NAPIER'S LOGARITHMS: THE DEVELOPMENT OF HIS THEORY.

By H. S. CARSLAW, Sc. D.

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Introductory.

§ 1. This paper deals with Napier's idea of a logarithm.[1] In my view there are three distinct stages in the development of this idea in his work. In the first he is concerned with a one-one correspondence between the terms of a Geometrical Progression and the terms of an Arithmetical Progression. There are traces of this in the *Constructio*[2] in his use of the series

$$10^7, 10^7\left(1 - \frac{1}{10^7}\right), 10^7\left(1 - \frac{1}{10^7}\right)^2, \text{etc.,}$$

and in the word logarithm itself, derived from λόγος ἀριθμός, and generally taken to mean "the number of the ratios." In the second he has passed from this correspondence, and his logarithms are given by the well known kinematical definition, which forms the foundation of the theory of the *Constructio*. In the third, referred to in the Appendix to the *Constructio*, he has reached the idea of a logarithm as *defined* by the property :—

[1] In a previous paper in these Proceedings:—The Discovery of Logarithms by Napier of Merchiston—Vol. 48, pp. 42–72, 1914—the question of the construction of the logarithms of the *Canon* has been discussed.

[2] The *Mirifici Logarithmorum Canonis Constructio* was published in 1619, two years after Napier's death, but had been written several years before his *Mirifici Logarithmorum Canonis Descriptio*, published in 1614. I shall refer to these works as the *Constructio* and the *Descriptio*. The *Descriptio* was translated into English by Wright (1616), and Filipowski (1857), the *Constructio* by Macdonald (1889). The former is a rare book, both in the original and in translation. Several of the more important pages are reproduced in the *Napier Tercentenary Memorial Volume*, Plates I–VI, (London, 1915).

The logarithms of proportional numbers have equal differences, with the additional condition that the logarithms of two numbers are given.

In the second and third stages he has obtained what we would now call a function of the independent variable—the number—, but the function of the third stage is more general than that of the second, which it includes as a special case.

If this view is correct, the statement that "Napier's *theory* rests on the establishment of a one-one correspondence between the terms of a geometric series and the terms of an arithmetic series"[1] should not be taken too literally. Further the custom of employing the term "Napier's logarithms" to describe only the logarithms of his *Canon* is unfortunate. It will be seen in the course of this paper that logarithms to the base 10—as we know them—are Napier's logarithms just as much as the logarithms of his *Canon.*

The First Stage.

§ 2. The idea that multiplication and division could be reduced to addition and subtraction by the correlation of a geometrical series and an arithmetical series was not a new one. Aristotle was familiar with it, and since his time many mathematicians had returned to it. If we take the series

1, 2, 3, 4, 5, 6, 7, 8, ... 15, ...
2, 4, 8, 16, 32, 64, 128, 256,.........32768,......

the product of 128 and 256 in the geometrical series can be read off as 32768, which corresponds to 15, the sum of 7 and 8 in the arithmetical series.

[1] Cajori, Napier's Logarithmic Concept: A Reply. *American Mathematical Monthly,* Vol, 23, p. 71, (1915).

The Swiss Bürgi in his *Arithmetische and Geometrische Progress Tabulen,*[1] constructed some time between 1603 and 1611, but first published in 1620, used the series

$$\begin{array}{llll} 10 \times 0,\ 10 \times 1 & ,\quad 10 \times 2 & ,\ldots 10 \times n, \ldots\ldots \\ 10^8, \quad 10^8\left(1+\frac{1}{10^4}\right), & 10^8\left(1+\frac{1}{10^4}\right)^2, \ldots & 10^8\left(1+\frac{1}{10^4}\right)^n, \ldots \end{array}$$

His tables cover the range 10^8 to 10^9, and for all practical purposes are as satisfactory as Napier's Table of Logarithms of 1614. If Napier had simply used the idea of the correspondence between the terms of a geometrical series and the terms of an arithmetical series, his work could not be regarded as so great an advance upon Bürgi's as it really is.

But it is clear that at the beginning of his labours, which extended over a period of twenty years, Napier's mind was working on the same lines as Bürgi's, and that at this stage he used the series

$$\begin{array}{lll} 0, \quad 1 & ,\quad 2 & ,\ldots\ldots \\ 10^7,\ 10^7\left(1-\frac{1}{10^7}\right), & 10^7\left(1-\frac{1}{10^7}\right)^2, & \ldots\ldots \end{array}$$

in a similar way. This geometrical series occurs in the *Constructio.* He employed it in the calculation of his logarithms, but neither then, nor later, are his logarithms the terms of the corresponding arithmetical series. His word logarithm, (See §1), is evidently a survival of the first stage of his work.

