Abstract.—Kepler's problem is reviewed from analytical and numerical standpoints, the region of usefulness of various solutions defined and formulae suggested for the nearly parabolic case. A bibliography is given.

The two body problem was solved (kinematically) by Kepler with the enunciation of his three laws of planetary motion which were later to be contained within the implications of the more general theory of gravitation. The Keplerian rules (regarded as derived from the gravitational equations) are still used in most derivations of the equations which connect the position variables of planetary motion with the time variable. The problem of finding the coordinates of a planet in the plane of its orbit in unperturbed motion is Kepler's problem.

If we express the equation of the solution path of planetary motion in a form using the parameters \( e \) (eccentricity) and \( q \) (perihelion distance) which apply equally to the three conies without involving infinite or imaginary values (as for example the major and minor axes would) and take \( x_0, y_0 \) as the rectangular coordinates in the plane of motion with the \( x_0 \) axis directed towards perihelion we obtain

\[
x_0^2 + y_0^2 = \left( q(1 + e) - ex_0 \right)^2
\]  

(1)

The equation for constancy of areal velocity is

\[
x_0 y'_0 - x_0' y_0 = e
\]

that is

\[
2x_0 y_0' - (x_0 y_0)' = e
\]

where the dashes indicate differentiation with regard to time.

Now if we write

\[
\lambda = \frac{x_0}{q}, \quad \mu = \frac{y_0}{q} \quad \text{and} \quad \varepsilon = \frac{1-e}{1+e}
\]  

(2)

and take the initial point at perihelion this becomes

\[
\int_0^{\mu_1} 2q^2 \lambda d\mu - q^2 \lambda_1 \mu_1 = ct
\]  

(3)

and equation (1) gives

\[
\lambda = \frac{1}{1-e} \{ -e \pm (1 - \varepsilon \mu^2)^{1/2} \}
\]  

(4)

In substituting (4) into (3) in the elliptic case the branch of \( \lambda \) corresponding to the upper sign must be used from perihelion (\( \mu = 0, \lambda = 1 \)) to \( \mu = e^{-1}, \lambda = -e/(1 - e) \) and that corresponding to the lower sign from this point to \( \mu = 0, \lambda = -e^{-1} \). In the hyperbolic case the lower sign corresponds to the non-solution branch of the hyperbola. The case \( e = 1 \) is obvious.

On integrating (3) in this way, putting \( e = kq^{1/2}(1 + e)^{1/2} \) from the dynamical theory where \( k \) is the Gaussian constant, we obtain

\[
k(1 + e)^{3/2} q^{-3/2} t = \varepsilon^{-3/2} \left\{ \pm \sin^{-1} \varepsilon^{1/2} \mu - \varepsilon^{1/2} \mu - C \right\}
\]  

(5)
where the subscript on \( \mu \) may now be dropped without ambiguity. In the elliptic case we can without loss of generality take the positive sign and \( C=0 \) if we put \( \sin^{-1}e^{1/2}\mu > \pi/2 \) in the interval in which \( \mu \) is decreasing. The hyperbolic case requires only the upper sign.

From (4) we obtain
\[
\lambda = 1 - \frac{\mu^2}{1+e} (1-\varepsilon\mu^{1/3})^{1/3},
\]
with the same remarks about signs. This will be found a convenient form for calculation.

Equation (5) will be taken as the general expression of Keplerian motion. If we put \( \varepsilon^{1/2}\mu = \sin E \)
\[
M = ka^{-3/2}t = E - e \sin E,
\]
where \( a \) is the semi-major axis, \( M \) the mean anomaly and \( E \) the eccentric anomaly.

When \( \varepsilon \) is negative put \( \varepsilon = -(e-1)/(e+1) \), \( \sinh F = \varepsilon^{1/2}\mu \) and we obtain (since \( \sin^{-1}(e^{1/2}\mu^2) = i \sinh^{-1}(e^{1/2}\mu) \))
\[
M = ka^{-3/2}t = e \sinh F - F.
\]

Equation (5) may be written
\[
\sqrt{2kq^{-3/2}t} = 12\mu + \mu^3(1+e)6\left(\frac{\sin^{-1}(e^{1/2}\mu - e^{1/2}\mu^3)}{e^{3/2}\mu^3}\right),
\]
which will be found a convenient expression for dealing with the nearly parabolic case (\( e \) near 1).

