A STUDY OF THE REFLECTION OF LIGHT IN THE CASE OF THREE HOMOGENEOUS, ISOTROPIC, NON-CONDUCTING MEDIA IN SUCCESSIVE CONTACT.

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In recent years considerable development has taken place in the production of anti-reflection films on such surfaces as glass. By means of the evaporation process, films, having a variety of properties and a high degree of uniformity, can be deposited. As a consequence, evaporated films have assumed a great importance, both in the world of commerce and in the research laboratory.

These films are produced from a number of metallic fluorides. For instance, calcium fluoride will give a soft film with a refractive index $n_D$ equal to 1.26 ca. (Bannon, 1945); cryolite a rather soft film having $n_D$ equal to 1.34 ca., and magnesium fluoride a very hard and water-resistant film, with $n_D$ varying from 1.36 to 1.38 ca., according to the method of evaporation and treatment. A film, developed by the author, called "crycal" (a mixture of cryolite and calcium fluoride) has a value of $n_D$ equal to 1.35 ca., is very hard and is about as resistant as cryolite to water (Bannon, 1944).

The fraction of monochromatic light energy reflected from a film coated glass surface depends on the refractive indices of the film and the glass and on the optical thickness of the film for that particular wavelength. In this paper a mathematical study is presented of the reflection of light, where there are three homogeneous, isotropic and non-conducting media in successive contact. The theory will be applicable to the case of metallic fluoride films on glass surfaces, if the films are homogeneous, isotropic and non-conducting.

From their great transparency and a consideration of the materials from which they are produced, as well as from direct tests, there is little doubt that these films are non-conducting. The author is at present carrying out research on the structures of some metallic fluoride films, and the evidence, so far, indicates that these films are isotropic, as they appear to be made up of minute crystals in random orientation. There may be an orientation in the first few molecules of thickness deposited on the glass, as in the case of the monomolecular built-up films of Blodgett (Holley and Bernstein, 1936). But such an orientation, confined to a relatively very small thickness, will not affect the present treatment. Neither will a variation in film density affect the treatment, if this variation is confined to a thickness of a few molecules at the film glass boundary. From measurements of refractive indices, using films of varying thicknesses, it does not seem likely that there is much departure from homogeneity throughout the film thickness, at least in the case of crown glasses.

A paper will be presented later, by the author, on the structure and homogeneity of evaporated metallic fluoride films.

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1 In a recent letter to Nature the author describes the production of "hard" calcium fluoride films.

H—August 1, 1945.
The general expression for a plane wave, which propagates itself in a homogeneous, isotropic, non-conducting medium, in the direction of the positive x axis of a rectangular system, is

\[
\begin{align*}
E_x &= 0 \\
E_y &= \frac{1}{\varepsilon} f\left(t - \frac{x}{v}\right) \\
E_z &= \frac{1}{\varepsilon} g\left(t - \frac{x}{v}\right) \\
H_x &= 0 \\
H_y &= -g\left(t - \frac{x}{v}\right) \\
H_z &= f\left(t - \frac{x}{v}\right)
\end{align*}
\]

where E and H are, respectively, the electric and magnetic field intensities, f and g are arbitrary functions of a single argument, and ε is the dielectric constant of the medium.

Suppose there are two infinitely thick, homogeneous, isotropic, non-conducting media 1 and 2, in contact, with a plane boundary between them, and suppose a plane wave is travelling in the direction x, which is parallel to AB. (See Figure 1.) Let AB make an angle θ₁ with the normal N₁N₂ to the boundary.

In general, when the wave strikes the boundary, it will be partly reflected back into medium 1 and partly refracted into medium 2. The reflected wave will move in the direction BC, which makes an angle θ₁ with BN₁, while the refracted wave will travel in the direction BD, which is inclined to BN₂ at an angle θ₂. BC and BD will be parallel to the plane, which contains AB and BN₁; that is, parallel to the plane of incidence.

If, for the media 1 and 2, the refractive indices are n₁ and n₂, the wave velocities are v₁ and v₂, and the dielectric constants are ε₁ and ε₂ respectively, then the following relations hold:

\[
\frac{n_2 v_1}{n_1 v_2} = \frac{\varepsilon_2}{\varepsilon_1} \frac{\sin \theta_1}{\sin \theta_2}
\]

If the direction of the y axis of the incident wave is parallel to the plane of incidence, the following relations exist between the wave functions f and g.
of the incident wave, and the wave functions \( f' \) and \( g' \) and \( f_1 \) and \( g_1 \) of the reflected and refracted waves respectively:

\[
\begin{align*}
    f' &= \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_1 \cos \theta_2 + n_2 \cos \theta_1} \quad f = 12m_1f \\
    f_1 &= \frac{2n_2 \cos \theta_1}{n_1 \cos \theta_2 + n_2 \cos \theta_1} \quad f = 12m_4f \\
    g' &= \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \quad g = 12s_1g \\
    g_1 &= \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \quad g = 12s_4g
\end{align*}
\]

The terms \( 12m, etc., \) above may be called absorption coefficients. If a wave, travelling through medium 2, was incident at an angle \( \theta_2 \) at the \( (21) \) boundary, then, the absorption coefficient, for the refracted \( f \)-wave function in medium 1, would be designated by \( 21m_1 \). The other absorption coefficients would be similarly designated.

