VIII.—*On Supplementary Curves.*

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The following paper contains, it is hoped, some contributions to the Theory of Imaginaries in Geometry. It was originally intended to include only such points in that theory as appeared to me to be new; but, having proceeded some length in drawing up the paper, it occurred to me that its utility might be increased by including such an outline of what has already been published on the subject as would make the paper intelligible in itself without perpetual references to other sources of information. Accordingly, the first section contains some matter which is not absolutely new; the principal definitions having been given by M. Poncelet, in his splendid work "*On the Projective Properties of Figures*," published in 1822, and several of the preliminary theorems having been published in the *Mathematician*, early in 1846, in some papers of mine "*On the Application of Algebra to the Modern Geometry*." *

The first idea of writing the paper originated in the perusal of a long note at the end of Mr. Salmon's new book "*On the Higher Plane Curves.*" In this note he has combated the two principal theories of imaginaries, as proposed respectively by M. Poncelet, and the late Mr. Gregory, of Cambridge. An able defence of Gregory's theory has just been published in No. XXX. of the *Cambridge and Dublin Mathematical Journal.* In this paper I have confined myself to Poncelet's theory exclusively, but my object is not so much to defend this theory as to illustrate and extend it.

SECTION I.—PRELIMINARY PROPOSITIONS.

I.

If the circle $x^2 + y^2 = a^2$ be cut by a straight line $x = c$, the ordinates of the points of intersection will evidently be

$$+ \sqrt{(a^2 - c^2)} \text{ and } - \sqrt{(a^2 - c^2)},$$

which are both imaginary when c is greater than the radius a. Thus we see that *any straight line in the plane of the circle, and at a distance from its centre greater than the radius, cuts the circle in two imaginary points.* Now it has been proposed by M. Poncelet, (*Traité des Propriétés Projectives des Figures*, p. 29.) that the real points

$$x = c, y = \sqrt{(c^2 - a^2)} \text{ and } x = c, y = - \sqrt{(c^2 - a^2)},$$

obtained by changing the sign of the quantity under the radical in the imaginary expressions given above, should be taken as the representatives of the imaginary points in question. Again, since $2\sqrt{(a^2 - c^2)}$ may be considered as the imaginary chord which the circle intercepts on the given line, M. Poncelet has proposed that the real expression $2\sqrt{(c^2 - a^2)}$ should be considered as the *ideal chord* intercepted by the circle $x^2 + y^2 = a^2$ on the straight line $x = c$. He has also applied the term *ideal chord* to the indefinite straight line $x = c$, when its intersections with the circle are imaginary: hence any straight line in the plane of a circle may be considered as the chord of the circle; but, for the sake of distinction, it is called a *real* chord when it cuts the circle in two real points, and an *ideal* chord when its intersections with the circle are imaginary, or when it lies wholly without the circle.

These principles being admitted, if an arbitrary series of straight lines be drawn, each of which cuts the circle in two imaginary points, the corresponding real points will not in general lie upon any particular locus; but when the lines succeed one another according to any given law, the representatives of the imaginary points in which they meet the circle will lie upon a regular curve, which may be considered as the locus of the imaginary points in which the circle is cut by any straight line subject to that law. Now since what has been said of the circle may readily be extended to any given curve, the following problem is naturally suggested:—

Any plane curve being given, to find the locus of the imaginary points in which it is cut by a series of straight lines drawn in the plane of the curve so as to succeed one another according to any given law.

In the present state of algebraical science there is little hope of obtaining a solution to this problem, except in particular cases. In this paper I shall confine myself to the two simplest cases, viz.,—1st. The case in which the system of secants is parallel to a given straight line;—2nd. That in which all the secants pass through the same point: and I propose to show, by a few examples, that the locus in question is intimately connected with the given curve in many important properties.

But before proceeding to the solutions of these problems it may be expedient to establish some general theorems, by means of which the subsequent investigations will be greatly simplified and abridged.

II.

Let $u = Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + 1 = 0 \;......(1)$
be the equation to any conic section,

$$v = \alpha y + \beta x + 1 = 0 \;............(2),$$

$$v' = \alpha' y + \beta' x + 1 = 0 \;............(3),$$

the equations of two straight lines; then the equation

$$u_1 = u + k v v' = 0 \;............(4),$$

where k is an arbitrary constant, will represent a system of conic sections passing through the four points in which the curve u is cut by the straight lines v and v'. For, since any values of x and y which satisfy the simultaneous equations (1) and (2) will also satisfy equation (4), it follows that the curve u_1 must pass through the two points in which the curve u is cut by the straight line v; and in a similar manner it can be shown that the curve u_1 passes through the intersections of u and v'.

(a.) When the straight line v is without the curve u, the points in which it intersects that curve will be imaginary; and since the curve u_1 passes through these points, it follows that in this case v will be an ideal common secant of u and u_1. But when the straight line v cuts the curve u in two real points, the curve u_1 will pass through these points, and v will be a real common secant of u and u_1.

(b.) When $\alpha' = 0$ and $\beta' = 0$, the straight line v' passes to infinity, because the points in which it meets the axes of x and y are at infinity. In this case we have $v' = 1$, and equation (4) becomes

$$u_2 = u + kv = 0 \text{ ....(5).}$$

Hence we see that this equation represents a system of conic sections having with the curve u a common secant at infinity. When the curve u is an ellipse, this secant at infinity must evidently be ideal.

Since the terms containing the squares and product of x and y are the same in equations (1) and (5), the curves u and u_2 are evidently similar and similarly placed. Hence any two curves of the second degree which are similar and similarly placed have a common secant at infinity.

(c.) If $\alpha' = \alpha$ and $\beta' = \beta$, the straight lines v and v' will coincide, and the four points of intersection of the curves u and u_1 will coalesce into two points of contact. In this case equation (4) becomes

$$u_3 = u + kv^2 = 0 \text{ ....(6),}$$

which therefore represents a conic section having a double contact with the curve u, the equation to the chord of contact being $v = 0$. When the straight line v is without the curve u, the two points of contact are imaginary, and the chord of contact is ideal.

(*d.*) When $\alpha = 0$ and $\beta = 0$, the straight line v passes to infinity, and equation (6) becomes

$$u_4 = u + k = 0 \text{ ....(7).}$$

Now since this differs from equation (1) only in its absolute term, it is evident that the curves u and u_4 are concentric, similar, and similarly placed; and thus we see that any two curves of the second degree which are concentric, similar, and similarly placed, have a double contact at infinity. From this it follows that any two concentric circles have a double contact at infinity. In this case, as well as that of two similar and concentric ellipses, the points of contact must evidently be imaginary, and the chord of contact ideal.