Napier meant his tables to be used in calculations involving the trigonometrical ratios. In his time, the sine, cosine, etc., were lines—or, more exactly, the measures of lines—in a circle of given radius. Napier took the radius

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[1] A facsimile of the title page of Bürgi's work and of one of the pages of the Tables will be found in the *Napier Tercentenary Memorial Volume* (Plates XII and XIII). Comparison with the references in Cantor's *Geschichte der Mathematik,* Tropfke's *Geschichte der Elementar-Mathematik,* and Braunmühl's *Geschichte der Trigonometrie* will show that in none of these works is the title quoted correctly.

as 10^7. It may be that Bürgi chose 10^8 in his tables for a similar reason. With our notation Napier's sines would correspond to 7-Figure Tables of Natural Sines, etc. If greater accuracy were required, the radius was taken as 10^{10}, and sometimes even a higher power of 10 was used. These sines, etc., following Glaisher,[1] we shall refer to as line-sines, etc.

The Second Stage.

§ 3. Napier opened out entirely fresh ground, when he passed to his kinematical definition of the logarithm of a sine or number. By this definition he associated with the sine, as it continually diminished from 10^7 for 90° to zero for 0°, a number which he called its logarithm; and the logarithm continually increased from 0, for the sine of 90°, to infinity, for the sine of 0°.

The fundamental proposition in Napier's theory in the *Descriptio* (1614) and the *Constructio* (1619) is to be found in Prop. I of the *Descriptio:*

The logarithmes of proportionall numbers and quantities are equally differing.[2]

And in Section 36 of the *Constructio* it appears as *the logarithms of similarly proportioned sines are equidifferent.*

Glaisher has introduced a convenient notation $nl_r\, x$ for Napier's logarithm, in this system, when the radius is 10^r. He also uses $\mathrm{Sin}_r\, x$ for the line-sine of the angle x, when the radius is 10^r, and he keeps the symbol $\sin x$ for the sine in the modern sense of the term. With this notation we have

$$\sin x = \frac{\mathrm{Sin}_r x}{10^r}.$$

[1] *Quarterly Journal*, Vol. 46, p. 125 (1916). To this paper I am indebted, not only for a most convenient notation for the different systems of logarithms, but also for an account of Speidell's work, hitherto inaccessible to me.

[2] In quoting the *Descriptio* I follow Wright's version, and for the *Constructio* I adopt Macdonald's.

In this paper I follow his notation, and $\log_e x$ is used in its modern sense for the logarithm of x to the base e, the system commonly called hyperbolic logarithms.

The fundamental theorem, referred to above, can now be stated as follows:—

If $a : b = c : d$, then $nl_r a - nl_r b = nl_r c - nl_r d$.........(1)
Also we are given that $nl_r 10^r = 0$. ....(2)
Napier's *Canon* consists of a Table of Logarithms in which (1) and (2) are satisfied. His definition of the logarithm by means of the velocities of two points moving in two different lines leads to the formula

$$nl_r x = 10^r \log_e\left(\frac{10^r}{x}\right).$$

But, of course, neither this, nor the fact that his function $nl_r x$ has -1 for its differential coefficient, when $x = 10^r$, could be known in his time.

The Third Stage.

§ 4. Since $uv : u = v : 1$,
we have $nl_r(uv) - nl_r u = nl_r v - nl_r 1$.

Thus $nl_r(uv) = nl_r u + nl_r v - nl_r 1$,
and it must be remembered that $nl_r 1$ is not zero.

When $r = 7$, $nl_r 1 = 161180896{\cdot}38$ (Cf. *Constructio*, Section 53).[1]

Similarly $nl_r(u/v) = nl_r u - nl_r v + nl_r 1$.

Thus multiplication and division are changed into addition and subtraction. But the logarithms of numbers with the same figures in the same order cannot be read off from one another, since, in this system,

$$nl_r(10^m a) = nl_r a - m(nl_r 1 - nl_r 10),$$

[1] The error in Napier's Second Table affects the accuracy of his Canon and this number should be 161180956·51. The alteration can be made from the corrected result given by Macdonald in his English translation of the *Constructio* pp. 94-5, for it is not difficult to show that $nl_7 1 = 7\,nl_7 10^6$.

and $nl_7 1 - nl_7 10 = 23025842{\cdot}34$ (Cf. *Constructio*, Section 53).[1]

It is obvious that if a system of logarithms could be devised in which the logarithm of unity is zero and the logarithm of 10 is unity, the calculations would be immensely simplified, and the table curtailed; because one of the chief defects of Napier's *Canon*, as well as of Bürgi's Tables, was that, if the numbers did not come within the range covered by it, more or less awkward calculations were needed to overcome this difficulty.