It is evident, from the above four equations, that the \( f \) and \( g \) functions are independent, and that the magnitude of an \( f \) absorption coefficient is, in general, not equal to the magnitude of the corresponding \( g \) absorption coefficient.

**THREE SURFACES IN SUCCESSIVE CONTACT.**

Suppose there are three homogeneous, isotropic, non-conducting media in successive contact, and suppose the boundaries between them are plane and parallel. (See Figure 2.) Let media 1 and 3 be infinitely thick, and
medium 2 be of thickness $D_2$. Let light (represented by a ray) travel through medium 1, and strike the (12) boundary at an angle of incidence $\theta_1$. This ray will be partly reflected along $A_1B_1$, and partly refracted, at an angle $\theta_2$, into medium 2 along $A_1C_1$. In medium 2 multiple reflections and refractions will occur. Through medium 1 an infinite number of parallel rays of rapidly diminishing amplitude will travel, and the wave function of the whole of the light returning through medium 1 can be estimated in the way set out below. Medium 2 will be assumed to be sufficiently thin for absorption effects to be neglected.

As in the case above for the two media, let the incident light, which strikes the (12) boundary, be plane. The two wave functions $f$ and $g$ can be treated separately. Considering the incident wave $f\left(t - \frac{x}{v_1}\right)$, let $x$ represent the wave normal at the point $A_1$, and $v_1$ the wave velocity in medium 1. If a wave front $B_1B_2B_3 \ldots$ be selected sufficiently far from the (12) boundary, and if $A_1B_1$, $A_2B_2$, $A_3B_3$, etc., be denoted by $r_1$, $r_2$, $r_3$, etc., respectively, then, the wave function at $B_1$ of the ray reflected at $A_1$ along $r_1$, is

$$12m f\left(t - \frac{x}{v_1} - \frac{r_1}{v_1}\right).$$

The wave function at $B_2$ of the ray travelling along $r_2$ is

$$12m_{12} m_{23} m_1 f\left(t - \frac{x}{v_1} - \frac{2d}{v_1} - \frac{r_2}{v_2} v_1\right),$$

where $v_2$ is the wave velocity in medium 2, and

$$d = A_1C_1 - C_1A_2.$$  \hfill(9)

The wave function at $B_{p+1}$ of the ray travelling along $r_{p+1}$ is

$$12m_{12} m_{23} m_{p} m_1 f\left(t - \frac{x}{v_1} - \frac{2pd}{v_2} - \frac{r_{p+1}}{v_1}\right).$$

Therefore, the wave function, for the whole of the reflected light, whose electric vector vibrates in the plane of incidence, is given by the expression

$$12m f\left(t - \frac{x}{v_1} - \frac{r_1}{v_1}\right) + \sum_{p=1}^{\infty} 12m_{12} m_{23} m_{p} m_1 f\left(t - \frac{x}{v_1} - \frac{2pd}{v_2} - \frac{r_{p+1}}{v_1}\right).$$  \hfill(10)

From equations (5) and (6) it follows that

$$12m = -2m; \quad 12m_{12} = 1 + 12m; \quad 2m_1 = 1 - 12m.$$  \hfill(11)

If $r_1 - r_{p+1} = e$, \hfill(12)

then $r_1 - r_{p+1} = pe$. \hfill(13)

Substituting from equations (4), (11) and (13), expression (10) reduces to

$$12m f(x) + \sum_{p=1}^{\infty} \left(1 - 12m^2\right) m_{p} m_1 m_{p-1} f(x - pe)$$  \hfill(14)

where $x = t - \frac{x}{v_1} - \frac{r_1}{v_1}$ \hfill(15)

and $\beta = \frac{2m_1 d - e}{v_1}$. \hfill(16)

To carry out the summation in expression (14), let $f$ be expanded in the Fourier series

$$\sum_{n=-1}^{\infty} a_n \cos (n\beta + \varphi_n).$$  \hfill(17)
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As the m's in expression (14) vary with the frequency \( \omega \), it will be necessary to deal with monochromatic light. This can be done by making \( a_n = 0 \) for all values of \( n \) except 1. By a suitable choice of the time, expression (17) will reduce to

\[ a_1 \cos \omega t. \]

As it is easier to work with exponential functions, \( f \) can be expressed as follows:

\[ f(t) = a_1 e^{i \omega t} \quad \text{................................. (18)} \]

where the real vibration is regarded as the real part of the complex vibration.

By using equation (18), expression (14) reduces to

\[ 12m a_1 e^{i \omega x} + \sum_{p=1}^{\infty} (1 - 12m^2) 23m^p (-12m)^{p-1} a_1 e^{i \omega (x - p \beta)}. \]

This reduces to

\[ \frac{a_1 e^{i \omega x} (12m + 23m e^{-i \beta})}{1 + 12m 23m e^{-i \omega \beta}}. \]

The square of the absolute value of expression (19) will give the intensity of the light returning through medium 1 (reflected light).