(*e.*) When the quadratic function u can be resolved into two factors v_1 and v_2 of the first degree, equation (6) becomes

$$u_5 = v_1 v_2 + k v^2 = 0 \text{ ....(8).}$$

In this case the conic section u breaks up into two straight lines v_1 and v_2, and the curve u_5 touches these lines at the points in which they are cut by the straight line v. Hence it is evident that the point of intersection of v_1 and v_2 is the pole of the straight line v in relation to the curve u_5. Now when the lines v_1 and v_2 are imaginary, their point of intersection is real, and continues to be the pole of the straight line v; but the points in which v_1 and v_2 touch the curve u_5 are imaginary, and the chord of contact v is ideal.

III.

The theorems given in the last number relative to the conic sections may readily be generalized. Thus, if $u = 0$ be a curve of the nth degree, and if $v = 0$ and $v' = 0$ be two curves of the mth degree, where m is not greater than $\frac{1}{2} n$, then $u + k v v' = 0$ will represent a system of curves of the

nth degree, passing through the points in which the curves v and v' intersect the curve u. Again, when the curves v and v' coincide, the $2mn$ points of intersection unite into mn points of contact; so that the equation $u + kv^2 = 0$ represents a curve of the nth degree touching the curve u at mn points which lie on the curve of contact v. It is evident also that all this reasoning holds good when any of the curves u, v, v' breaks up into a system of straight lines.

IV.

Let $u = Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0$ ......(1), and $u' = A'y^2 + 2B'xy + C'x^2 + 2D'y + 2E'x + F' = 0$ ......(2), represent any two conic sections; then, if k denote a given constant, the equation

$$u_1 = u + k u' = 0 \text{ ....(3)}$$

will evidently represent a conic section passing through the four points of intersection of the curves u and u'. Putting $A + k A' = A_1$, $B + k B' = B_1$, &c., equation (3) may be written in the form

$$A_1 y^2 + 2B_1 xy + C_1 x^2 + 2D_1 y + 2E_1 x + F_1 = 0 \text{ ......(4).}$$

Now if k be so assumed that the first member of this equation can be resolved into two factors of the first degree, the curve u_1 will break up into two straight lines, which will be conjugate common secants of the curves (1) and (2). In this case, let x_1 and y_1 be the co-ordinates of the point of intersection of the two lines in question; then, by transferring the origin of co-ordinates to this point, equation (4) becomes

$$A_1 y^2 + 2B_1 xy + C_1 x^2 + 2D_2 y + 2E_2 x + F_2 = 0 \text{ ......(5);}$$

where, for the sake of brevity, we assume

$$\left.\begin{aligned} D_2 &= A_1 y_1 + B_1 x_1 + D_1 \\ E_2 &= B_1 y_1 + C_1 x_1 + E_1 \end{aligned}\right\} \text{............(6),}$$

$$F_2 = A_1 y_1^2 + 2B_1 x_1 y_1 + C_1 x_1^2 + 2D_1 y_1 + 2E_1 x_1 + F_1 \text{ ......(7).}$$

Because the two straight lines represented by equation (5) pass through the new origin, we must evidently have

$$D_2 = 0,\ E_2 = 0,\ F = 0;$$

and by eliminating x_1 and y_1 from these three equations we shall have

$$A_1E_1^2+C_1D_1^2+F_1B_1^2-2B_1D_1E_1-A_1C_1F_1=0\ \ldots\ldots(8),$$

which is an equation of the third degree with respect to k, since A_1, B_1, C_1, &c., are linear functions of k. Now since every equation of the third degree has at least one real root, it follows that one of the values of k deduced from equation (8) must be real; and if this value be substituted in equations (6) we shall obtain the corresponding values of x_1 and y_1, which will also be real, since equations (6) are of the first degree. Hence we see that *any two conic sections* (1) *and* (2) *described in the same plane have at least one system of conjugate common secants, the point of intersection of which is necessarily real, although the secants themselves may be imaginary.*

V.

If a straight line AP be cut by a given curve of the nth degree in the points Q_1, Q_2, $Q_3\ldots\ldots Q_n$, it is required to express the ratio

$$PQ_1 \cdot PQ_2 \cdot PQ_3 \ldots\ldots PQ_n : AQ_1 \cdot AQ_2 \cdot AQ_3 \ldots\ldots AQ_n,$$

which is compounded of the ratios of the segments of AP, in terms of the co-ordinates of P referred to any rectangular axes passing through A.

Let x', y' be the co-ordinates of P, and put $AP=r$, $AQ_1=\rho_1$, $AQ_2=\rho_2$, $AQ_3=\rho_3\ldots\ldots AQ_n=\rho_n$; then, if we adopt the notation

$$(PQ_n)=PQ_1 \cdot PQ_2 \cdot PQ_3 \ldots\ldots PQ_n,$$

we shall have $\dfrac{(PQ_n)}{(AQ_n)}=\dfrac{(r-\rho_1)(r-\rho_2)(r-\rho_3)\ldots\ldots(r-\rho_n)}{\rho_1 \cdot \rho_2 \cdot \rho_3 \ldots\ldots \rho_n}$;

and when n is even this reduces to

$$\frac{(PQ_n)}{(AQ_n)}=1-\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}+\text{\&c.}\right)r+\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_1\rho_3}+\frac{1}{\rho_2\rho_3}+\text{\&c.}\right)r^2 - \text{\&c.}\ \ldots\ldots\ldots\ldots(1).$$

Let $u=0$ be the equation of the given curve; then

$$u = 1 + f_1(x, y) + f_2(x, y) + f_3(x, y) \ldots + f_n(x\, y) = 0 \ldots\ldots (2),$$

where $f_m(x, y)$ denotes a complete homogeneous function of x and y of the mth degree, such as

$$A\, x^m + B\, x^{m-1} y + C\, x^{m-2} y^2 + \&c.;$$

then if ρ denote the radius vector of the curve u, θ and ϕ the cosines of the angles which ρ makes with the axes of x and y, we shall have

$$x = \rho\, \theta,\ y = \rho\, \phi;$$

and by substituting these values of x and y in equation (2) we obtain

$$1 + r\,.\,f_1(\theta, \phi) + r^2\,.\,f_2(\theta, \phi) + r^3\,.\,f_3(\theta, \phi) \ldots + r^n\,.\,f_n(\theta, \phi) = 0,$$

which is the polar equation to the given curve. Now since $\rho_1, \rho_2, \rho_3, \ldots\ldots \rho_n$ are the roots of this equation, we obtain, by the theory of equations,

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \&c. = -f_1(\theta, \phi),$$

$$\frac{1}{\rho_1 \rho_2} + \frac{1}{\rho_1 \rho_3} + \frac{1}{\rho_2 \rho_3} + \&c. = f_2(\theta, \phi),\ \&c.;$$

hence, by substituting these expressions in equation (1), we get

$$(PQ_n) : (AQ_n) = 1 + r\,.\,f_1(\theta, \phi) + r^2\,.\,f_2(\theta, \phi) \ldots + r^n\,.\,f_n(\theta, \phi);$$

and since $r\,\theta = x'$ and $r\,\phi = y'$, this equation can be written in the form

$$(PQ_n) : (AQ_n) = 1 + f_1(x', y') + f_2(x', y') \ldots + f_n(x', y') = u',$$

where u' denotes the same function of x' and y' as u is of x and y.