Napier's *Canon* was first printed in the *Descriptio* (1614). After his death in 1617 the *Constructio* was published by the care of his son. It had been written several years before the *Descriptio*. To this work was added an Appendix, by the hand of Napier himself, "On the Construction of another and *better kind of* Logarithms, *namely* one in which the Logarithm of unity is 0." This Appendix begins with the words:—

"*Among the various improvements of Logarithms, the more important is that which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one-tenth of unity or ten times unity. Then, these being once fixed, the Logarithms of all other numbers necessarily follow.*"

It is clear from Napier's words that, when he wrote the Appendix, not only did he see the advantage of such a system, but he was in a position to draw up a Table of Logarithms in which these conditions would be satisfied. Indeed he gives three distinct methods of finding these logarithms. The kinematical definition of the logarithm was superseded, and the correspondence between the terms

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[1] This number should be $23025850{\cdot}93$, since it is easy to show that $nl_7 1 - nl_7 10 = nl_7 10^6$, and Macdonald gives the corrected logarithm of 10^6 (*loc. cit.*, pp. 94-5).

of a geometrical series and the terms of an arithmetical series was left far behind. This is the third and final stage of his work.

Briggs and Napier.

§ 5. In the change from the logarithms of the Canon to this "better kind of logarithms" Briggs was associated with Napier; but, chiefly because of the unsatisfactory account of the matter given by Hutton in his *History of Logarithms*,[1] the share of the former in the discovery has been exaggerated. The fault is not due to Briggs; and, though his reference to the question in the preface to the *Arithmetica Logarithmica* (1624) is familiar, I reproduce it again here:—

"I myself, when expounding publicly in London their doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 should stand for the logarithm of the whole sine, as in the Canon Mirificus, but that the logarithm of the tenth part of the whole sine, that is to say, 5 degrees 44 minutes 21 seconds, should be 10,000,000,000. Concerning that matter I wrote immediately to the author himself; and as soon as the season of the year and the vacation time of my public duties of instruction permitted, I took journey to Edinburgh, where, being most hospitably received by him, I lingered for a whole month. But as we held discourse concerning this change in the system of logarithms, he said that for a long time he had been sensible of the same thing, and had been anxious to accomplish it, but that he had published those he had already prepared, until he could construct tables more convenient, if other weighty matters and his frail health would permit him to do. But he conceived that the change ought to be affected in this manner, that 0 should become the logarithm of unity, and 10,000,000,000 that of the whole sine; which I could not but admit was by far the most convenient of all. So, rejecting those which I had already

[1] Hutton's *Tracts on Mathematical and Philosophical Subjects*, Vol. I, Tract 20.

prepared, I commenced, under his encouraging counsel, to ponder seriously about the calculation of these tables."

Napier also mentions his discovery of the new system in the dedication of his *Rabdologia* (1617) in a passage quoted in my previous paper.[1]

It will be seen from Briggs' own words, that the modification which he suggested to Napier was to keep the logarithm of the radius as zero, but to take the logarithm of one-tenth of the radius as 10,000,000,000. His reference to the *Canon* is sufficient to show that he does not look upon the radius as unity. In the construction of the Table of Logarithms, after Napier's death, he takes it as 10^{10}, and it is for this reason that the characteristics 9, 8, etc., are to be found in the logarithms of the sines, etc.

Using the notation $bl_r x$ for the logarithm of x in the system suggested by Briggs when the radius is 10^r, we have

$$bl_r\, a - bl_r\, b = bl_r\, c - bl_r\, d,$$

when $a : b = c : d$.

Also $bl_r 10^r = 0$, and $bl_r 10^{r-1} = 10^{10}$.

In this system we have

$$bl_r\,(uv) = bl_r u + bl_r v - bl_r 1,$$
$$bl_r(u/v) = bl_r u - bl_r v + bl_r 1.$$

$$\text{Also } bl_{10} 10^{10} = 10\, bl_{10} 10 - 9\, bl_{10} 1 = 0$$
$$bl_{10} 10^{9} = 9\, bl_{10} 10 - 8\, bl_{10} 1 = 10^{10}.$$

Thus $bl_{10} 10 = 9 \times 10^{10}$ and $bl_{10} 1 = 10 \times 10^{10}$.