For the parabolic case (\( e = 1 \))
\[
12\sqrt{2kq^{-3/2}t} = 12\mu + \mu^3
\]
and if we place
\[
\tan \frac{\nu}{2} = \tau = \frac{1}{\mu},
\]
we obtain
\[
6k(2q)^{-3/2} = 3\tau + \tau^3
\]
which is the usual expression for parabolic motion, where \( \nu \) is the true anomaly.

Equation (7) is the one which now bears Kepler’s name. It is an early example of a transcendental equation occurring in applied mathematics and very few men even as eminent as Kepler can have such an enduring memorial as this equation. The necessity for its frequent solution and the difficulties, numerical and analytical, which it presents have kept alive interest in the equation during the whole of the 300 years since its discovery. The analytical points involved if we wish to express the implicit function \( E = E(e, M) \) of equation (7) explicitly as a series in \( M \) and \( e \) are of interest and consideration of them has been important in the development of the theory of analytic functions (see references in Wintner, 1941). The interval of convergence as a power series in \( e \) must depend on the singularities of the function \( E = E(e, M) \) in the complex plane for \( e \).

The function
\[
e = \frac{E - M}{\sin E}
\]
is a meromorphic function with simple poles at the points \( \sin E = 0 \) except when \( E = M \).

The inverse function \( E = E(e, M) \) must be multiple valued and the branch for which \( e = 0 \) implies \( E = M \), that is the branch in which we are interested,
is regular at \( e = 0 \). The singularities of the inverse function are given by zeros of the derivative, that is by

\[
\frac{1 - e \cos E}{\sin E} = 0
\]

The singularities at \( E = \infty \) correspond to connected paths to infinity (see Hurwitz, 1906) in the \( E \) plane for which \( e \), given by (11), approaches a finite point. This only occurs for \( e = 0 \) and therefore does not affect the branch in which we are interested, which is regular at this point. Hence the singularities are determined by (11) and the equation

\[
M - E + \tan E = 0
\]  

(12)

where \( M \) is real (in the elliptical case).

Every point on the real axis of \( e \) for which \( |e| > 1 \) is a branch point. Let us put \( E = a + ib \) into equation (12) and equate real and imaginary parts giving

\[
b = \frac{(1 + \tan^2 a)}{1 + \tan^2 a} \tanh b
\]

\[
-M + a = \frac{(1 - \tan^2 b)}{1 + \tan^2 a} \tanh^2 b
\]

from which

\[
\tan^2 a = \frac{\tanh b - b}{(b - \tanh b - \tanh b)}
\]  

(13)

and

\[
(a - M)^2 = (\tanh b - b)(b - \coth b)
\]  

(15)

Equation (15) shows that \( b \) must lie between \(-\beta\) and \(+\beta\) where \( \beta \) is given by

\[
\beta = \coth \beta
\]

If, using equations (11) and (12), we write

\[
\xi + i\eta = e = \sec E,
\]

equate real and imaginary parts and use equation (13) we obtain

\[
\begin{align*}
\xi &= b \cos a \cosech b, \\
\eta &= \sin a \sech b.
\end{align*}
\]

(16)

If we now use \( b \) as a parameter equations (14), (15) and (16) give the curve of singularities.

For the case of the hyperbola \( M \) is replaced by \( iN \) (with \( N \) real); equation (12) becomes

\[
-iN + E - \tan E = 0
\]

and any real value of \( e \) for which \( |e| < 1 \) gives a solution (that is a singularity).

If as before we let \( E = a + ib, \ e = \xi + i\eta \) the equations corresponding to (14), (15) and (16) are

\[
\begin{align*}
\tanh^2 b &= \frac{\tan a - a}{(1 + \tan a) \tan a}, \\
(b - N)^2 &= (\tan a - a)(\cot a + a)
\end{align*}
\]  

(17)

and

\[
\begin{align*}
\xi &= (b - N) \cos a \cosech b, \\
\eta &= (b - N) \sin a \sech b.
\end{align*}
\]  

(18)

The curves (symmetrical about both axes) of singularities for the ellipse and the hyperbola are shown in Figure 1, where the dashed line refers to the
ellipse and the dotted line to the hyperbola. The curve for the ellipse was first discussed by T. Levi-Civita (1904) and C. V. L. Charlier (1904) (the work of the latter not having been available to me) and the hyperbola by H. G. Block (1904). Other discussions are given by H. Andoyer (1923) and A. Wintner (1941).