When n is odd it may be demonstrated in the same manner that $(PQ_n) : (AQ_n) = -u'$; hence, generally, we have

$$\frac{PQ_1\,.\,PQ_2\,.\,PQ_3 \ldots\ldots PQ_n}{AQ_1\,.\,AQ_2\,.\,AQ_3 \ldots\ldots AQ_n} = \pm u' \ldots\ldots\ldots\ldots (3),$$

where the upper or lower sign must be taken according as n is even or odd.

(a.) When the proposed equation is of the first degree, the demonstration still holds good, and equation (3) becomes

$$PQ_1 : AQ_1 = -u'.$$

In this case, if PM and AN be drawn perpendicular to the given line u, the triangles AQ_1N and PQ_1M will evidently be similar, and we shall have

$$PQ_1 : AQ_1 = PM : AN;$$

in virtue of which the preceding equation becomes

$$PM : AN = -u' \text{............} (4).$$

(*b.*) It is evident that the demonstration given above will hold good when the proposed equation $u = o$ is resolvable into factors. Let it be resolvable into n factors of the first degree, in which case it will represent a system of n straight lines. Let PM_1, PM_2, PM_3...... PM_n and AN_1, AN_2, AN_3, ... AN_n be the perpendiculars drawn from P and A respectively to these lines; then, by similar triangles,

$$PQ_1 : AQ_1 = PM_1 : AN_1,\ PQ_2 : AQ_2 = PM_2 : AN_2, \text{ \&c.},$$

and therefore equation (3) may be written in the form

$$\frac{PM_1 . PM_2 . PM_3 \text{......} PM_n}{AN_1 . AN_2 . AN_3 \text{......} AN_n} = \pm u' \text{............} (5).$$

(*c.*) The last equation fails when each of the n straight lines represented by the equation $u = o$ passes through the point A; since, in that case, each of the perpendiculars AN_1, AN_2, &c. is zero. In this case equation (2) takes the form

$$u = A_1y^n + A_2y^{n-1}x + A_3y^{n-2}x^2 \text{......} + A_nx^n = o \text{............} (6),$$

and if we assume $y = mx$ this becomes

$$A_1m^n + A_2m^{n-1} + A_3m^{n-2} \text{......} + A_n = o \text{............} (7).$$

Now if m_1, m_2, m_3, ... m_n denote the roots of this equation, these will evidently be the direction indices of the system of straight lines represented by equation (6), so that the equations of these lines will be

$$y = m_1x,\ y = m_2x, \text{......}\ y = m_nx \text{............} (8).$$

Let x' and y' be the co-ordinates of P, then, by the theory of the straight line, we shall have

$$PM_1 = \frac{y'-m_1x'}{\sqrt{(1+m_1^2)}},\ PM_2 = \frac{y'-m_2x'}{\sqrt{(1+m_2^2)}},\ \&c.,$$

$$\therefore PM_1 . PM_2 ... PM_n = \frac{(y'-m_1x')\,(y'-m_2x')...(y'-m_nx')}{\sqrt{\{(1+m_1^2)\,(1+m_2^2)...(1+m_n^2)\}}}$$

$$= \frac{u'}{A_1\sqrt{\{(1+m_1^2)\,(1+m_2^2)\,(1+m_n^2)\}}} ...(9).$$

This equation shows that the function u' has a fixed ratio to the continued product of the perpendiculars PM_1, PM_2, PM_3 ...... PM_n. Denoting this ratio by M, equation (9) gives

$$M^2 = A_1^2\,(1+m_1^2)\,(1+m_2^2)\,(1+m_3^2)......(1+m_n^2).........(10);$$

or, by performing the multiplication

$$M^2 = A_1^2\,\{1 + (m_1^2 + m_2^2 + m_3^2 + \&c.)$$
$$+ (m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2 + \&c.)$$
$$+ (m_1^2m_2^2m_3^2 + \&c.) + \&c.\}............(11).$$

The values of these symmetrical functions of the roots m_1, m_2, &c. may be very easily obtained by constructing the equation whose roots are m_1^2, m_2^2, m_3^2, &c. For the sake of simplicity let us consider the case of $n = 5$; then equation (7) becomes

$$A_1m^5 + A_2m^4 + A_3m^3 + A_4m^2 + A_5m + A_6 = 0.$$

If we assume $m^2 = z$ this becomes

$$z^{\frac{1}{2}}\,(A_1z^2 + A_3z + A_5) + A_2z^2 + A_4z + A_6 = 0;$$

hence, by clearing the radical, and arranging according to the powers of z, we get

$$A_1^2z^5 + (2A_1A_3 - A_2^2)\,z^4 + (A_3^2 + 2A_1A_5 - 2A_2A_4)\,z^3$$
$$+ (2A_3A_5 - 2A_2A_6 - A_4^2)\,z^2 + (A_5^2 - 2A_4A_6)\,z - A_6^2 = 0.$$

Now since m_1^2, m_2^2, m_3^2, &c. are the roots of this equation, we shall have

$$m_1^2 + m_2^2 + \&c. = (A_2^2 - 2A_1A_3) : A_1^2,$$
$$m_1^2m_2^2 + m_1^2m_3^2 + \&c. = (A_3^2 - 2A_2A_4 + 2A_1A_5) : A_1^2,$$
$$m_1^2m_2^2m_3^2 + \&c. = (A_4^2 + 2A_2A_6 - 2A_3A_5) : A_1^2,\ \&c.\ \&c.$$

and in virtue of these equation (11) becomes

$$M^2 = A_1^2 + A_2^2 - 2A_1A_3 + A_3^2 - 2A_2A_4 + 2A_1A_5 + A_4^2$$
$$+ 2A_2A_6 - 2A_3A_5 + A_5^2 - 2A_4A_6 + A_6^2,$$

and this may evidently be written in the form

$$M^2 = (A_1 - A_3 + A_5)^2 + (A_2 - A_4 + A_6)^2 \text{.........} (12).$$

In the general case we should evidently arrive at a similar result, and hence equation (9) gives

$$u' = M \times PM_1 . PM_2 . PM_3 \ldots\ldots PM_n \text{............} (13),$$

where $M^2 = (A_1 - A_3 + A_5 - \&c.)^2 + (A_2 - A_4 + A_6 - \&c.)^2 \ldots (14)$.

