ON CONTACT TRANSFORMATIONS ASSOCIATED WITH THE SYMPLECTIC GROUP.

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1. In this note I shall give a new proof for a slight generalisation of the following well-known theorem of Frobenius (1879).

A linear transformation

\[
\begin{align*}
y &= Ax + Bp \\
q &= Cx + Dp
\end{align*}
\]

in the \(2n\) independent variables \(x, p\), which leaves invariant the skew-symmetric bilinear form

\[
x^\top p - x^\prime p = \sum_{\nu=1}^{n} (x_\nu p_\nu - \bar{x}_\nu \bar{p}_\nu)
\]

in the \(4n\) variables \(x, p, \bar{x}, \bar{p}\), has its determinant

\[
\delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = +1.1
\]

There is no difficulty in proving that \(\delta^2 = 1\). In order to prove the more precise statement (2), which is important in the theory of canonical transformations, formerly a representation of the determinant of a skew-symmetric matrix by means of the so-called Pfaffian aggregates was used, and also later proofs of the theorem are not all as simple as may be desired. The proof given recently by Radon (1939) is based on a certain general lemma concerning the rank of matrices, proved by making use of continuity. In his paper Radon suggests that a proof avoiding these considerations may be found; apart from a minor addition this will be done here (§5). Thus the lemma which may be useful in other parts of matrix calculus appears in its most general form as to the algebraic domain over which it is valid.

The (unimportant) generalisation of the abovementioned theorem which may be justified by the short consideration in §§2 and 3 was suggested to me by a paper of A. Wintner (1934) where the whole matter is discussed from a different point of view. As to the proof, we begin with some particular cases (§4) to which we reduce the most general case by means of Radon’s lemma (§6).

1 The usual notations of matrix calculus are employed here. Capitals \(A, B \ldots\) are \(n\)-rowed square matrices, \(E\) is the unit matrix, \(A^\top\) the transpose of \(A\). Small Latin letters \(x, p \ldots\) without indices are columns, \(x^\top, p^\top \ldots\) the corresponding rows.

2 Cf. Caratheodory (1935), Radon (1939), Siegel (1939), Weyl (1939), Williamson (1939). The name “symplectic group” has been introduced by Weyl instead of the formerly used “complex group”.

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2. We consider the transformation in \(2n + 1\) variables \(x, p, x_0\), which arises from a linear transformation of the form (1) if we add the equation
\[
y_0 = h(x, p) + h_0 x_0
\]
and only require that the transformation represented by (1) and (3) is a contact transformation in \(x, p, x_0\):
\[
q'dy - dy_0 = \delta(p'dx - dx_0).
\]
From here it follows that the function \(h(x, p)\) has to be a quadratic form in the \(2n\) variables \(x, p\):
\[
h(x, p) = x'H_1 x + 2x'H_2 p + p'H_2 p = (x', p') \begin{pmatrix} H_1 & H_2 \\ H_1' & H_2 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}
\]
where \(H_1, H_2, H\) are square matrices with \(n\) rows, \(H_1, H_2\) symmetric.\(^3\) Further we find from (3) and (4)
\[
\delta = h_0 = \text{const.}
\]
Putting the expressions (1) and (3) in (4), we get the relations
\[
\begin{align*}
(A'C - 2H_1)x + (A'D - 2H - h_0 E)p &= 0 \\
(B'C - 2H')x + (B'D - 2H_2)p &= 0
\end{align*}
\]
whence we have
\[
\begin{align*}
2H_1 &= A'C = C'A \\
2H_2 &= B'D = D'B \\
2H' &= C'B = A'D - h_0 E
\end{align*}
\]
If, on the other side, we state the fact that (1) is a symplectic transformation, i.e.
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} O & -E \\ E & O \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} O & -E \\ E & O \end{pmatrix},
\]
we get easily the same relations between the matrices \(A, B, C, D\) with the only restriction \(h_0 = 1\).\(^4\)

3. By means of the relations (7) we can invert the transformation (1) if and only if \(h_0 \neq 0\); we get
\[
\begin{align*}
x &= \frac{1}{h_0} D'y - \frac{1}{h_0} B'q \\
p &= -\frac{1}{h_0} C'y + \frac{1}{h_0} A'q.
\end{align*}
\]
Further we have, from (3) and (8),
\[
\begin{align*}
x_0 &= -\frac{1}{h_0} h(x, p) + \frac{1}{h_0} y_0 = k(y, q) + \frac{1}{h_0} y_0 \\
&= y'K_1 y + 2y'Kq + q'K_2 q + k_0 y_0.
\end{align*}
\]
\(^3\) Not every quadratic form \(h(x, p)\) comes into question here. In another paper to be published elsewhere I have established necessary and sufficient conditions for these forms. From here a new parametrization of the symplectic group is derived such that there are no exceptional elements (cf. H. Weyl (1939)).

