THE

LAWS OF STATICAL EQUILIBRIUM

ANALYTICALLY INVESTIGATED.

BY

MR. JOHN GOUGH.

COMMUNICATED BY DR. HOLME.

THE theorem, which is frequently called the parallelogram of forces, contains the fundamental principles of dynamics. On this account the young mathematician ought to use his best endeavours to acquire clear notions relative to the proposition in question; because it is of the first importance in the science of mechanics. Many writers on this subject, more especially those on the continent, object to the common demonstration of the theorem, because motion is made use of in demonstrating it, whereas the idea of Statical Equilibrium is in direct opposition to the idea of motion. In conformity with this objection the French mathematicians assume as an axiom, a proposition, which shall be pointed out and demonstrated in the course of the present essay, and then have recourse to an analytical process, in the perusal of which the reader loses all sight of the idea of force and has his mind totally engaged with algebraic symbols. I shall on the contrary endeavour, in the following train of reasoning, to keep the idea of force in view as much as possible; for which purpose, the geometrical analysis will be used, in preference to algebra; because the diagrams of the former method, constantly recall the attention of the reader to the elementary principles of his subject.

Article 1.—DEF. The term statical equilibrium is used by certain modern writers on mechanics, to denote an opposition of forces, which does not produce motion.

Art. 2.—AXIOM. If a number of causes act in conjunction, their joint effect differs from the effect which would be produced by any one of the number acting separately. Should this be objected to as an axiom, we may observe that the contrary proposition supposes causes to act without effects, consequently such are no causes at all.

Art. 3.—AXIOM. Every force acts in a right line, which line is called its direction.

Art. 4.—AXIOM. Equal forces acting in

opposite directions, in the same right line, produce an equilibrium, by counteracting each other's effects: but no two forces, whether equal or unequal, can maintain an equilibrium, the directions of which include an angle; for in the latter hypothesis the direct opposition is wanting, which alone gives the title of an axiom to the preceding assertion.

Art. 5. Suppose F and G to represent two forces in magnitude, which act conjointly upon a material point B, in the directions (or right lines) AB and CB, Fig. 1. including the angle ABC. These things being stated, it follows that if F or G were to act separately, B would be urged in AB or CB by Art. 3, but F and G act conjointly: therefore B is urged in neither of these directions by Art. 2, moreover AB and CB include an angle ABC; consequently F and G do not keep the point in equilibrio by Art. 4.; hence B is urged in a right line by Art. 3; let this be DB. Draw SC meeting DB at right angles at D; now since F and G urge B towards different parts, while this point remains in DB, it is evident from Art. 4, that B is retained in DB by two equal and contrary forces, acting at the angle B parallel to AC, in the op posite directions AD, CD; therefore DB is in the plane BAC; and it divides the angle

ABC. For if BD be not in the plane BAC, then BACD is a pyramid, and the solid angle D is contained by three triangles; of which ADC is one; therefore the sides AD, CD include an angle; consequently the point B cannot remain in BD; but it has been shewn to remain in BD.

Art. 6. Let K denote in magnitude the force, with which B is urged by the joint action of F and G in the right line DB; then F, G, are called the components, and K their equivalent. These things being stated, it will appear evident from Art. 4, that if a force equal to K be exerted at any point of DB produced or not, in opposition to the joint effect of F and G, action and reaction will take place, without the production of motion, which is a statical equilibrium, by Art. 1. It is easily proved in like manner, that if F and K be two forces acting in the right lines AB, DB, and a third force G be opposed to their joint effect in the line CB or CB produced, statical equilibrium will ensue; viz. G is the equivalent of F and K; hence any one of the three forces F, G and K, is the equivalent, and the remaining two are its components.

Art. 7. It appears from Art. 5, that the point B is retained in DB by two equal and contrary forces acting at B perpendicular to

DB; but F and G are the only forces which affect the state of B; consequently parts of these forces are destroyed in maintaining this equilibrium, and the remaining parts constitute a quantity K; hence the sum of the components is greater than their equivalent.

Art. 8. All that has been concluded respecting F, G and K, relates to abstract forces having no particular ratios or differences; consequently if three right lines be taken proportionate to these magnitudes, a triangle may be formed of them; because any two are greater than the third: conversely the three sides of any triangle may represent three forces in magnitude; any two of which are the components of the remaining one.

Art. 9. Let AB, CB, fig. 2, including the angle ABC, denote two forces, F and G, in magnitude and direction; also, suppose BM to be the direction of their equivalent: join AC meeting BM in M; through B draw PR perpendicular to BM; and make AP, CR perpendicular to PR. Then as F is to the force in PB, so is AB to PB; and as G is to the force in RB so is CB to RB; but as F is to G, so is AB to CB by hyp.; therefore as the force in PB is to force RB, so is PB to RB. Now these forces are equal and contrary by Art. 7, hence PB = RB; but BM is parallel to AP and CR, therefore as PB is to BR, so is AM to MC; hence AM = MC. (Eucl. VI. and 2.) complete the parallelogram ADCB, and BM produced will evidently pass through the angle D; therefore the diameter BD, gives the direction of the equivalent K. Again make AT perpendicular to BD, and as BTAP is a parallelogram BT = AP, but AB represents F, therefore AP or BT represents that part of F which acts in the direction DB by Art. 7 and 8; for the same reason CR represents the part of G which acts in DB; but the triangle CBR and DAT are similar and equal, consequently BT + TD or BD expresses K in magnitude, which it also represents in direction.