(*d.*) The formulas demonstrated in this number are of great importance in reducing geometrical expressions to algebraic form, and the converse. Let us consider, for instance, the equation

$$u_1 = u + kvv' = o, \text{ (II, 4.)}$$

Let AP be a straight line drawn from the origin A to any point P in the curve u_1, and let this line meet the conic section u in the points Q_1 and Q_2, and the straight lines v and v' in R_1 and R_2 respectively; then, by equations (3) and (4),

$$u = \frac{PQ_1 . PQ_2}{AQ_1 . AQ_2}, \; v = -\frac{PR_1}{AR_1}, \; v' = -\frac{PR_2}{AR_2};$$

$$\therefore u_1 = \frac{PQ_1 . PQ_2}{AQ_1 . AQ_2} + k\frac{PR_1 . PR_2}{AR_1 . AR_2} = o.$$

Hence we see that the curve u_1 may be considered as the locus of a point P, such that if a straight line be drawn from it to a fixed point A, meeting the curve u in Q_1 and Q_2, and the straight lines v and v' in R_1 and R_2, the compound ratio $PQ_1 . PQ_2 : AQ_1 . AQ_2$ shall have to $PR_1 . PR_2 : AR_1 . AR_2$ a fixed ratio $-k$.

SECTION II.—OF CURVES WHICH ARE SUPPLEMENTARY IN RELATION TO A GIVEN STRAIGHT LINE.

VI.

A curve of the second degree being given, to find the locus of the imaginary points in which it is cut by a system of straight lines parallel to a given line.

If the given line be taken as the axis of y, the equation of the given curve will be of the form

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0.$$

By solving this with respect to y, we get

$$Ay = -(Bx+D) \pm \sqrt{\{(Bx+D)^2 - A(Cx^2+2Ex+F)\}} \ldots (1),$$

and by changing the algebraic sign of the quantity under the radical sign in this expression, we obtain

$$Ay = -(Bx+D) \pm \sqrt{\{A(Cx^2+2Ex+F) - (Bx+D)^2\}} \ldots (2),$$

which is the equation of the required locus.

(*a*.) Since every value of x which gives real values of y in equation (2) will give imaginary values of y in equation (1), it is evident that the original curve (1) is the locus of the imaginary points in which the curve (2) is cut by a series of straight lines parallel to the axis of y. Hence we see that *any straight line, parallel to the axis of* y, *will meet one of the curves in two real points, which may also be considered as imaginary points on the other curve.* On account of this remarkable relation, the curves (1) and (2) have been called *supplementary conics in relation to the axis of* y. But since the property does not hold good with respect to straight lines drawn in any other direction, the curves (1) and (2) are not supplementary, except in relation to the system of parallel straight lines in question.

(*b*.) Similarly, any two curves A and B may be said to be *supplementary in relation to a given straight line*, when any straight line parallel to the given one meets the curve A in

p real and q imaginary points, while the same straight line meets the curve B in q real and p imaginary points.

(c.) It is evident from equations (1) and (2) that the diameter

$$Ay + Bx + D = o \ldots\ldots\ldots (3),$$

which bisects chords parallel to the axis of y, is common to the curves (1) and (2); hence, *if the two conics are supplementary in relation to a given straight line, the diameters of the two curves which bisect chords parallel to that line are coincident in direction.*

(d.) Equations (1) and (2) may evidently be written in the form

$$(Ay + Bx + D)^2 = (Bx + D)^2 - A\,(Cx^2 + 2\,Ex + F) \ldots (4),$$

$$(Ay + Bx + D)^2 = -\,(Bx + D)^2 + A\,(Cx^2 + 2\,Ex + F) \ldots (5),$$

from which we see (II, 8) that the supplementary conics (1) and (2) touch the two straight lines represented by the equation

$$(B^2 - AC)\,x^2 + 2(BD - AE)\,x + D^2 - AF = o \ldots\ldots (6)$$

at the points in which these lines are cut by the common diameter (3).

When the given curve is a parabola, we have $B^2 - AC = o$; hence, in this case, equation (6) represents only one straight line, which touches each of the supplementary conics at the point in which it is cut by the common diameter (3).

(e.) From equations (4) and (5) we obtain by addition

$$(Ay + Bx + D)^2 = o,$$

hence (II, c) the curves (4) and (5) have a double contact, the straight line (3) being the chord of contact. Thus we see that *two conic sections which are supplementary in relation to a given straight line have a double contact, the chord of contact being the diameter of each curve which bisects chords parallel to the given line.*

(f.) If equation (5) be denoted by

$$A'y^2 + 2B'xy + C'x^2 + 2D'y + 2E'x + F' = 0,$$

we shall have $A' = A^2$, $B' = AB$, $C' = 2B^2 - AC$,

$$\therefore B'^2 - A'C' = -A^2(B^2 - AC);$$

hence, *if the original curve be an ellipe the supplementary curve will be an hyperbola, but if the original curve be a parabola the supplementary one will also be a parabola.*

(*g.*) If any straight line be drawn through the origin A, meeting the curve (4) in a real point P (x, y), the straight line (3) in a point Q, and the two straight lines (6) in R_1 and R_2, we shall have (V, 3)

$$PQ : AQ = -(Ay + Bx + D) : D$$

$$PR_1 . PR_2 : AR_1 . AR_2 = \{(B^2 - AC)x^2 + 2(BD - AE)x + D^2 - AF\} : (D^2 - AF)$$

and in virtue of these equation (4) becomes

$$D^2\left(\frac{PQ}{AQ}\right)^2 = (D^2 - AF)\frac{PR_1 . PR_2}{AR_1 . AR_2} \quad \text{.........(7).}$$

If the same line meet the curve (5) in a real point P, we shall have

$$D^2\left(\frac{PQ}{AQ}\right)^2 = (AF - D^2)\frac{PR_1 . PR_2}{AR_1 . AR_2} \quad \text{.........(8).}$$

Hence we see that each of the supplementary curves (4) and (5) may be considered as the locus of a point P, such that $\frac{PR_1 . PR_2}{AR_1 . AR_2}$ has a fixed ratio to $\left(\frac{PQ}{AQ}\right)^2$; the algebraic sign of this ratio being different for the two curves, but its absolute magnitude being the same for both.