\(^4\) According to a remark due to Herglotz (1932) the matrix equations (6) and (10) can be interpreted as the so-called “reciprocal theorems” for linear dynamical systems, which are involved in the “bracket conditions” of Poisson and those of Lagrange respectively. Cf. Whittaker (1927).
The equations (8), (9) representing also a contact transformation, the relations (6), (7) between the matrices 
\[ A, B, C, D, H_1, H_2, H \]
will continue to be true if we replace them respectively by
\[ \frac{D'}{h_0}, \frac{B'}{h_0}, \frac{C'}{h_0}, \frac{A'}{h_0}, K_1, K_2, K \quad \left( k_0 = \frac{1}{h_0} \right). \]
This leads immediately to the following set of relations, evidently equivalent to (6), (7)
\[ -2h_0^2K_1 = DC' = CD' \]
\[ -2h_0^2K_2 = BA' = AB' \]
\[ 2h_0^2K = CB' = DA' - h_0E \]
By the relations (6) or (10) the matrix is characterised as a Hamiltonian matrix (cf. Wintner, 1934).

4. We shall now calculate the determinant \( \delta \) of this matrix. From (8) we know that \( \delta \neq 0 \) if \( h_0 \neq 0 \). First we suppose \( |A| \neq 0 \). Then it is possible to determine two matrices \( P, Q \) with \( n \) rows which satisfy the equation
\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E & O \\ P & Q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \]
indeed this equation means
\[ C = PA \text{ and } D = PB + Q \]
whence
\[ P = CA^{-1}, \quad Q = D - CA^{-1}B. \]
Thus by (12) we have
\[ \delta = \left| \begin{array}{cc} A & B \\ C & D \end{array} \right| = |Q| \cdot |A|. \]
In virtue of (10) and (11) we have
\[ QA' = DA' - CA^{-1}BA' = DA' - CA^{-1}AB' = DA' - CB' = h_0E \]
and thus
\[ \delta = |Q| \cdot |A| = |Q| \cdot |A'| = |QA'| = h_0^n. \]
For \( h_0 = 1 \) this is the theorem stated in §1.

In the same way we can prove the theorem if \( |B| \neq 0 \). Then it is possible to find two matrices \( U, V \) which satisfy the equation
\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E & O \\ U & V \end{bmatrix} \begin{bmatrix} A & B \\ E & O \end{bmatrix} = \begin{bmatrix} A & B \\ UA + V & UB \end{bmatrix} \]
whence
\[ U = DB^{-1}, \quad V = C - DB^{-1}A. \]
By (10) we have
\[ VB' = CB' - DB^{-1}AB' = CB' - DA' = -h_0E; \]
\[ \delta = (-1)^n |V| \cdot |B| = (-1)^n \cdot h_0E = h_0^n. \]
5. Generally we cannot suppose $|A| \neq 0$ or $|B| \neq 0$. We can only suppose that the $n \times 2n$ matrices $(A, B)$ and $(C, D)$ have the rank $r$, because otherwise the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ could not be regular. To establish the theorem also in this case we shall generalise the product formula (12) or (15) by means of the following lemma of Radon, which we shall prove here algebraically:

**Lemma:** If the $n \times 2n$ matrix $(A, B)$ with elements in the infinite integral domain $\Delta$ has the rank $r \leq n$, then there is a diagonal matrix $S = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \\ 0 & & & s_n \end{pmatrix}$ with $s_v$ in $\Delta$, such that the matrix $AS + B$ has the rank $r$.

**Proof:** First we shall suppose $r = n$ (the only case we need actually here). Let $A = (a^{(1)}, a^{(2)}, \ldots, a^{(n)}), B = (b^{(1)}, b^{(2)}, \ldots, b^{(n)})$ where $a^{(v)}, b^{(v)}$ are the columns of $A$ and $B$. If $S$ is a diagonal matrix with indeterminate diagonal elements $s_v$, we have

$$AS + B = (s_1 a^{(1)} + b^{(1)}, s_2 a^{(2)} + b^{(2)}, \ldots, s_n a^{(n)} + b^{(n)})$$

and hence the determinant

$$|AS + B| = s_1 s_2 \ldots s_n |a^{(1)}, a^{(2)}, \ldots, a^{(n)}| + \sum_{v=1}^{n} s_v s_{v+1} \ldots s_n \times$$

$$\begin{vmatrix} a^{(1)}, \ldots, a^{(v-1)}, b^{(v)}, a^{(v+1)}, \ldots, a^{(n)} \end{vmatrix} + \ldots$$

$$+ \sum_{v=1}^{n} s_v |b^{(1)}, \ldots, b^{(v-1)}, a^{(v)}, b^{(v+1)}, \ldots, b^{(n)}|$$

