NOTE ON RECURRENTS RESOLVABLE INTO A SEQUENCE
OF ODD INTEGERS.

By Sir Thomas Muir, LL.D.

(1) In an examination of the properties of the series
1, 3, 10, 35, 126, . . . .
the r\textsuperscript{th} term of which is C\textsubscript{2r-1}, r-1, a determinant made its appearance which
for the 4th order is
\[
\begin{vmatrix}
1 & 1 & \cdots & \\
2 & 3 & 1 & 1 \\
4 & 2 & 5 & 1 & 1 \\
6 & 4 & 2 & 7 & 1 \\
\end{vmatrix}
\]
and has the value — 9 11 13. The fact that the value was the product of
a sequence of odd integers led to the discovery that under another form it
had in effect been already studied. The alternative form in the case of the
4th order is
\[
\begin{vmatrix}
1 & 2 & \cdots & \\
3 & 1 & 4 & \\
5 & 3 & 1 & 6 \\
7 & 5 & 3 & 1 \\
\end{vmatrix}
\]
and is clearly the more convenient of the two. It was brought forward in
1892 in a paper* connected with the theory of integers, where it had for a
companion the determinant
\[
\begin{vmatrix}
1 & -2 & \cdots & \\
3 & 1 & -4 & \\
5 & 3 & 1 & -6 \\
7 & 5 & 3 & 1 \\
\end{vmatrix}
\]
These for the n\textsuperscript{th} order, which we may denote by R\textsubscript{n} and R'\textsubscript{n}, their dis-
coverer had quite skilfully evaluated, his results being
\[
R\textsubscript{n} = (-1)^{n-1}(2n + 1) (2n + 3) \ldots (4n - 3),
R'\textsubscript{n} = (2n + 3) (2n + 5) \ldots (4n - 1).
\]

(2) The main object of the present paper is to establish a pair of much
more general equalities, and to do so not by a mere extension of Reich's

(2), xi, pp. 176–193.
method, but by a quite different and more rapidly effective procedure. Before entering on this, however, we shall, in introducing four fresh results, establish them by the old method, in order that a knowledge of both may be available.

(3) Increasing the last row of $R_n$, namely,

$$(2n-1)_{n-1}, (2n-3)_{n-2}, \ldots, 5, 3, 1,$$

by

$$row_{n-1}2 + row_{n-2}2 + row_{n-3}4 + \ldots,$$

where the $r$th of the multipliers

$$2, 2, 4, 10, 28, \ldots$$

is \( \frac{r}{C_{n-3}} \), we find that the new $n$th row is divisible by $4n-3$, and that this factor being removed the row becomes

$$2, 2, 2, 2, \ldots, 5, 3, 1.$$ 

It thus follows that, if we denote the resulting determinant by $U_n$, we have

$$U_n = \frac{R_n}{4n-3} = (-1)^{n-1} (2n + 1) (2n + 3) \ldots (4n - 5); \quad \text{(I)}$$

for example,

$$U_4 = \begin{vmatrix} 1 & 2 & \ldots & \ldots \\ 3 & 1 & 4 & \ldots \\ 10 & 3 & 1 & 6 \\ 5 & 2 & 1 & 1 \end{vmatrix} = -911.$$

Note should be taken of the relation between the the series of multipliers and the row which ultimately comes of using them, namely, that the multipliers when halved give the elements of the said $n$th row in reverse order.

(4) Again, by adding to 3 times the $n$th row of $R_n$

$$1\cdot row_{n-1} + 2\cdot row_{n-2} + 5\cdot row_{n-3} + \ldots,$$

we obtain a new $n$th row which is divisible by $2n + 1$ and which on removal of the said factor becomes

$$\frac{2}{n+1} (2n-1)_{n-1}, \frac{2}{n} (2n-3)_{n-2}, \ldots, \frac{2}{4} 3, \frac{2}{3} 1, \frac{2}{2} 1;$$

so that, if the resulting determinant be denoted by $V_n$, we have

$$V_n = (-1)^{n-1} 3 (2n + 3) (2n + 5) \ldots (4n - 3). \quad \text{(II)}$$

Here the multipliers are

$$1, 2, 5, 14, \ldots, \frac{2}{n} (2n - 3)_{n-2}$$

and the last row of $V_n$ is

$$\frac{2}{n+1} (2n - 1)_{n-1}, \ldots, 14, 5, 2, 1.$$
(5) Similarly we may treat $R_n'$ as $R_n$ has been treated in §3, the difference being that the addition of the sum of multiples of rows is made to $-3$ times the $n^{th}$ row, and that the factor removed is $-(4n - 1)$. Calling the new determinant $U_n'$, which, be it noted, has the same last row as $U_n$, we have