(*h.*) Let it be required to find the locus of a point P, such that if a straight line be drawn from it to a fixed point A, meeting the given straight line (3) in Q, and the two parallel straight lines (6) in R_1 and R_2, then $\frac{PR_1 . PR_2}{AR_1 . AR_2}$ shall have a given ratio to $\left(\frac{PQ}{AQ}\right)^2$.

To obtain the most general solution of this question, it would seem that the locus must include not only all the points between the straight lines (6) which satisfy the proposed condition, but also all points external to these lines on either side of them which satisfy the condition; and in a geometrical point of view the exterior and interior points in question seem to have perfectly equal claims to be considered as points of the locus. Now it is evident from what has been advanced above (g), that, when the given ratio is considered as a positive quantity, the equation to the required locus is (4), which represents the internal points; but when the ratio is considered as negative, the equation to the required locus is (5), which represents the external points. Thus we see that neither of the equations will represent the whole locus of external and internal points, unless we include the imaginary values of the variables; but that, with this assumption, either of the equations will represent the entire locus. Thus we have a remarkable instance, in which the science of algebra seems to be at fault, in point of generality, as compared with the kindred science of geometry; and it is not possible to remedy the defect except by admitting imaginary values of the algebraic symbols. This seems to afford a strong argument for considering the two supplementary curves (4) and (5) as branches of the same general curve; but I must admit that the authority even of CHASLES himself is opposed to this view of the subject.*

(i.) If the direction of the straight lines (6) be supposed to change, while the curve (4) remains invariable, the magnitude and position of the curve (5) will be changed; and conversely, if the straight lines (6 change their direction, while the curve (5) remains unchanged, the magnitude and position of the curve (4) will vary. In fact, we have seen (g) that the two supplementary conics (4) and (5) can be viewed as intimately related to the three straight lines (3) and (6), and of course

* See his *History of Geometry*, Note XXVI., page 396 of the German edition.

one or both of the conics may vary with any variation in the fundamental data. This consideration may perhaps remove Mr. Salmon's objection to this theory, which he has founded on the fact that any conic section has an infinite number of supplementary conics.

(*k.*) *If a system of conic sections have an ideal common secant, it will be a real common secant of the system of conics which is supplementary to the given one in relation to any straight line parallel to the common secant.* For, since every curve of the given system cuts the ideal chord in the same two imaginary points, these will evidently be real points on every curve of the supplementary system.

VII.

If two conic sections have a double contact, and if their supplementary conics be taken in relation to the chord of contact, the supplementary conics will have a double contact at the same points as the first two conics; so that, if the chord of contact be real for the one system it will be ideal for the other.

Let the axis of y be parallel to the chord of contact, then if the equation to one of the conics be

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0 \ldots\ldots(1),$$

the equation of the other conic will be (II, c)

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = m(b-x)^2 \ldots(2),$$

where $b - x = 0$ denotes the chord of contact. Now from equation (2) we readily obtain

$$Ay = -(Bx + D)$$
$$\pm\sqrt{\{(Bx+D)^2 - A(Cx^2 + 2Ex + F) + Am(b-x)^2\}},$$

from which it is evident (VI, a) that the equation of the conic supplementary to the curve (2) will be

$$(Ay + Bx + D)^2 = -(Bx + D)^2 + A(Cx^2 + 2Ex + F)$$
$$- Am(b-x)^2 \ldots\ldots\ldots\ldots(3).$$

The equation of the conic supplementary to the curve (1) may be found from the last equation by taking $m = o$, which gives

$$(Ay + Bx + D)^2 = -(Bx + D)^2 + A(Cx^2 + 2Ex + F) \ldots (4);$$

and it is evident (II, c) that the curves (3) and (4) have a double contact, the chord of contact being $b - x = o$.

VIII.

To find the curve which is supplementary to any given curve of the third degree in relation to a straight line parallel to an asymptote.

If the given straight line be taken as the axis of y, the equation to the given curve will be of the form

$$By^2x + 2Cyx^2 + Dx^3$$
$$+ Ey^2 + 2Fxy + Gx^2 + 2Hy + Kx + L = o.$$

By solving this equation for y, we obtain

$$(1) \ldots\ldots\ldots\ldots (Bx + E)y = -(Cx^2 + Fx + H)$$
$$\pm\sqrt{\{(Cx^2 + Fx + H)^2 - (Bx + E)(Dx^3 + Gx^2 + Kx + L)\}},$$

and therefore the equation to the required curve is

$$(2) \ldots\ldots\ldots\ldots (Bx + E)y = -(Cx^2 + Fx + H)$$
$$\pm\sqrt{\{-(Cx^2 + Fx + H)^2 + (Bx + E)(Dx^3 + Gx^2 + Kx + L)\}}.$$

(a) It is evident from the forms of these two equations that the hyperbola

$$(Bx + E)y = -(Cx^2 + Fx + H) \ldots\ldots\ldots\ldots (3)$$

bisects every chord of either curve which is parallel to the axis of y. Hence we see that *the curvilinear diameter which bisects chords parallel to the axis of* y *is common to the two supplementary curves.*

(b) By clearing the radicals from equations (1) and (2), we get

$$\{(Bx+E)y+Cx^2+Fx+H\}^2=(Cx^2+Fx+H)^2$$
$$-(Bx+E)(Dx^3+Gx^2+Kx+L)\,.........(4),$$
$$\{(Bx+E)y+Cx^2+Fx+H\}^2=-(Cx^2+Fx+H)^2$$
$$+(Bx+E)(Dx^3+Gx^2+Kx+L)\,.........(5);$$

from which we see (III) that each of the supplementary curves touches the four straight lines represented by the equation

$$(C^2-BD)x^4+(2CF-BG-DE)x^3+(F^2+2CH-BK-GE)x^2$$
$$+(2FH-BL-KE)x+H^2-LE=o\,......(6)$$

at the eight points in which these lines are cut by the hyperbola (3).

(*c*) By adding together equations (4) and (5), we get

$$\{(Bx+E)y+Cx^2+Fx+H\}^2=o,$$

from which it is evident (III) that *the two supplementary curves touch each other at eight points, which lie on the hyperbola* (3).