This intensity \( I_f \) is as follows:

\[ I_f = \frac{a_1 e^{i \omega x} (12m + 23m e^{-i \beta}) 	imes a_1 e^{-i \omega x} (12m + 23m e^{i \beta})}{1 + 12m 23m e^{-i \omega \beta}}. \]

\[ I_f = \frac{a_1^2 (12m^2 + 212m 23m \cos \omega \beta + 23m^2)}{1 + 212m 23m \cos \omega \beta + 12m^2 23m^2}. \]

Since the intensity of the incident light is obviously \( a_1^2 \), therefore the ratio \( R_f \) of the intensity of the reflected light to that of the incident light is

\[ R_f = \frac{12m^2 + 212m 23m \cos \omega \beta + 23m^2}{1 + 212m 23m \cos \omega \beta + 12m^2 23m^2} \quad \text{......................... (20)} \]

Since the thickness of medium 2 is \( D_2 \), it follows from equations (4), (9), (12) and (16) that

\[ \beta = \frac{2n_2 D_2 \cos \theta_2}{n_1 v_1} = \frac{2D_2 \cos \theta_2}{v_2}. \]

\[ \frac{\omega}{2\pi} \lambda_2 = v_2 \quad \text{........................................ (22)} \]

where \( \lambda_2 \) is the wavelength of the light in medium 2

\[ \therefore \omega \beta = \frac{4\pi D_2 \cos \theta_2}{\lambda_2} = \psi. \]

But the path retardation \( 1_2 \), between any two consecutive rays reflected back into medium 1, on account of medium 2, is given by

\[ 1_2 = 2D_2 \cos \theta_2 \quad \text{........................................ (24)} \]

\[ \therefore \omega \beta = \frac{2\pi 1_2}{\lambda_2} = \psi. \]

Therefore \( \omega \beta \) is the difference in phase between any two such consecutive rays.

Using equation (25), \( R_f \) may be expressed as follows:

\[ R_f = \frac{(12m - 23m)^2 + 412m 23m \cos^2 \psi / 2}{(1 - 12m 23m)^2 + 412m 23m \cos^2 \psi / 2}. \]

\[ \quad \text{........................................ (26)} \]
By similar reasoning, the ratio \( R_g \) of the intensity of the reflected light to that of the incident light for the g-wave is

\[
R_g = \frac{(18^g - 23^g)^2 + 4\cdot 18^g \cdot 23^g \cos^2 \phi/2}{(1 - 12^g - 23^g)^2 + 4\cdot 12^g \cdot 23^g \cos^2 \phi/2}
\]

Substituting in equations (26) and (27) from equations (5) and (7) and from two similar equations with respect to a (23) boundary, the following relations are obtained:

\[
R_f = \frac{(n_2^2 \cos \theta_1 \cos \theta_3 - n_1 n_3 \cos^2 \theta_2)^2 +}{(n_2^2 \cos \theta_1 \cos \theta_3 + n_1 n_3 \cos^2 \theta_2)^2 +} \frac{(n_2^2 \cos^2 \theta_1 - n_1^2 \cos^2 \theta_2)(n_3^2 \cos^2 \theta_2 - n_2^2 \cos^2 \theta_3) \cos^2 \phi/2}{(n_3^2 \cos^2 \theta_1 - n_1^2 \cos^2 \theta_2)(n_3^2 \cos^2 \theta_2 - n_2^2 \cos^2 \theta_3) \cos^2 \phi/2}
\]

and

\[
R_g = \frac{(n_1 n_3 \cos \theta_1 \cos \theta_3 - n_2^2 \cos^2 \theta_2)^2 +}{(n_1 n_3 \cos \theta_1 \cos \theta_3 + n_2^2 \cos^2 \theta_2)^2 +} \frac{(n_1^2 \cos^2 \theta_1 - n_2^2 \cos^2 \theta_2)(n_3^2 \cos^2 \theta_2 - n_2^2 \cos^2 \theta_3) \cos^2 \phi/2}{(n_3^2 \cos^2 \theta_1 - n_2^2 \cos^2 \theta_2)(n_3^2 \cos^2 \theta_2 - n_2^2 \cos^2 \theta_3) \cos^2 \phi/2}
\]

Let equation (28) be written in the form

\[
R_f = \frac{a_f^2 + c_f \cos^2 \phi/2}{b_f^2 + c_f \cos^2 \phi/2}
\]

Since the media are non-conducting, \( n_1, n_2 \) and \( n_3 \) are both positive and real, and, consequently, \( a_f^2 \) is less than \( b_f^2 \). Also, when negative, \( c_f \) is less in magnitude than \( a_f^2 \), except when \( n_1 = n_3 \). It then equals \( a_f^2 \) in magnitude.

Hence, if \( c_f \) is positive, the minimum value of \( R_f \), as \( D_2 \) varies, will be \( a_f^2/b_f^2 \), and the maximum \( (a_f^2 + c_f)/(b_f^2 + c_f) \).

If \( c_f \) is negative, the maximum value of \( R_f \), as \( D_2 \) varies, will be \( a_f^2/b_f^2 \), and the minimum \( (a_f^2 + c_f)/(b_f^2 + c_f) \).

