{P} + \begin{vmatrix} w & -v & u \\ & z & -y \\ & & x \end{vmatrix}^2 ,$$

where P is the bilinear whose square array is that of the 2nd compound of the 4-line axisymmetric determinant, and whose laterals are

$$x,\ y,\ z,\ u,\ v,\ w.$$

On account of the existence of the said additional term

$$(xw - yv + zu)^2$$

it might fairly be expected that the deduced theorem would no longer hold, or, at least, not in the same form as before. Considerable importance therefore attaches to the value which this additional term assumes when

* 'Trans. R. Soc. Edinburgh,' xxxix (1897), p. 222.

x, y, z, u, v, w are given the values which they hold in our theorem above. Now, even where the basic determinant is not axisymmetric but is $|\, a_1b_2c_3d_4 \,|$, as at the close of §4 above, we have

$$xw - yv + zu$$

$$= |\, a_3b_4c_5d_6 \,|\,|\, a_1b_2c_5d_6 \,| - |\, a_2b_4c_5d_6 \,|\,|\, a_1b_3c_5d_6 \,| + |\, a_2b_3c_5d_6 \,|\,|\, a_1b_4c_5d_6 \,|,$$

and this being an extensional of

$$|\, a_3b_4 \,|\,|\, a_1b_2 \,| - |\, a_2b_4 \,|\,|\, a_1b_3 \,| + |\, a_2b_3 \,|\,|\, a_1b_4 \,|$$

must vanish identically as the latter is known to do.

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