(*d*) Let a straight line be drawn from the origin A to any point P (x, y) on the curve (4), cutting the hyperbola (3) in the points Q_1 and Q_2, and meeting the four straight lines (6) in the points R_1, R_2, R_3, R_4; then, (V),

$$\frac{PQ_1\,.\,PQ_2}{AQ_1\,.\,AQ_2}=\frac{(Bx+E)y+Cx^2+Fx+H}{H},$$

$$\frac{PR_1.PR_2.PR_3.PR_4}{AR_1.AR_2.AR_3.AR_4}=\frac{(C^2-BD)x^4+\&c......+(H^2-LE)}{H^2-LE};$$

and in virtue of these equation (4) becomes

$$H^2.\left(\frac{PQ_1\,.\,PQ_2}{AQ_1\,.\,AQ_2}\right)^2=(H^2-LE)\frac{PR_1.PR_2.PR_3.PR_4}{AR_1.AR_2.AR_3.AR_4}...(7).$$

If the same straight line meet the curve (5) in a real point P, we should find, in a similar manner

$$H^2.\left(\frac{PQ_1\,.\,PQ_2}{AQ_1\,.\,AQ_2}\right)^2=(LE-H^2)\frac{PR_1.PR_2.PR_3.PR_4}{AR_1.AR_2.AR_3.AR_4}...(8).$$

Hence we see that *each of the supplementary curves* (4) *and* (5) *can be considered as the locus of a point* P, *such that*

$$\frac{PR_1 \,.\, PR_2 \,.\, PR_3 \,.\, PR_4}{AR_1 \,.\, AR_2 \,.\, AR_3 \,.\, AR_4} \textit{ is to } \left(\frac{PQ_1 \,.\, PQ_2}{AQ_1 \,.\, AQ_2}\right)^2$$

in a fixed ratio; and thus, in a geometrical point of view, the curves (4) and (5) can be considered as two branches of the same curve.

IX.

If the equation to a given curve be of the form

$$y^2 - 2y \,.\, \phi x + (\psi x)^2 = 0,$$

where ϕ and ψ denote any given functions, it is required to find the supplementary curve in relation to a system of straight lines parallel to the axis of y.

By solving the equation as a quadratic, we obtain

$$y = \phi x \pm \sqrt{\{(\phi x)^2 - (\psi x)^2\}} \ldots\ldots\ldots\ldots (1),$$

from which it is evident (VI, *b*) that the equation of the required curve will be

$$y = \phi x \pm \sqrt{\{(\psi x)^2 - (\phi x)^2\}} \ldots\ldots\ldots\ldots (2).$$

(*a*) It is evident from the last two equations that the curve whose equation is

$$y = \phi x \ldots\ldots\ldots\ldots (3)$$

bisects all chords of the curves (1) and (2) which are parallel to the axis of y. Hence *the curvilinear diameter which bisects chords parallel to the axis of* y *is common to the supplementary curves* (1) *and* (2).

(*b*) By clearing the equations (1) and (2) of radicals, we obtain

$$(y - \phi x)^2 = (\phi x)^2 - (\psi x)^2 \ldots\ldots\ldots\ldots (4),$$

$$(y - \phi x)^2 = (\psi x)^2 - (\phi x)^2 \ldots\ldots\ldots\ldots (5);$$

from which it is evident (III) that *each of the supplementary curves touches the system of straight lines represented by the equation*

$$(\phi x)^2 - (\psi x)^2 = 0 \ldots\ldots\ldots\ldots (6)$$

at the points in which these lines are cut by the diametral curve (3).

(*c*) By taking the sum of equations (4) and (5) we obtain

$$(y - \phi x)^2 = 0,$$

from which we see (III) that *the supplementary curves* (4) *and* (5) *touch each other at each of the points in which they are cut by the curve* (3).

(*d*) Let a straight line be drawn from the origin A to any point P in the curve (4), meeting the curve (3) in the points Q_1, Q_2, Q_3, &c., and meeting the system of straight lines (4) in the points R_1, R_2, R_3, &c.; then if ϕx be an algebraic function of x of the nth degree, and ψx one of the mth degree, where m is not greater than n, we shall have (V)

$$\frac{PQ_1 \,.\, PQ_2 \,.\, PQ_3 \ldots\ldots PQ_n}{AQ_1 \,.\, AQ_2 \,.\, AQ_3 \ldots\ldots AQ_n} = \pm \frac{y - \phi x}{M},$$

$$\frac{PR_1 \,.\, PR_2 \,.\, PR_3 \ldots\ldots PR_{2n}}{AR_1 \,.\, AR_2 \,.\, AR_3 \ldots\ldots AR_{2n}} = \frac{(\phi x)^2 - (\psi x)^2}{T},$$

where M and T are the absolute terms of the functions ϕx and $(\phi x)^2 - (\psi x)^2$ respectively. Hence, by substituting these geometrical expressions in equation (4), it becomes

$$M^2 \left(\frac{PQ_1 \,.\, PQ_2 \ldots PQ_n}{AQ_1 \,.\, AQ_2 \ldots AQ_n}\right) = T \frac{PR_1 \,.\, PR_2 \ldots PR_{2n}}{AR_1 \,.\, AR_2 \ldots AR_{2n}} \ldots\ldots (7).$$

Similarly, if the same straight line meet the curve (5) in any point P, we shall have

$$M^2 \left(\frac{PQ_1 \,.\, PQ_2 \ldots PQ_n}{AQ_1 \,.\, AQ_2 \ldots AQ_n}\right) = -T \frac{PR_1 \,.\, PR_2 \ldots PR_{2n}}{AR_1 \,.\, AR_2 \ldots AR_{2n}} \ldots . (8).$$

From the last two equations it is evident that *each of the supplementary curves can be considered as the locus of a point* P *such that*

$$\frac{PR_1 . PR_2 \ldots\ldots PR_{2n}}{AR_1 . AR_2 \ldots\ldots AR_{2n}} \text{ is to } \left(\frac{PQ_1 . PQ_2 \ldots\ldots PQ_n}{AQ_1 . AQ_2 \ldots\ldots AQ_n}\right)^2$$

in a fixed ratio; and thus we see that the two curves (4) and (5) can be considered as two branches of a geometrical locus, which is determined by a unique geometrical condition.

X.

In some cases the equation to the supplementary curve can be obtained with great facility. Thus, the equation to the *Conchoid of Nicomedes* being

$$m^2 x^2 = (p - x)^2 (x^2 + y^2) \ldots\ldots\ldots\ldots (1),$$

it is evident that the equation of the curve which is supplementary to it in relation to the axis of y will be

$$m^2 x^2 = (p - x)^2 (x^2 - y^2 \ldots\ldots\ldots\ldots (2),$$

and the supplementary curves (1) and (2) will possess properties analogous to those which have been noticed in the preceding examples.