By the use of equations such as (4),

\[
c_f = \frac{n_1^2 n_2^2}{16 \sin^2 \theta_2 \sin^2 \theta_3} \left( \sin^2 2\theta_1 - \sin^2 2\theta_2 \right) \left( \sin^2 2\theta_2 - \sin^2 2\theta_3 \right)
\]

We shall examine the changes of \( R_f \) and \( R_g \), with variation of \( D_2 \), in the following cases:

**Case 1.** \( n_1 < n_2 < n_3 \).

Here, \( c_f \) is zero if

\[
\theta_1 + \theta_3 = 90^\circ
\]

or

\[
\theta_2 + \theta_3 = 90^\circ
\]

Combining equations (32) and (33) with equations such as (4), \( c_f \) is thus zero if

\[
\tan \theta_1 = \frac{n_2}{n_1}
\]

or

\[
\tan \theta_2 = \frac{n_3}{n_2}
\]

Equation (35) can be expressed in the form

\[
\sin \theta_2 = \frac{n_3}{(n_2^2 + n_3^2)^{1/4}}
\]
By using equation (4), this reduces to
\[ \sin \theta_1 = \frac{n_2 n_3}{n_1 (n_2^2 + n_3^2)^{1/2}} \] .................................. (36)

Consequently, equation (33) will be satisfied if
\[ n_2 n_3 = \text{or} \left( \frac{n_1 (n_2^2 + n_3^2)^{1/2}}{} \right) \]

That is, if
\[ n_2^2 = \text{or} \left( \frac{n_1^2 n_3^2}{n_3^2 - n_1^2} \right) \] .................................. (37)

There will, of course, always be a value of \( \theta_1 \) to satisfy equation (32).

\( c_f \) is positive, then, when \( \theta_1 \) lies between
\[ 0 \text{ and } \sin^{-1} \left( \frac{n_2}{(n_1^2 + n_2^2)^{1/2}} \right) \]
\( c_f \) is negative when \( \theta_1 \) lies between
\[ \sin^{-1} \left( \frac{n_2}{(n_1^2 + n_2^2)^{1/2}} \right) \text{ and } \sin^{-1} \left( \frac{n_2 n_3}{n_1 (n_2^2 + n_3^2)^{1/2}} \right) \]
\( c_f \) is again positive for values of \( \theta_1 > \sin^{-1} \left( \frac{n_2 n_3}{n_1 (n_2^2 + n_3^2)^{1/2}} \right) \) if such values are possible.

Hence, for varying values of \( D_2 \), \( R_f \) is a maximum, for a constant value of \( \theta_1 \), which lies between \( \sin^{-1} \left( \frac{n_2}{(n_1^2 + n_2^2)^{1/2}} \right) \) and \( \sin^{-1} \left( \frac{n_2 n_3}{n_1 (n_2^2 + n_3^2)^{1/2}} \right) \) ........................ (38)
when \( \psi = (2n + 1)\pi \), \( n = 0, 1, 2, \ldots \) ........................ (39)
and a minimum when
\[ \psi = 2n\pi, \quad (n = 0, 1, 2, \ldots) \] .................................. (40)

When \( \theta_1 \) lies outside the limits given in (38), then \( R_f \) is a minimum when
\[ \psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots) \]
and a maximum when
\[ \psi = 2n\pi, \quad (n = 0, 1, 2, \ldots) \]

In the case of \( R_g \), equation (29) may be expressed in the form
\[ R_g = \frac{a^2 g + c_g \cos^2 \psi/2}{b^2 g + c_g \cos^2 \psi/2} \] .................................. (41)

The quantity \( c_g \) is always positive, and, hence, \( R_g \) is a minimum, for a constant value of \( \theta_1 \), when
\[ \psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots) \]
and a maximum when
\[ \psi = 2n\pi, \quad (n = 0, 1, 2, \ldots) \]

**Case 2.** \( n_1 < n_3 < n_2 \).

By similar reasoning to the above, in this case, when \( D_2 \) varies, \( R_f \) is a minimum, for a constant value of \( \theta_1 \), which lies between the limits given in (38), when
\[ \psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots) \]
and a maximum when
\[ \psi = 2n\pi, \quad (n = 0, 1, 2, \ldots) \].
When \( \theta_1 \) lies outside the limits given in (38), \( R_f \) is a maximum when

\[
\phi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots)
\]

and a minimum when

\[
\phi = 2n\pi, \quad (n = 0, 1, 2, \ldots).
\]

In the case of \( R_g \), the quantity \( c_g \) is always negative and less in magnitude than \( a_g^2 \); hence, for a constant value of \( \theta_1 \), \( R_g \) is a maximum when

\[
\phi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots)
\]

and a minimum when

\[
\phi = 2n\pi, \quad (n = 0, 1, 2, \ldots).
\]

Case 3. \( n_1 > n_2 > n_3 \).

Here, \( R_f \) and \( R_g \) attain maximum and minimum values under the same conditions as in case 1; also, equations (32) and (33) will always be satisfied.

Case 4. \( n_1 > n_3 > n_2 \).

Here, \( R_f \) and \( R_g \) attain maximum and minimum values under the same conditions as in case 2, and equations (32) and (33) will always be satisfied.