Art. 10. The angle ADB = angle CBD; therefore their sines are equal; and we have AB × sine ABD = BC × sine CBD; hence as AB : CB :: sine CBD : sine ABD; but as AB : CB :: F : G; hence as F : G :: sine CBD : sine ABD; consequently if F = G, DB bisects the angle ABC; which is assumed as an axiom by the French analysts.

Art. 11. Sine of angle BAD or of BCD = sine of ABC; therefore BD = $\frac{\text{BA} \times \text{sine ABC}}{\text{sine DBC}} = \frac{\text{CB} \times \text{sine ABC}}{\text{sine ABD}}$; hence

$K=\frac{F\times\text{sine }ABC}{\text{sine }DBC}=\frac{G\times\text{sine }ABC}{\text{sine }DBA}$. Moreover $AP=AB\times\cos.\ PAB$; and $CR=CB\times\cos.\ RCB$; hence $\cos.K=F\times PAB+G\times\cos.RCB$. Now as $\angle PAB+\angle RCB=\angle ABC$, it follows that if the directions of F and G converge to a point infinitely distant from A, C, the angle ABC will become evanescent, and the cosine of PAB become $=\cos.\ RCB=$ Radius $=$ Unity; hence in this case $K=F+G$; from which we easily infer that whatsoever supports a body acted upon by gravity, sustains the whole weight of it. This is also assumed as an axiom by some writers on mathematics. (vide Emerson.)

Art. 12. Let AB, CB be the directions of two forces F, G, fig. 1; and let DB denote the direction of their equivalent K; through any point L in the line BD, draw any right line ST meeting AB, CB in S, T; also draw LV any how meeting CB in V; then $LB\times\text{sine }LBS=LS\times\text{sine }LSB$; and $LB\times\text{sine }LBT=LT\times\text{sine }LTB=LV\times\text{sine }LVB$; hence as sine LBT : sine LBS :: $LT\times\text{sine }LTB$: $LS\times\text{sine }LSB$; but as sine LBT : sine LBS :: G : F, Art. 11; therefore as $LS\times\text{sine }LSB$: $LT\times\text{sine }LTB$ :: G : F; for the same reason as $LS\times\text{sine }LSB$: $LV\times\text{sine }LVB$ :: G : F.

Art. 13. Suppose ST to be an inflexible right line, at the extremities of which S, T, the forces F and G act in the directions SB, TB; also, let ST be sustained at L by a force K acting in BD, and a statical equilibrium will be produced by Art. 6. Now let the lines of direction SB, TB converge to a point infinitely distant from ST; then will the angle SBT be evanescent; and we have seen the angles LSB, LTB, are supplemental to each other; as SL : LT :: G : F; which is a known property of the straight lever; and if SLV represent a crooked lever, the proportion stated above is universal, viz. as LS×sine LSB : LV×sine LVB :: G : F.

Art. 14. *Vide* fig. 3. Let AB, CB and DB be the direction of two forces F, G, and their equivalent K; from any point P, not in DB, draw PS, PT and PV perpendicular to AB, CB and DB respectively; then F×SP, G×TP and K×VP are called the momenta of F, G and K referred to the point or centre P. Now no force whatever applied to the point P, can keep the material point B in equilibrio, when acted upon as in the figure, by Art. 4; because the direction of such a force will either be parallel to DB the direction of K, or it will form an angle with it; therefore if PV be an inflexible line, capable of revolving about P, the force acting in the line DB perpendicular to PV, will cause it so to revolve.

Art. 15. Through PI draw P parallel to DB, meeting AB produced in I and make IK parallel to CB, meeting PT produced in R; also from D, draw DM, DN, perpendicular to AB, CB; then the triangles DMB, PSI are similar, as are also DNB, PRI; therefore as BD : IP :: DM : PS; and as BD : IP :: DN : PR; hence as DM : PS :: DN : PR; and multiplying by F and G we have, as $F \times DM : F \times PS :: G \times DN : G \times PR$; but $F \times DM = G \times DN$, by Art. 12; consequently $F \times PS = G \times PR$; therefore $F \times PS — G \times PT = G \times PR - G \times PT = G \times TR$; but by trigonometry $TR = IB \times$ sine RIB or ABC; and $PV = IB \times$ sine PIB or of DBA; hence as PV:RT :: sine∠ABD : sine CBA; but as sine ABD : sine CBA :: G : K, by Art. 11; consequently as PV : RT :: G : K; and $PV \times K = RT \times G = PS \times F - PT \times G$; that is, the momentum of K=the difference of the momenta of F and G. Had P been situated out of the angle ABC, the momentum of K=the sum of the momenta of F and G.— If *a*B, *c*B, &c. be the direction of forces F′, G′, *d*B the direction of their equivalent K′, and *h*B the direction of L the equivalent of K and K′, draw P*s*, P*t*, P*v*, P*w*, perpendicular to Ba Bc, Bd and Bs; then by last Art. $K \times PV + K' \times Pv = L \times Pw$; but $K \times PV = F \times PS - G \times PT$; for the same reason $K' \times Pv = F' \times Ps + G' \times Pt$, therefore $L \times Pw = F \times PS - G \times PT + F' \times Ps + G' \times Pt$, &c.; that is, if any number of forces F, G, F′, G′ act in the directions BA, BC, Ba, Bc, &c. lying in the same plane, and Bh be the direction of their common equivalent L, the momentum of L referred to any point P=the sum of momenta of forces F′ and G′ without the angle of whose direction P is situated+the difference of momenta of F and G within the angle of whose directions the same point lies.