For elliptic motion the nearest singular point to the origin is on the imaginary axis at a point given by

$$\beta = \coth \beta$$

$$\eta = \beta \sech \beta = 0.6627$$.

The power series expression for $E$ in elliptic motion is convergent uniformly with $M$ only for $0 < e < 0.6627$. In both cases the curve is incipient with real axis at an angle of $\pi/3$ and in the hyperbolic case the curve is asymptotic to the line $\xi = \pi/2$.

![Fig. 1.—Curves of singularities in the complex e plane (first quadrant) for solutions of elliptical and hyperbolic motion.](image)

In the case of hyperbolic motion the curve of singularities shows that no development in $e$ or $1/e$ converges for all values of $M$ and even the development in powers of $e - e_0$ ($e_0 > 1$) is of no practical use for the circle of convergence (uniformly with $M$) is limited by the curve and cannot include the region near $e = 1$, the cusp of the curve.

Turning to equation (10) for parabolic motion $\mu$ is seen to be a three-valued function of $D = 12\sqrt{2kq^{-3/2}}$. For $D$ real there are two complex roots and one real, the real root increasing monotonically with $D$. The zeros of the first derivative of $12\mu + \mu^3 + D$ occur at $\mu = \pm 2i$; the Riemann surface of $\mu = \mu(D)$ has branch points at $D = \pm 16i$ and there are no further finite singularities. These singularities limit the radius of convergence and the general usefulness of any development of (the real branch of) $\mu$ according to powers of $D - D_0$. 
Turning to the nearly parabolic case write equation (9) in the form 
\[ F(\varepsilon, \mu, D) = 0. \]
Expanding \( \sin^{-1} \varepsilon^{1/2} \mu \) as a series we obtain
\[ 0 = -D + 12\mu + \mu^2(1 + \varepsilon)6 \left( \frac{1}{2.3} + \frac{1.3}{2.4} \frac{\varepsilon \mu^2}{5} + \ldots \right), \quad (19) \]
where the right-hand member is an expression of part of the branch of 
\[ F(\varepsilon, \mu, D) \]
for which \( \mu = 0 \) implies \( D = 0 \) (for \( |\varepsilon \mu^2| < 1 \)). For this branch of 
\[ F(\varepsilon, \mu, D) \] the origin is a regular point but it is a pole for every other branch.

Also we have
\[
\frac{\partial F}{\partial \mu} = 12 + (1 + \varepsilon)6 \left( (1 - \varepsilon \mu^2)^{-1/2} - 1 \right)
= 12 + \mu^2(1 + \varepsilon)6 \left( \frac{1}{2} + \frac{1.3}{2.4} \varepsilon \mu^2 + \ldots \right). \quad (20)
\]
This shows that for \( D, \varepsilon \) and \( \mu \) real we have

(i) When \( 0 < \varepsilon < 1 \), the elliptical case, i.e. \( 1 > \varepsilon > 0 \), \( F \) is monotonic increasing
with \( \mu \) for \( \varepsilon \mu^2 < 1 \),

(ii) When \( \varepsilon > 1 \), the parabolic (see equation 10) and hyperbolic cases, \( 0 > \varepsilon > -1 \) \( F \) is monotonic increasing with \( \mu \) and there is a unique real solution
of (9) for any real value of \( D \). We also see that for small values of \( \varepsilon \) there are always solutions of \( \partial F/\partial \mu = 0 \) near the points \( \mu^2 = -4 \) on the imaginary axis of \( \mu \).
The existence of these singularities limits the radius of convergence and usefulness
of the development of \( \mu \) as a power series in \( D \) with coefficients in \( \varepsilon \) (fixed).
The attempt to express the coefficient of \( \mu^2 \) in (9) as a development in series in
order to evaluate the coefficient and transform the equation for solution into an
ordinary cubic meets with the same difficulty.

However, let us consider \( \mu = \mu(D, \varepsilon) \) as a function of \( \varepsilon \) (\( D \) real constant)
and wish to develop the solution as a Taylor’s series proceeding in powers of \( \varepsilon \)
with coefficients functions of \( \nu = \mu(D, 0) \). It will be necessary to see how the
matter is affected by singularities of \( \mu = \mu(D, \varepsilon) \) for \( \varepsilon \) in the complex domain in
the vicinity of the solution \( \varepsilon = 0, \mu = \nu \) on the real axis.