From these cases it is evident that, for the \( f \)-wave, the value of \( c_f \) is zero, when the light rays are incident at the (12) or the (23) boundary at the Brewster angle corresponding to that boundary. For these two particular directions, \( R_f \) is independent of \( D_2 \). \( R_f \), for the one direction, is equal to reflection at a (13) boundary, and, for the other direction, at a (12) boundary.

For the \( g \)-wave, \( c_g \) never becomes zero, and, consequently, \( R_g \) is never independent of \( D_2 \).

By referring back to equation (23), it is seen that \( \psi \) varies with the wavelength of the incident light. Consequently, the conditions expressed by equations (39) and (40) will hold for one particular wavelength only, at any one time. Graphs showing the variations of reflectivity with wavelength in the case of cryolite coated glass surfaces appear in a paper by the author published in 1943 (Bannon, 1943). The angle of incidence \( \theta_1 \) was taken as zero.

In Figure 3, the percentage reflected energies associated with the \( f \)- and \( g \)-waves, as well as the percentage total energy, are given in the case where \( D_2 \) is constant and equal to \( \frac{1}{2} \Delta_2 \). In other words, at vertical incidence, \( \psi \) equals \( \pi \). At an angle of incidence \( \theta_1 \), then, \( \psi \) is equal to \( \pi \cos \theta_2 \). The graphs in Figure 3 are of especial interest, because they are the theoretical curves for dense flint glass surfaces (a) uncoated and (b) coated with film such as magnesium fluoride. It is evident that, in either case, the total amount of light energy reflected varies very little until the angle of incidence \( \theta_1 \) exceeds 30°. For this particular case, \( R_f \) for the coated glass decreases to a minimum and then increases, while \( R_g \) steadily increases for increasing values of \( \theta_1 \).

In the case of three media, such as we are considering, under certain conditions \( R_f \) or \( R_g \) may become zero.

\( R_f \) Equals Zero.

For \( n_1 \neq n_2 \neq n_3 \), \( R_f \) is zero, if the numerator of equation (30) is zero. As \( c_f \neq 0 \), except when equation (32) or (33) is satisfied and as \( c_r \), when negative, is less in magnitude than \( a_r^2 \), except when \( n_1 = n_3 \), \( R_f \) is, in general, zero only when, simultaneously,

\[
\psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots)
\]

and

\[
n_2^2 \cos \theta_1 \cos \theta_3 - n_1 n_3 \cos^2 \theta_2 = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (42)
\]
Squaring both sides and substituting for \( \sin \theta_2 \) and \( \sin \theta_3 \) their values in terms of \( \sin \theta_1 \) and the refractive indices, we have

\[
n_2^4(1 - \sin^2 \theta_4) \left( 1 - \frac{n_1^2}{n_3^2} \sin^2 \theta_1 \right) = n_1^2 n_3^2 \left( 1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_1 \right) \quad (43)
\]

On expansion, this equation becomes

\[
(n_2^8 n_1^2 - n_1^4 n_3^4) \sin^4 \theta_1 - n_2^2(n_2^6(n_1^2 + n_3^2) - 2n_1^4 n_3^4) \sin^2 \theta_1 + n_2^4 n_3^2(n_2^4 - n_1^2 n_3^2) = 0 \quad (44)
\]

Hence

\[
\sin^2 \theta_1 = \frac{1}{2n_1^2(n_2^8 - n_1^4 n_3^4)} \left[ n_2^2(n_2^6(n_1^2 + n_3^2) - 2n_1^4 n_3^4) + n_2^4(n_2^8(n_3^2 - n_1^2)^2 + 4n_1^4 n_3^4(n_2^2 - n_1^2)(n_2^2 - n_3^2))^{\frac{1}{2}} \right] \quad (45)
\]

**Fig. 3.**—Theoretical curves of intensity of reflected monochromatic light vs. angle of incidence \( \theta_1 \). Curves \( a, b \) and \( c \) apply to media 1 and 3 in contact. Curves \( x, y \) and \( z \) apply to media 1, 2 and 3 in successive contact, where \( D_1 = \frac{1}{2} \theta_2 \). All curves intersect the 90 degree ordinate at the point 100.
Let equation (45) be written in the form
\[ \sin^2 \theta_1 = \frac{x \pm y^\frac{1}{3}}{z} \]  \hspace{1cm} (46)
when \( n_2^2 = n_1 n_3 \)
\[ x = n_2^6 (n_3 - n_1)^2, \]
\[ y = n_2^{16} (n_3 - n_1)^4 \]
and
\[ z = 0. \]

It is easily seen that when \( n_2^2 > n_1 n_3 \), \( x \), \( y \), and \( z \) are each positive.

As \( n_2^2 \) decreases in value below \( n_1 n_3 \), \( x \) decreases to zero and then becomes negative. \( y \) decreases to zero for a value of \( n_2^2 \) equal to \( (n_1 n_3 - \delta) \). Below this value \( y \) becomes negative, reaches a minimum and becomes zero again when \( n_2^2 \) equals \( (n_1 n_3 - \beta) \). Below this value of \( n_2^2 \), \( y \) is positive. \( (n_1 n_3 - \beta) \) is greater than the lesser of the two quantities \( n_1^2 \) and \( n_3^2 \).