The advantage of the new system consists in the fact that the logarithms of numbers with the same figures in the same order could be read off from each other, since we have
$$bl_r(10^m a) = bl_r a - m \times 10^{10}.$$

§ 6. The change upon which Napier had resolved, previous to Briggs' visit, was a much more important one. He "conceived that the change ought to be affected in this

[1] See also Macdonald's English translation of the *Constructio*, p. 88.

manner, that 0 should become the logarithm of unity, and 10,000,000,000 that of the whole sine." And finally in the Appendix we see that he often passes from logarithms of sines, and now drops all reference to the radius.

In the new system, logarithms were to be *defined* by the relations:—

$$\text{If } a : b = c : d, \text{ then } nl\, a - nl\, b = nl\, c - nl\, d,$$
$$\text{with } nl\, 1 = 0 \text{ and } nl\, 10 = 10^{10}.$$

It need hardly be added that 10^{10} was taken for the logarithm of 10 instead of unity, for the same reason that 10^7 (or 10^{10}) was taken for the radius in dealing with the trigonometrical ratios.

Later Briggs takes the logarithm of 10 as unity, and introduces the notation of decimal fractions in his Tables, a notation employed, probably for the first time, by Napier himself.

If this account of the growth of the idea of a logarithm in Napier's work is correct,[1] it seems unfortunate that the term Napier's logarithms is usually confined to the logarithms of his *Canon.* His "better kind of logarithms" actually consists of the logarithms now in daily use—the logarithms which we call logarithms to the base 10. In some textbooks they receive the awkward name Briggsian logarithms. Certainly Briggs calculated them, and the rapidity and industry with which he performed this immense work in computation will always be the admiration of mathematicians. But the discovery of the system was Napier's, and the logarithms are as much Napier's logarithms as those of his *Canon.*

Speidell's New Logarithmes (1619).

§ 7. In most accounts of the discovery of logarithms reference is made to Speidell's *New Logarithmes* (London,

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[1] See also Gibson's paper in the *Napier Tercentenary Memorial Volume,* pp. 111 – 137.

1619), and it is stated that they contain the first table of logarithms to the base e.[1] Attention is also usually called to the fact that, while logarithms to the base e are frequently spoken of as Napierean logarithms, they are quite different from the logarithms of Napier's *Canon;* and it is pointed out that the place of the number e in the theory of logarithms and the possibility of defining logarithms as exponents were discoveries of a much later day. These two statements, at first sight, seem inconsistent. A word or two regarding Speidell's system will make the matter clearer, and will also confirm the view I have taken above as to Napier's final conception of the logarithm.

Speidell's *New Logarithmes*, like Napier's *Canon*, refer to the trigonometrical ratios. Using Glaisher's notation $sl_r x$ for Speidell's logarithm of x when the radius is 10^r, we have
$$sl_r x = 10^{r+1} - nl_r x.$$

It follows that
$$sl_r(uv) = sl_r u + sl_r v - sl_r 1,$$
$$sl_r(uv) = sl_r u - sl_r v + sl_r 1,$$
and $sl_r 1$ is not zero.

The sole advantages of this system was that it avoided the use of negative quantities in calculation with logarithms. Such quantities were then outside the range of the "vulgar and common arithmetic."

Since $nl_r x = 10^r \log_e \left(\dfrac{10^r}{x}\right)$,

we have $sl_r x = 10^{r+1} + 10^r \log_e \left(\dfrac{x}{10^r}\right)$.

$$\text{Thus } sl_r \operatorname{Sin}_r x = 10^{r+1} + 10^r \log_e \left(\frac{\operatorname{Sin}_r x}{10^r}\right)$$
$$= 10^r (10 + \log_e \sin x).$$

[1] In Glaisher's paper already referred to, he published the interesting discovery that an Appendix (1618) to Wright's English translation of the *Descriptio* contains a table of hyperbolic logarithms by an anonymous author, whom he identifies with Oughtred.

In a sense Speidell's *New Logarithmes* may be said to be hyperbolic logarithms, but the sense is the same as that in which the logarithms of Napier's *Canon* are sometimes said to be logarithms to the base e^{-1}. However this is a misuse of the term.[1] Still Speidell's logarithms of sines, from the accident that the sine is now used in a different sense, have actually the same figures as our hyperbolic logarithms of sines.