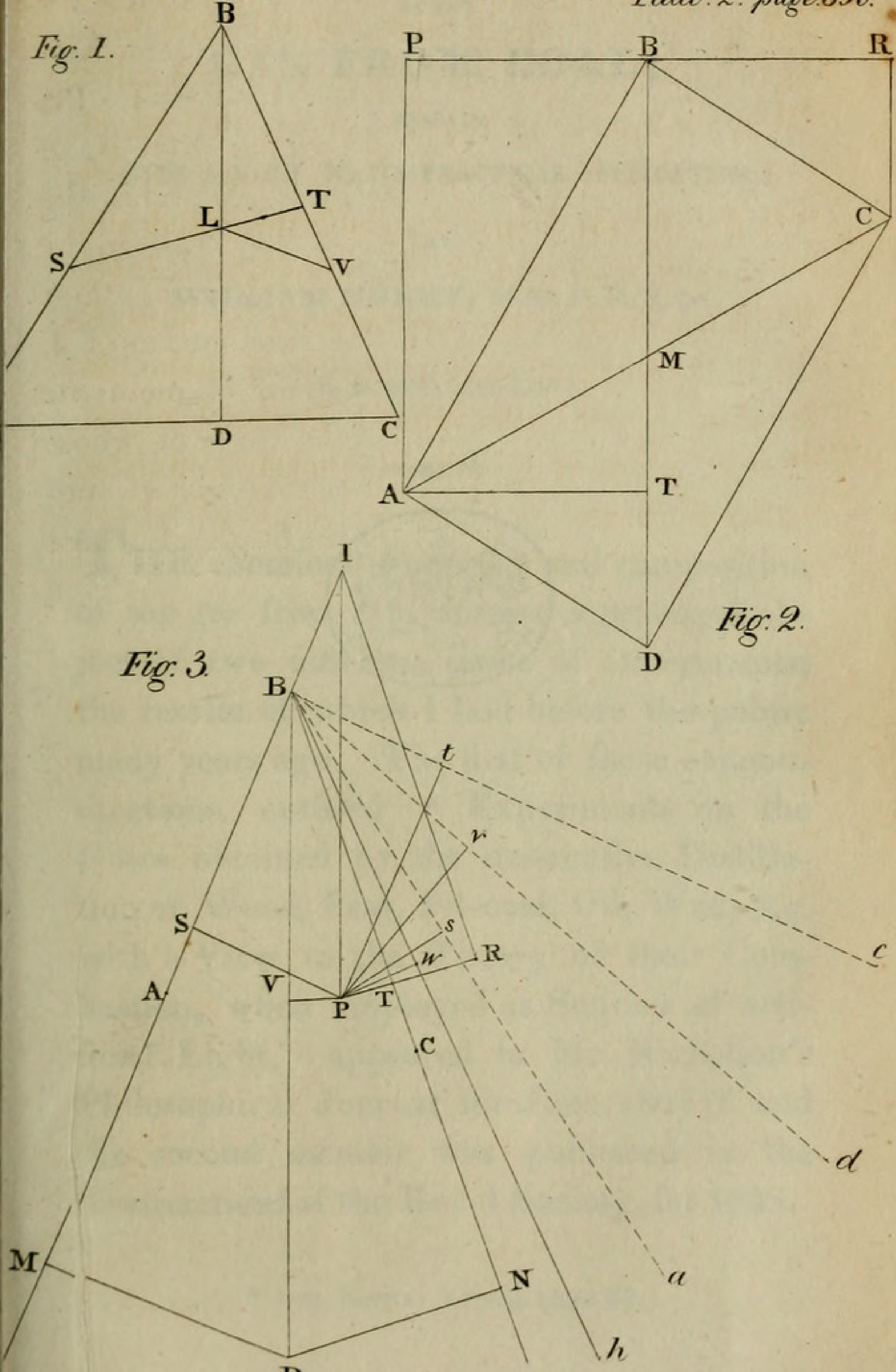
Fig. 1.
B
T
L
S
V
D
C
P
B
R
C
M
A
T
D
Fig. 2.
I
Fig. 3.
B
t
v
S
s
w
R
A
V
P
T
C
c
d
M
N
a
D
h