When \( n_2^6 = \frac{2n_1^4 n_2^4}{n_1^2 + n_3^2} \)
\[ x = 0 \]
and
\[ y = 4n_1^2 n_2^4 n_3^2 (n_2^8 - n_1^4 n_3^4) (n_1^2 n_2^2 - n_2^4), \]
\hspace{1cm} (47)
which is negative.

Therefore, when \( y \) becomes zero, for \( n_2^2 = (n_1 n_3 - \delta) \), \( x \) is still positive. But \( x \) is negative when \( y \) becomes positive again for values of \( n_2^2 \) less than \( (n_1 n_3 - \beta) \).

Also,
\[ x^2 - y = 4n_1^2 n_2^4 n_3^2 (n_2^8 - n_1^4 n_3^4) (n_2^4 - n_1^2 n_3^4) \]
\hspace{1cm} (48)
Therefore \( x^2 = y \) when \( n_2^2 = n_1 n_3 \), and \( x^2 > y \) in all other cases.

When \( n_2^2 > n_1 n_3 \), let us consider the equation
\[ \sin^2 \theta_1 = \frac{x \pm y^\frac{1}{3}}{z} \]  \hspace{1cm} (49)
If \( x > z \), then no value of \( \theta_1 \) can satisfy this equation.
If \( x < z \), then there is a value of \( \theta_1 \) if \( (z - x) > y^\frac{1}{3} \).
On substitution, this condition demands that \( n_1 n_3 > n_2^2 \), which by assumption is not so. Hence the equation (49) is not tenable.

Now consider the equation
\[ \sin^2 \theta_1 = \frac{x - y^\frac{1}{3}}{z} \]  \hspace{1cm} (50)
The numerator \( (x - y^\frac{1}{3}) \) is positive on account of equation (48).
If \( x < z \), there obviously is a value of \( \theta_1 \), which satisfies equation (50).
If \( x > z \), then there is a value of \( \theta_1 \) if
\[ (x - z) < y^\frac{1}{3}; \]
that is, if
\[ n_1 n_3 < n_2^2 \]
which by assumption is the case.

Hence equation (50) is tenable.
Similarly equation (50), but not equation (49), is tenable when \( n_2^2 = n_1 n_3 \).
When \( n_2^2 < n_1 n_3 \), a number of cases must be considered.

**Case 1.** \( n_1 n_3 > n_2^2 < (n_1 n_3 - \delta) \).
Here \( x \pm y^\frac{1}{3} \) is positive and \( z \) is negative; therefore there can be no value of \( \theta_1 \).
Case 2. \((n_1 n_3 - \alpha) > n_2 > (n_1 n_3 - \beta)\).

Here, \(y\) is negative and, therefore, there can be no value of \(\theta_i\).

Case 3. \(n_2 = \text{or} < (n_1 n_3 - \beta)\).

Here, \(x\) is negative, \(y\) is zero or positive, and \(z\) is negative.

Let \(x = -x'\) and \(z = -z'\), where \(x'\) and \(z'\) are positive.

Then \(z' - x' = n_2^8 (n_3^2 - n_1^2) - 2n_1^4 n_3^4 (n_2^2 - n_1^2)\).

When \(n_3 > n_1\), and \(n_2^2 = n_1 n_3\), \((z' - x')\) is positive. For values of \(n_2^2\) between \(n_1 n_3\) and \(n_1^2\), \((z' - x')\) will reduce to zero if

\[n_2^8 (n_3^2 - n_1^2) - 2n_1^4 n_3^4 (n_2^2 - n_1^2) = 0.\]

Substituting the value of \(n_2^6\) from this equation in the expression for \(y\) appearing in equation (45), we have

\[y = -2n_1^4 n_2^8 n_3^4 (n_2^2 - n_1^2)(n_1^2 - 4n_2^2 + 3n_3^2).\]

Since \(n_1 n_3 > n_2 > n_1^2\), \(y\) is negative.

Therefore, the values of \(n_2^2\), between \(n_1 n_3\) and \(n_1^2\), for which \((z' - x')\) can be zero, must lie between \((n_1 n_3 - \alpha)\) and \((n_1 n_3 - \beta)\).

In a similar way it can be shown that \((z' - x')\) is positive for values of \(n_2^2\) for which \(y = 0\).

Also, for \(n_2 = \text{or} < n_1\), \((z' - x')\) is positive.

Therefore, for all values of \(n_2 = \text{or} < (n_1 n_3 - \beta)\), \((z' - x')\) is positive.

When \(n_3 < n_1\), and \(n_2^2 < n_1 n_3\), \((z' - x')\) is obviously always positive.

Hence, in all cases \((z' - x')\) is positive for values of \(n_2^2 = \text{or} < (n_1 n_3 - \beta)\).

Let us consider equation (49). This equation can be written in the form

\[\sin^2 \theta_1 = \frac{-x' + y^\frac{1}{2}}{-z'}.\]

\[\therefore \sin^2 \theta_1 = \frac{x' - y^\frac{1}{2}}{z'}.\]

As \((x' - y^\frac{1}{2})\) is positive by equation (48), and as \(z' > x'\), there is a value of \(\theta_1\) which satisfies this equation.