(*a*) It is evident from the form of these equations, that the origin is a double point on each of the supplemementary curves (1) and (2). For the curve (1) the tangents at this point are represented by the equation

$$p^2 y^2 + (p^2 - m^2) x^2 = 0 \ldots\ldots\ldots\ldots (3),$$

and for the curve (2) they are represented by the equation

$$-p^2 y^2 + (p^2 - m^2) x^2 = 0 \ldots\ldots\ldots\ldots (4).$$

(*b*) When $m > p$, the tangents (3) are real and the tangents (4) are imaginary, so that the origin is a double point of the curve (1) and a conjugate point of the curve (2); but, when $m < p$, the tangents (4) are real, and the tangents (3) are imaginary, so that the origin is a double point of the curve (2) and a conjugate point of the curve (1). Thus we see that *a double point on one of the curves is a conjugate point on the supplementary curve; and that the real and imaginary tangents, applied at that point to the two curves, form two supplementary systems of straight lines.*

(*c*) When $m = p$, each of the equations (3) and (4) becomes $y^2 = 0$; hence, in this case, the axis of x is a double tangent to each of the supplementary curves (1) and (2) at the origin. Hence we see that *when a curve has a cusp, this point will also be a cusp on the curve which is supplementary to the given one in relation to the double tangent at its cusp.*

SECTION III.—OF CURVES WHICH ARE SUPPLEMENTARY IN RELATION TO A GIVEN POINT.

XI.

A curve of the second degree being given, to find the locus of the imaginary points in which it is cut by a system of straight lines which pass through a given point A.

When the given point A is without the curve, so that two real tangents can be drawn from it to the curve, it is evident that innumerable straight lines can be drawn through the point so that each of them shall cut the curve in two imaginary points; but when the given point is on the curve, or within it, it is plain that no straight line can be drawn through A to meet the curve in any imaginary point. Hence it will be sufficient to consider the case in which the given point A is without the given curve, since the locus cannot exist in any other case.

If the given point A be taken as the origin of rectangular co-ordinates, the equation to the given curve will be of the form

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + F = 0.$$

Let ρ and θ denote the polar co-ordinates of the point x, y; then, by dividing the preceding equation by ρ^2, we obtain

$$A\sin^2\theta + 2B\sin\theta\cos\theta + C\cos^2\theta + 2(D\sin\theta + E\cos\theta)\tfrac{1}{\rho} + F(\tfrac{1}{\rho})^2 = 0;$$

$$\therefore\ F : \rho = -(D\sin\theta + E\cos\theta)$$

$$\pm\sqrt{\{(D\sin\theta + E\cos\theta)^2 - F(A\sin^2\theta + 2B\sin\theta\cos\theta + C\cos^2\theta)\}} \ldots(1)$$

is the polar equation of the given curve, and by changing the algebraic sign of the quantity under the radical sign, we shall have

$$F : \rho = -(D\sin\theta + E\cos\theta)$$

$$\pm\sqrt{\{F(A\sin^2\theta + 2B\sin\theta\cos\theta + C\cos^2\theta) - (D\sin\theta + E\cos\theta)^2\}} \ldots(2)$$

which is the polar equation of the required locus.

(*a*) Since any value of θ which gives real values of ρ in equation (2) will give imaginary values of ρ in equation (1), it follows that the given curve (1) may be considered as the locus of the imaginary points in which the curve (2) is cut by any system of straight lines passing through the given point. Hence if any straight line passing through A cut either of the curves in two real points, these may also be considered as imaginary points on the other curve. From the analogy of this property to the one obtained above (VI, *a*), the curves (1) and (2) may be considered as *supplementary in relation to the given point* A. And, in general, *any two curves may be said to be supplementary in relation to a given point*, when any straight line drawn through the point meets one of the curves in p real and q imaginary points, which can also be considered as p imaginary and q real points on the other curve.

(*b*) By passing from polar to rectangular co-ordinates, and by clearing the radicals, equations (1) and (2) become

$$(Dy + Ex + F)^2 = -F(Ay^2 + 2Bxy + Cx^2) + (Dy + Ex)^2 \ldots(3),$$

$$(Dy + Ex + F)^2 = F(Ay^2 + 2Bxy + Cx^2) - (Dy + Ex)^2 \ldots(4),$$

hence we see (II, 8) that each of the curves (1) and (2) touches the two straight lines represented by the equation

$$(D^2 - AF)y^2 + 2(DE - BF)xy + (E^2 - CF)x^2 = o \ldots (5),$$

the equation of the chord of contact being

$$Dy + Ex + F = o \ldots\ldots\ldots\ldots (6),$$

which is therefore the polar of the given point A taken in relation to each of the given curves. Hence *if two conic sections be supplementary in relation to a given point, the polars of the point taken in relation to the two conic sections are coincident in direction.*

(*c*) By adding together equations (3) and (4), we obtain

$$(Dy + Ex + F)^2 = o;$$

from which we see that the curves (3) and (4) have a double contact, the straight line (6) being the chord of contact. Hence, *two conic sections which are supplementary in relation to a given point have a double contact, the chord of contact being the polar of the given point in relation to each of the curves.*

(*d*) When the given curve is a hyperbola, having its centre at the given point, we have $D = o$ and $E = o$, so that equations (3) and (4) become

$$Ay^2 + 2Bxy + Cx^2 + F = o \ldots\ldots\ldots\ldots (7),$$

$$Ay^2 + 2Bxy + Cx^2 - F = o \ldots\ldots\ldots\ldots (8).$$

In this case the curves have been called conjugate hyperbolas; hence we see that *any two conjugate hyperbolas are supplementary in relation to their common centre.*

(*e*) From any point $P(x, y)$ in the curve (3) let PQ be drawn perpendicular to the straight line (6), and let PR_1 and PR_2 be perpendicular to the straight lines (5); then (V, c) we shall have

$$Dx + Ey + F = PQ \,.\, \sqrt{(D^2 + E^2)},$$

$$(D^2 - AF)y^2 + 2(DE - BF)xy + (E^2 - CF)x^2$$
$$= M \,.\, PR_1 \,.\, PR_2,$$

where $M^2 = \{D^2 - E^2 - F(A - C)\}^2 + 4(DE - BF)^2$...(9);
and by substituting these expressions in equation (3) we get

$$(D^2 + E^2) . PQ^2 = M . PR_1 . PR_2 \text{...........(10)}.$$

Similarly, if perpendiculars be drawn from any point P in the curve (4) to the straight lines (5) and (6), and if the same notation be adopted, we shall have

$$(D^2 + E^2) . PQ^2 = - M . PR_1 . PR_2 \text{...........(11)}.$$

Thus we see that *each of the supplementary curves* (3) *and* (4) *may be considered as the locus of a point, such that if perpendiculars be drawn from it to the three straight lines* (5) *and* (6), *the product of the perpendiculars drawn to the straight lines* (5) *shall have a given ratio to the square of the third perpendicular; the algebraic sign of this ratio being different for the two curves, but its absolute magnitude being the same for both.* From this theorem it is evident (VI, *h*) that the curves (3) and (4) may be considered as branches of the same geometrical curve, although the whole curve cannot be represented by either of these equations unless we take in the imaginary values of the variables.