$$U_n' = 3 \ (2n + 3) \ (2n + 5) \ldots \ (4n - 3) \quad \ldots \ (III)$$

(6) Lastly, we may treat $R_n'$ as $R_n$ has been treated in §4, the difference being that the addition of the sum of multiples of rows is made to $-5$ times the $n^{th}$ row, and that the factor removed is $-(2n + 3)$; so that if the resulting determinant be denoted by $V_n'$ we have

$$V_n' = 5 \ (2n + 5) \ (2n + 7) \ldots \ (4n - 1) \quad \ldots \ (IV)$$

for example,

$$\begin{vmatrix} 1 & -2 & \ldots & \ldots & \ldots & \ldots \\ 3 & 1 & -4 & \ldots & \ldots & \ldots \\ 10 & 3 & 1 & -6 & \ldots & \ldots \\ 35 & 10 & 3 & 1 & -8 & \ldots \\ 42 & 14 & 5 & 2 & 1 & \ldots \\
\end{vmatrix} = -5 \cdot 15 \cdot 17 \cdot 19.$$  

(7) Comparing the equalities obtained in §§4, 5, we reach the otherwise curious result

$$U_n' = (-1)^{n-1} V_n' \quad \ldots \quad \ldots \quad \ldots \quad (V)$$

for example,

$$U'_5 \equiv \begin{vmatrix} 1 & -2 & \ldots & \ldots & \ldots & \ldots \\ 3 & 1 & -4 & \ldots & \ldots & \ldots \\ 10 & 3 & 1 & -6 & \ldots & \ldots \\ 35 & 10 & 3 & 1 & -8 & \ldots \\ 42 & 14 & 5 & 2 & 1 & \ldots \\
\end{vmatrix} = \begin{vmatrix} 1 & 2 & \ldots & \ldots & \ldots & \ldots \\ 3 & 1 & 4 & \ldots & \ldots & \ldots \\ 10 & 3 & 1 & 6 & \ldots & \ldots \\ 35 & 10 & 3 & 1 & 8 & \ldots \\ 42 & 14 & 5 & 2 & 1 & \ldots \\
\end{vmatrix}.$$  

(8) The four equalities upon which depends the finding of the new $n^{th}$ rows in §§3, 4, 5, 6 are properties of the series

$$1, 3, 10, 35, 126, \ldots$$

referred to in §1. They are all, however, reducible to one fundamental equality, namely,

$$(2m + 1)w + \frac{1}{2} \ (2m - 1)_{w-1} \cdot 1_0 + \frac{1}{3} \ (2m - 3)_{w-2} \cdot 3_1 + \ldots + \frac{1}{m + 2} \ (2m + 1)_{w} = \frac{3}{2} \ (2m + 1)w$$

or, if we call the members of the series $t_1, t_2, \ldots$

$$(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m + 2}, t_{w+1}, t_w, \ldots, t_1, 1) \ (1, t_1, \ldots, t_{w+1}) = \frac{3}{2} t_{w+1}.$$  

It is not essentially different from Reich's principal equality (p. 181) which he proves gradationally, that is to say, by proceeding from one value of $m$ to the next higher.
(9) Let us now consider a determinant differing from $R_n$ in the last row, which instead of being

$$\begin{align*}
(2n - 1)_{n-1}, (2n - 3)_{n-2}, \ldots, 5, 3, 1
\end{align*}$$

is

$$\begin{align*}
(2n + m - 4)_{n-1}, (2n + m - 6)_{n-2}, \ldots, (m + 2)_{m}, m, 1.
\end{align*}$$

This determinant may appropriately be denoted by $R_n(m)$, which implies, of course, that what we have hitherto called $R_n$ would now be denoted by $R_n(3)$.

On account of the equality

$$r_s = (r - 1) + (r - 1),$$

the last row of $R_n(m)$ may be partitioned into two, namely, into

$$\begin{align*}
(2n + m - 5)_{n-1}, (2n + m - 7)_{n-2}, \ldots, (m + 1)_{m}, 1, \\
(2n + m - 5)_{n-2}, (2n + m - 7)_{n-3}, \ldots, (m + 1), 1, 0;
\end{align*}$$

so that we have at once the recurrent law of formation

$$R_n(m) = R_n(m - 1) - 2(n - 1) R_{n-1}(m + 1). \quad (VI)$$

If now we view $R_n(3)$ as known, namely,

$$R_n(3) = (-1)^{n-1} (2n + 1) (2n + 3) \ldots (4n - 3)$$

we can readily evaluate $R_n(m)$. For, when $m$ is 1 the $n^{th}$ row of the determinant is identical with the $(n - 1)^{th}$ row save in the $n^{th}$ place, so that we have

$$R_n(1) = - (2n - 3) R_{n-1}(3)$$

whence by substitution

$$R_n(1) = (-1)^{n-1} (2n - 3) (2n - 1) \ldots (4n - 7);$$

and, in the next place, putting $m = 2$ in (VI) we have

$$R_n(2) = R_n(1) - 2(n - 1) R_{n-1}(3)$$

whence by substitution

$$R_n(2) = (-1)^{n-1} (2n - 1) (2n + 1) \ldots (4n - 5).$$

We thus have a formula for $R_n(m)$ which holds for three consecutive values of $m$; and, this being the case, the recurrence formula (VI) enables us to show that it holds generally, namely,

$$R_n(m) = (-1)^{n-1} (2n + 2m - 5) (2n + 2m - 3) \ldots (4n + 2m - 9). \quad (VII)$$