Now consider equation (50). This equation can be written in the form

\[\sin^2 \theta_1 = \frac{-x' - y^\frac{1}{2}}{-z'}.\]

\[\therefore \sin^2 \theta_1 = \frac{x' + y^\frac{1}{2}}{z'}.\]

A value of \(\theta_1\) satisfies this equation if

\((z' - x') > y^\frac{1}{2}\)

that is, if \(n_1 n_3 > n_2^2\), which is the case.

Hence for case 3

\[\sin^2 \theta_1 = \frac{x + y^\frac{1}{2}}{z}.\]

\[R_g \text{ EQUALES ZERO.}\]

For \(n_1 \neq n_2 \neq n_3\), \(R_g\) is zero if, simultaneously,

\[\psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots)\]

and

\[n_1 n_3 \cos \theta_1 \cos \theta_3 - n_2^2 \cos^2 \theta_2 = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots (51)\]
By substitution in equation (51) we have
\[ \sin^2 \theta_1 = \frac{n_1^2 n_3^2 - n_2^4}{n_1^2(n_1^2 + n_3^2 - 2n_2^2)} \] \hspace{1cm} (52)

If \( n_2^2 < n_1 n_3 \), then both numerator and denominator, on the right-hand side of equation (52), are positive. Moreover, since \( (n_2^2 - n_1^2)^2 > 0 \),
\[ n_1^2(n_1^2 + n_3^2 - 2n_2^2) > (n_1^2 n_3^2 - n_2^4). \]
Therefore, for some value of \( \theta_1 \), between 0 and \( \pi/2 \), \( R_g \) becomes zero.

If \( n_2^2 > n_1 n_3 \), then the numerator is negative; and if the denominator is positive, \( R_g \) cannot be zero.

If the denominator is negative, then, for \( R_g \) to become zero,
\[ -n_1^2(n_1^2 + n_3^2 - 2n_2^2) \text{ must be } > (n_1^2 n_3^2 - n_2^4) \]
i.e. \( -(n_1^2 - n_2^2)^2 \text{ must be } > 0. \)
As this is impossible, \( R_g \) cannot be zero when \( n_2^2 > n_1 n_3 \).

\[ R_f \text{ AND } R_g \text{ SIMULTANEOUSLY EQUAL TO ZERO.} \]

In the general case when \( n_1 = n_2 = n_3 \), \( R_f \) and \( R_g \) cannot be simultaneously zero unless equations (39), (42) and (51) are all satisfied.

That is, unless
\[ \psi = (2n + 1)\pi, \] (n = 0, 1, 2, \ldots)
and
\[ \cos^2 \theta_2 = \cos \theta_1 \cos \theta_3 \] \hspace{1cm} (53)

Squaring both sides in equation (53), and substituting for \( \sin \theta_2 \) and \( \sin \theta_3 \) their values in terms of \( \sin \theta_1 \) and the refractive indices, we have
\[ n_1^2(n_2^4 - n_1^2 n_3^2) \sin^4 \theta_1 - n_2^2(n_1^2 n_2^2 + n_2^4 n_3^2 - 2n_2^2 n_3^2) \sin^2 \theta_1 = 0. \]
Therefore
\[ \sin^2 \theta_1 = 0 \] \hspace{1cm} (54)
or
\[ \sin^2 \theta_1 = \frac{n_2^2(n_1^2 - n_2^2 + n_2^2 n_3^2 - 2n_1^2 n_3^2)}{n_1^2(n_2^4 - n_1^2 n_3^2)} \] \hspace{1cm} (55)

Equation (55) is not tenable, as the right-hand side is always greater than unity.

Hence, if equation (39) is satisfied, \( R_f \) and \( R_g \) can be simultaneously zero only when \( n_2^2 = n_1 n_3 \), and the incidence is vertical.

In the particular case when \( n_1 = n_3 \), we see, by referring to equations (28), (29), (30) and (41), that
\[ c_f = -a_1^2 \]
and \[ c_g = -a_2^2. \]

Hence \( R_f = R_g = 0 \) for all angles of incidence provided that
\[ \psi = 2n\pi, \] (n = 0, 1, 2, \ldots)

In Figure 4 a series of graphs is given showing the variations, with angle of incidence \( \theta_1 \), of amplitudes of reflected monochromatic light for the \( f \) - and \( g \) - waves. The reflected amplitudes are given as percentages of the incident amplitude, and are calculated on the assumption that, for each angle of incidence \( \theta_1 \), the thickness \( D_2 \) is such that
\[ \psi = (2n + 1)\pi, \] (n = 0, 1, 2, \ldots).
Consequently, the reflected percentage amplitudes $A_f$ and $A_g$ can be expressed as follows:

$$A_f = \frac{100a_f}{b_f}$$

and

$$A_g = \frac{100a_g}{b_g}$$

Throughout the series, $n_1=1$ and $n_3=2$, while $n_2$ is given in turn the values $1.025, 1.2, 1.5$ and $3$. It is to be noted that, as $\theta_1$ approaches very closely to $90^\circ$, the reflected amplitudes, for both the $f$- and the $g$-waves, approach the incident amplitude in magnitude but not in sign. Graph 1 ($f$) is an instance of particular values of $n_1, n_2$ and $n_3$ for which equation (42) is satisfied by two values of $\theta_1$, while graph 1 ($g$) shows that equation (51) is satisfied by one value of $\theta_1$. For $n_2=1.2$, the reflected amplitude of the $f$-wave does not become zero for any angle of incidence, while, for $n_2=1.5$ and for $n_2=3$, the amplitude of the $g$-wave is never zero.