This polynomial in the $n$ indeterminates $s_1, s_2, \ldots, s_n$ over $\Delta$ cannot be identically zero because at least one of its coefficients is different from zero; since there are in $(A, B)$ exactly $n$ linearly independent columns, say $a^{(v_1)}, a^{(v_2)}, \ldots, a^{(v_k)}, b^{(v_{k+1})}, \ldots, b^{(vn)}$, the coefficient of $s_{v_1} s_{v_2} \ldots s_{vk}$ will be different from zero. Therefore one can choose values of $s_1, s_2, \ldots, s_n$ in $\Delta$ such that the square matrix $AS + B$ has the rank $n$.

If $r < n$ we shall apply the same procedure. Let $a^{(l_1)}, \ldots, a^{(l_k)}, b^{(l_{k+1})}, \ldots, b^{(l_r)}$ be a set of $r$ linearly independent columns of $(A, B)$. Then a certain square submatrix $(\bar{A}, \bar{B})$ formed by $r$ rows of the matrix $(a^{(l_1)}, \ldots, a^{(l_k)}, b^{(l_{k+1})}, \ldots, b^{(l_r)})$ has the rank $r$. Let

$$(\bar{A}, \bar{B}) = (\bar{a}^{(l_1)}, \ldots, \bar{a}^{(l_k)}, \bar{b}^{(l_{k+1})}, \ldots, \bar{b}^{(l_r)})$$

where the columns $\bar{a}^{(l_k)}, \bar{b}^{(l_k)}$ arise from $a^{(l_k)}, b^{(l_k)}$ by omitting certain $n - r$ elements, all in the same rows of $(A, B)$. Then as above we see that the submatrix

$$(s_{\mu_1} \bar{a}^{(l_1)} + \bar{b}^{(l_1)}, \ldots, s_{\mu_k} \bar{a}^{(l_k)} + \bar{b}^{(l_k)}, s_{\mu_{k+1}} \bar{a}^{(l_{k+1})} + \bar{b}^{(l_{k+1})}, \ldots, s_{\mu_r} \bar{a}^{(l_r)} + \bar{b}^{(l_r)})$$

of $AS + B$ with $r$ rows and $r$ columns has "generally" the rank $r$; for its deter-

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5 Radon (1939) only stated the existence of a symmetric matrix $S$ with this property, and only if $\Delta$ is the field of all real or complex numbers. In the paper mentioned in note (3) I have proved that if $(A, B)$ is of rank $n$, one can impose on $S$ the condition $SB = B'S$, i.e. $SB$ to be symmetric.
The determinant is a certain polynomial in $s_{\mu_1}, s_{\mu_2}, \ldots, s_{\mu_r}$, which has not all its coefficients equal to zero. At least we have $|a^{(l_1)}, \ldots, a^{(l_k)}; \bar{b}^{(l_{k+1})}, \ldots, \bar{b}^{(l_r)}| \neq 0$; this is the coefficient of $s_{\mu_1}, s_{\mu_2}, \ldots, s_{\mu_r}$.

The following generalisation of Radon's lemma which easily follows from our proof may be stated here. Let $A_\lambda, B_\lambda (\lambda = 1, 2, \ldots, l)$ be $l$ pairs of square matrices over $\Delta$ such that the $n \times 2n$ matrices $(A_\lambda, B_\lambda)$ have the ranks $r_\lambda < n$. Then one diagonal matrix $S$ over $\Delta$ can be found for which the $l$ matrices $A_\lambda S + B_\lambda$ have the ranks $r_\lambda$ (resp.).

6. According to the lemma we determine a diagonal matrix $S$ for which $AS + B$ has the rank $n$. Then to the matrix

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
E & S \\
O & E
\end{bmatrix}
= \begin{bmatrix}
A & AS + B \\
C & CS + D
\end{bmatrix}
= \begin{bmatrix}
A & B^* \\
C & D^*
\end{bmatrix},
\]

which has the same determinant as the matrix \[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]
we can apply the formula (15) with $B^*, D^*$ instead of $B, D$. By (16) we have

\[
V^* = C - D^*B^*-1A = C - (CS + D)(AS + B)^{-1}A
\]
and

\[
V^*B^* = CSA' - CB' - (CS + D)(AS + B)^{-1}(ASA' + AB').
\]

Now we make use of the conditions (10); thus we get

\[
V^*B'' = CSA' - CB' - (CS + D)(AS + B)^{-1}(AS + B)A' = CSA' - CB' - CSA' - DA' = -h_0E
\]
whence (17) gives again $\delta = h_0 n$.

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