(*f*) We have seen (IV) that any two conic sections described in a plane have at least one real point of intersection of conjugate common secants. Now when the four points of intersection of the given curves are all imaginary, if the curves be constructed which are supplementary to the given ones in relation to a real point of intersection of conjugate common secants, since each of these curves passes through all the imaginary points of its supplementary curve, it follows that the supplementary curves will cut each other in four real points, which are the imaginary points of intersection of the two given curves. Hence when two conic sections are entirely exterior to each other, or when one of them is entirely within the other, their imaginary points of intersection may be obtained by constructing the curves which are supplementary to the given ones in relation to a point of intersection of conjugate common secants of the two given curves.

XII.

To find the curve which is supplementary to a given curve of the fourth degree, in relation to a double point A on the given curve.

If the point A be taken as the origin of rectangular co-ordinates, the equation to the given curve will be of the form

$$\begin{aligned} &Ay^4 + By^3x + Cy^2x^2 + Dyx^3 + Ex^4 \\ &\quad + 2(A'y^3 + B'y^2x + C'yx^2 + D'x^3) \\ &\qquad + A''y^2 + B''xy + C''x^2 = 0 \text{ ....(1).} \end{aligned}$$

Let r and θ be the polar co-ordinates of the point x, y; then, the preceding equation becomes

$$A_1 r^4 + 2B_1 r^3 + C_1 r^2 = 0 \text{ ....(2),}$$

where, for the sake of brevity, we assume

$$\left.\begin{aligned} A_1 &= A\sin^4\theta + B\sin^3\theta\cos\theta + \&c. \\ B_1 &= A'\sin^3\theta + B'\sin^2\theta\cos\theta + \&c. \\ C_1 &= A''\sin^2\theta + B''\sin\theta\cos\theta + C''\cos^2\theta \end{aligned}\right\} \text{......(3).}$$

Now, by rejecting the common factor r^2, and solving equation (2) as a quadratic, we get

$$A_1 r = -B_1 \pm \sqrt{(B_1^2 - A_1 C_1)} \text{ ....(4),}$$

which may be considered as the polar equation of the given curve; and by changing the algebraic signs of the quantities under the radical sign, we obtain

$$A_1 r = -B_1 \pm \sqrt{(A_1 C_1 - B_1^2)} \text{ ....(5),}$$

which is the polar equation of the required curve. It is evident from the values of A_1, B_1, C_1 given above, that the curve (5) is of the eighth degree, the origin being a multiple point of the sixth order.

(*a*) It is evident from the forms of the equations (4) and (5), that the curve whose equation is

$$A_1 r + B_1 = o \ldots\ldots\ldots\ldots(6)$$

bisects every chord of either curve which passes through the origin A. Hence *the locus of the points of bisection of all chords of the curves* (4) *and* (5) *which pass through the common multiple point* A, *is a curve of the fourth degree having* A *for a triple point.*

(*b*) By clearing the equations (4) and (5) of radicals, we obtain

$$(A_1 r + B_1)^2 = B_1^2 - A_1 C_1 \ldots\ldots\ldots\ldots(7),$$

$$(A_1 r + B_1)^2 = A_1 C_1 - B_1^2 \ldots\ldots\ldots\ldots(8);$$

from which it is evident (III) that *each of the supplementary curves touches the system of straight lines represented by the equation*

$$B_1^2 - A_1 C_1 = o \ldots\ldots\ldots\ldots(9)$$

at the points in which these lines are cut by the curve (6). Hence, also, the supplementary curves touch each other at the points in question.

(*c*) If a straight line be drawn from the origin A to any point P on either of the supplementary curves, cutting the curve (6) in the points Q_1, Q_2, Q_3, Q_4; and if PR_1, PR_2, PR_3, &c. be the perpendiculars drawn from P to the system of straight lines (9), we should find as in No. XI. that

$$PR_1 \,.\, PR_2 \ldots\ldots PR_6 \textit{ is to } \left(\frac{PQ_1 \,.\, PQ_2 \,.\, PQ_3 \,.\, PQ_4}{AQ_1 \,.\, AQ_2 \,.\, AQ_3 \,.\, AQ_4}\right)^2$$

in a fixed ratio; and therefore the two supplementary curves (4) and (5) may be considered as branches of a geometrical locus, all the points of which are determined by a unique geometrical condition.

XIII.

The results which have been obtained in the last two numbers can be extended to a certain class of algebraic curves of every degree. Thus, if a curve of the nth degree have a multiple point of the order $n-2$, and if this point be taken as the origin of rectangular co-ordinates, the polar equation of the curve will be of the form

$$A_1 r^n + 2 B_1 r^{n-1} + C_1 r^{n-2} = 0,$$

where A_1, B_1, C_1, are algebraic functions of $\sin\theta$ and $\cos\theta$ of the degrees n, $n-1$, and $n-2$ respectively. Rejecting the common factor r^{n-2}, and solving this equation as a quadratic, we obtain

$$A_1 r = -B_1 \pm \sqrt{(B_1^2 - A_1 C_1)} \;............(1);$$

and by changing the algebraic signs of the quantities under the radical sign we get

$$A_1 r = -B_1 \pm \sqrt{(A_1 C_1 - B_1^2)} \;............(2),$$

which is the polar equation of the curve supplementary to the given one in relation to the multiple point which we have taken as the origin of co-ordinates. Since these equations are of the same form as those obtained in the last two numbers, the same reasoning will lead to results in every respect similar to those which have been there developed.

XIV.

The reader will not fail to perceive that the theory which I have endeavoured to explain can be extended to surfaces with the utmost facility. In the case of supplementary surfaces, the general problem to be solved is as follows:—

Any surface being given, to find the locus of the imaginary points in which it is cut by a system of straight lines, drawn so as to succeed one another according to any given law.

For instance, if the given surface be one of the second degree, and if the system of straight lines be drawn parallel to a given line, we should find, as in No. VI., that the supplementary surface is also one of the second degree touching the given surface along the plane curve in which it is cut by the diametral plane which bisects chords parallel to the given line. And as the preliminary propositions in Section I. can readily be extended to surfaces, the two supplementary surfaces will be found to possess properties exactly analogous to those developed in the notes to No. VI. But having extended this paper rather beyond the prescribed limits, I must conclude by merely pointing out to the reader a fertile and interesting field of inquiry, which, as far as I know, has not been hitherto investigated.