(10) It is of importance, however, not to assume a knowledge of the value of $R_n(3)$, but to establish another recurrence-formula which is effective without it, namely,

$$R_n(m) = - (2n + 2m - 5) R_{n-1}(m + 2). \quad (VIII)$$

This is done by performing on $R_n(m)$ the operation

$$(2n - 2) \cdot \text{row}_{n-1} - \text{row}_{n-1}$$

$$- (m - 1) \left\{ \text{row}_{n-2} - \frac{1}{2} (m + 2) \cdot \text{row}_{n-3} + \frac{1}{3} (m + 4) \cdot \text{row}_{n-4} + \ldots \right\}$$
which enables us to remove from the last row the factor $2n + 2m - 5$ and leave the row

$$(2n + m - 4)_{n-2}, (2n + m - 6)_{n-3}, \ldots, (m + 2)_1, 1, 0$$

thus giving us

$$(2n - 2) R_n(m) = -(2n - 2) \cdot (2n + 2m - 5) R_{n-1}(m + 2)$$

from which we have only to remove the common factor.

By applying (VIII) to itself and repeating the operation we obtain

$$R_n(m) = (-1)^n(2n + 2m - 5) (2n + 2m - 3) R_{n-2} (m + 4)$$

$$= \ldots \ldots$$

$$= (-1)^{n-1} (2n + 2m - 5) \ldots \ldots (4n + 2m - 9) \cdot R_1(m + 2n - 2)$$

and $R_1$ is 1 for all arguments.

(11) The complicated operation which brings about the removal of the factor $2n + 2m - 5$ in the preceding paragraph is based on a property of the numbers 1, 3, 10, 35, \ldots $(2n - 1)_{n-1}$, which, on the other hand, is simple and orderly, namely,

$$\frac{(2p + 2 + q)_{p+1} - (2p + 3)_{p+1}}{q - 1}$$

$$= (2p + 1)^q \cdot q_0 + \frac{1}{2} (2p - 1)_{p-1} (q + 2) + \frac{1}{3} (2p - 3)_{p-2} (q + 4) + \ldots$$

$$+ \frac{1}{p + 1} (1)_p (q + 2p)_p \ldots \ldots (IX)$$

where the possibility of division on the left-hand side is attested by the vanishing of the numerator when $q$ is put equal to 1. It may be viewed as a generalisation of the theorem quoted in § 8: for on putting $q = -1$ the left-hand member here becomes

$$\frac{(2p + 3)_{p+1} - (2p + 1)_{p+1}}{2} \quad i.e. \quad \frac{3p + 4}{2(p + 2)^2(2p + 1)_p},$$

after which we have only got to add $\frac{1}{p + 2} (2p + 1)_p$ in order to complete the transformation.

(12) $R'$ may be generalised in its last row exactly as $R_n$ has been, the resulting determinant being denoted by $R'_n(m)$, and there being a series of companion theorems. For example, as analogues to (VI) and (VII) we have

$$R'_n(m) = R'_n(m - 1) + 2(n - 1)R'_{n-1}(m + 1) \quad (X)$$

$$R'_n(m) = (2n + 2m - 3) (2n + 2m - 5) \ldots \ldots (4n + 2m - 7) \ldots \ldots (XI).$$

(13) With the closely resembling values of $R_n(m)$ and $R'_n(m)$ before us it is easy to suggest relations the proofs of which would form interesting exercises in the transformation of determinants. Thus, since from § 9 we have
\( R_n(m + 1) = (-1)^{n-1}(2n + 2m - 3) (2n + 2m - 5) \cdots (4n + 2m - 7) \)

it follows with the help of (XI) that

\[ R_n(m + 1) = (-1)^{n-1}R_n(m) \]

\( i.e., \) when \( n \) is 5

\[
\begin{array}{cccc|cccc}
1 & 2 & . & . & 1 & -2 & . & . \\
3 & 1 & 4 & . & 3 & 1 & -4 & . \\
10 & 3 & 1 & 6 & . & 10 & 3 & 1 & -6 \\
35 & 10 & 3 & 8 & . & 35 & 10 & 3 & 1 & -8 \\
\end{array}
\]

\( (m+7)_4 (m+5)_3 (m+3)_2 (m+1)_1 \) \( (m+6)_4 (m+4)_3 (m+2)_2 \) \( m_1 \) \( 1 \)

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