Fig. 4.—Theoretical curves of amplitude of reflected monochromatic light vs. angle of incidence $\theta_1$. $D_2$ varies so that the condition $\psi=(2n+1)\pi$ is fulfilled for each angle of incidence $\theta_1$. 
CONCLUSION.

When there are two media 1 and 2, separated by a plane boundary and having refractive indices $n_1$ and $n_2$ respectively, then, for monochromatic light travelling through medium 1 and incident on the boundary at an angle $\theta_1$,

$$R_t = 0 \text{ when } \tan \theta_1 = \frac{n_2}{n_1},$$

while $R_g$ is never equal to zero.

When there are three media such as we have dealt with above, $R_t$ is equal to zero for one value of $\theta_1$ between 0 and $\pi/2$, if there simultaneously hold the equation (39), namely

$$\psi = (2n + 1)\pi, \quad (n = 0, 1, 2, \ldots)$$

and the inequality,

$$n_2^2 > n_1 n_3$$

When $n_2^2 = n_1 n_3$, the value of $\theta_1$ is zero.

$R_t$ is not equal to zero, for any value of $\theta_1$, if $n_2^2$ lies between $n_1 n_3$ and $(n_1 n_3 - \beta)$, where $(n_1 n_3 - \beta)$ is the smaller of the two real values of $n_2^2$, which satisfy the equation

$$n_2^4 (n_2^4 - n_1^2)^2 + 4 n_1^4 n_3^4 (n_2^4 - n_1^2) (n_2^4 - n_3^2) = 0.$$ 

$R_t$ is equal to zero, for two values of $\theta_1$ between 0 and $\pi/2$, if equation (39) is satisfied and, simultaneously,

$$n_2^2 < (n_1 n_3 - \beta).$$

$R_g$ is equal to zero, for some value of $\theta_1$ between 0 and $\pi/2$, if equation (39) is satisfied and, simultaneously,

$$n_2^2 < n_1 n_3.$$ 

When $n_2^2 = n_1 n_3$, the value of $\theta_1$ is zero.

When $n_1 = n_3$, both $R_t$ and $R_g$ are zero, when

$$\psi = 2n\pi, \quad (n = 0, 1, 2, \ldots).$$

If $n_1 < n_2 < n_3$ and, if $D_2$ only is variable, then $R_t$ is a maximum when $\psi = (2n + 1)\pi$, if $\theta_1$ lies between the limits $\sin^{-1} \frac{n_2}{(n_1^2 + n_2^2)^{1/2}}$ and $\sin^{-1} \frac{n_3 n_2}{n_1 (n_2^2 + n_3^2)^{1/2}}$, and a minimum, if $\theta_1$ lies outside those limits.

When $\psi = 2n\pi$, $R_t$ is a minimum, when $\theta_1$ is within the above limits, and a maximum, when $\theta_1$ is outside them.

On the other hand, $R_g$, for any particular value of $\theta_1$, is a minimum, when $\psi = (2n + 1)\pi$, and a maximum, when $\psi = 2n\pi$.

When $n_1 < n_3 < n_2$, the above statements are true, if everywhere the words maximum and minimum are interchanged.

For other relationships between $n_1, n_2$ and $n_3$, statements of a similar nature can be made, regarding $R_t$ and $R_g$.

$R_t$ is independent of the value of $D_2$, when the light rays are incident at the (12) or the (23) boundary at the Brewster angle corresponding to that boundary.

$R_g$ is never independent of $D_2$.

SUMMARY.

A mathematical study is presented of the reflection of light, where there are three homogeneous, isotropic, non-conducting media in successive contact. The boundaries between the media are assumed to be plane. Media 1 and 3, of refractive indices $n_1$ and $n_3$ respectively, are infinitely thick. Medium 2, of
refractive index $n_2$, is of thickness $D_2$, which is sufficiently thin for absorption effects to be neglected.

A plane wave of monochromatic light is assumed to pass through medium 1, and strike the (12) boundary at an angle $\theta_1$. Expressions are derived for the fractional intensities $R_f$ and $R_g$ of the light reflected into medium 1, in the case of the f- and g-waves respectively. The electric vector of the f-wave vibrates in the plane of incidence, while the electric vector of the g-wave vibrates at right angles to that plane.

$R_f$ and $R_g$ are functions of $n_1, n_2, n_3, \theta_1$ and $D_2$. The changes in these functions, with change in $D_2$, are examined. The nature of these changes is shown to depend on the relative values of $n_1$, $n_2$ and $n_3$ and, in the case of $R_f$, on the value of $\theta_1$ also. For certain discreet values of $D_2$, the functions are analysed, when $\theta_1$ is the variable. It is found that $R_f$ and $R_g$ then depend, in a very striking way, on the relative values of $n_1, n_2$ and $n_3$.

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