In the *New Logarithmes* (1619) he takes the radius as 10^5, so that these tables give

$$sl_5 \operatorname{Sin}_5 x = 10^5 (10 + \log_e \sin x).$$

§ 8. But subsequently Speidell *did* publish a table of hyperbolic logarithms of numbers, which gives the values of $10^6 \log_e x$ for numbers 1 to 1,000. This table probably appeared either separately, or attached to an impression of the *New Logarithmes*, in 1622 or 1623. In this system he takes

$$sl_r x = nl_r 1 - nl_r x.$$

It follows that

$$sl_r (uv) = sl_r u + sl_r v,$$
$$sl_r (u/v) = sl_r u - sl_r v;$$

and since

$$nl_r x = 10^r \log_e \left(\frac{10^r}{x}\right),$$

we have

$$sl_r x = 10^r \log_e x.$$

But it is clear that in both Speidell's systems of logarithms the connection with hyperbolic logarithms is accidental, and the same is true of the logarithms discovered by Glaisher, to which reference is made at the beginning of this section.

Like Napier and Briggs, Speidell sees that the fundamental property, that the logarithms of proportional numbers have equal differences, can be taken as the starting

[1] *Cf.* Glaisher, *loc. cit.*, p. 146, footnote.

point of the theory; and that, if the logarithm of unity is zero, the logarithms of the product and the quotient of two numbers are, respectively, the sum and difference of their separate logarithms.

§ 9. **The Differential Equation satisfied by the logarithm of x.**

We have seen that the theory of the different systems of logarithms described in the previous pages rests upon the fundamental property:—

If $a : b = c : d$, then $\lambda(a) - \lambda(b) = \lambda(c) - \lambda(d)$, where $\lambda(x)$ stands for the logarithm of x.

The function $\lambda(x)$, therefore, satisfies the equation

$$\lambda(x+h) - \lambda(x) = \lambda\left(1+\frac{h}{x}\right) - \lambda(1).$$

$$\therefore \quad \frac{\lambda(x+h)-\lambda(x)}{h} = \frac{1}{x}\left\{\frac{\lambda\left(1+\frac{h}{x}\right)-\lambda(1)}{\frac{h}{x}}\right\}.$$

Proceeding to the limit $h \longrightarrow 0$, of course keeping x fixed, we have

$$\lambda'(x) = \frac{A}{x}\text{, where } A = \lambda'(1).$$

Therefore $\lambda(x) = A \log_e x + B$,

and the system is made definite by adding two other conditions.

In Napier's *Canon*, writing ρ for the radius, we have

$$nl\, x = A \log_e x + B,$$

with $\quad nl\, \rho = 0$, and $nl'\, \rho = -1$.

Therefore $nl\, x = \rho \log_e \left(\frac{\rho}{x}\right)$.

In Briggs' modification of the system, we have

$$bl\, x = A \log_e x + B,$$

with $\quad bl\, \rho = 0$ and $bl\left(\frac{\rho}{10}\right) = 10^{10}$.

Thus $bl\, x = 10^{10}\dfrac{\log_e\left(\frac{\rho}{x}\right)}{\log_e 10} = 10^{10}\log_{10}\left(\frac{\rho}{x}\right)$.

And Napier's final form is, of course,

$$nl\ x = 10^{10} \log_{10} x.$$

Bürgi's *Arithmetische und Geometrische Progress Tabulen* also come under the same law. If the terms in the Arithmetical Progression are taken as the logarithms of the terms in the Geometrical Progression, and $Bl\ x$ stands for what I may call Bürgi's logarithm of x, we have

$$Bl\ x = 10\frac{\log_e\left(\dfrac{x}{10^8}\right)}{\log_e\left(1+\dfrac{1}{10^4}\right)} = 10 \log_{1+\frac{1}{10^4}}\left(\frac{x}{10^8}\right)$$

for $x = 10^8\left(1+\dfrac{1}{10^4}\right)^s$, s being any positive integer.

Finally, treating Napier's series

$$0,\quad 1,\quad 2\ldots\ldots$$

$$10^7,\ 10^7\left(1-\frac{1}{10^7}\right),\ 10^7\left(1-\frac{1}{10^7}\right)^2,\ldots\ldots$$

in the same way, and denoting this logarithm by $Nl\ x$, we have

$$Nl\ x = \frac{\log_e\left(\dfrac{x}{10^7}\right)}{\log_e\left(1-\dfrac{1}{10^7}\right)} = \log_{1-\frac{1}{10^7}}\left(\frac{x}{10^7}\right),$$

for $x = 10^7\left(1-\dfrac{1}{10^7}\right)^s$, s being any positive integer.