Let us write equation (20) in the form
\[
\frac{1}{12} \frac{\partial F}{\partial \mu} = 1 + \frac{1}{2} \left( \frac{1 - (1 - \varepsilon \mu^2)^{1/2}}{1 - \varepsilon \mu^2} \right) + \frac{\mu^2}{4} \left( \frac{2[1 - (1 - \varepsilon \mu^2)^{1/2}]}{1 - \varepsilon \mu^2 - 1} \right)
= 1 + f_1(\varepsilon \mu^2) + \frac{\mu^2}{4} f_2(\varepsilon \mu^2)
\]
Now for the points \( \varepsilon = 0, \mu = \nu, 1 + \mu^2/4 \neq 0 \) and \( f_1(\varepsilon, \mu) = 0, f_2(\varepsilon, \mu) = 0 \). Then if \( m \) is the lower bound of \( |1 + \mu^2/4| \) on the circle \( |\mu - \nu| = r \) about \( \mu = \nu \) it is necessarily possible to choose \( r \), so that \( m > 0 \). It is also possible to choose \( p \) such that \( |\varepsilon| < p \) makes \( f_1(\varepsilon, \mu) + f_2(\varepsilon, \mu) < m \). Hence by Rouché’s theorem
\( \partial F/\partial \mu \) has within the domains \( |\mu - \nu| < r, |\varepsilon| < p \) the same number of zeros as
\( 1 + \mu^2/4 \) and since \( \nu \) is a point on the real axis with no zeros in its vicinity this
can be made no zeros. The radius of convergence of the series for \( \mu \) in \( \varepsilon \) thus
shown to exist can be proved at least sufficient for practical needs.

Choose the circle round \( \nu \) to be
\[
\mu = \nu(1 + e^{\varphi}/5) \quad (21)
\]
(\( e \) here and for the rest of this paragraph is the exponential). Then the real
part of \( 1 + \mu^2/4 \)
\[
= 1 + \frac{\nu^2}{4} + \frac{\nu^2}{10} \cos \varphi + \frac{\nu^2}{100} \cos 2\varphi
> 1 + \frac{14}{100} \nu^2.
\]
On this circle \( f_1(\epsilon \mu^2) + \frac{\mu^2}{4} f_2(\epsilon \mu^2) \)

\[
= f_1(\epsilon \mu^2) + \frac{\nu^2}{4} \left( 1 + \frac{2}{5} \epsilon \mu^2 + \frac{1}{25} \epsilon^2 \mu^2 \right) f_2(\epsilon \mu^2),
\]

the modulus of which

\[
\leq |f_1(\epsilon \mu^2)| + \frac{\nu^2}{4} \frac{36}{25} |f_2(\epsilon \mu^2)|.
\]

Now consider \( \epsilon \mu^2 \) as the other variable (instead of \( \epsilon \)). Since the coefficients of \( \epsilon \mu^2 \) in the series for \( f_1 \) and \( f_2 \) are positive numbers the maxima of the moduli of the functions for \( |\epsilon \mu^2| = \rho \) occur on the positive real axis—that is for \( \epsilon \mu^2 = \rho \).

Take \( \rho = 0.35 \), then

\[
|f_1| < 0.12, \quad |f_2| < 0.38
\]

whence

\[
\left| \frac{f_1 + f_2}{|1 + \frac{\mu^2}{4}|} \right| < 1.
\]

So that \( \partial F/\partial \mu \) has the same number of zeros in the circle \((21)\) with \( \epsilon \mu^2 \) in the domain \( |\epsilon \mu^2| < 0.35 \) as has \( 1 + \frac{\mu^2}{4} \)—that is none—and there are no singularities for the values of \( \epsilon \) and \( \mu \) satisfying the given conditions. We shall see later that the nearly parabolic solution is not required for \( |\epsilon \mu^2| > 0.58 \)—that is \( |\epsilon \mu^2| > 0.34 \).

The solution of \((19)\) may now be obtained as a Taylor’s series of the form

\[
\mu = C_0 + C_1 \epsilon + C_2 \epsilon^2 + \ldots
\]

either by calculating the necessary differential coefficients or by equating coefficients taking \( D = 12\nu + \nu^3 \), \( \nu \) being the solution for \( \epsilon = 0 \). If this is done we find

\[
C_0 = \nu
\]

\[
C_1 = \frac{\nu^3}{1 + \frac{\nu^2}{4}} \left( \frac{1}{2^2.3 + 24.5} \nu^2 \right)
\]

\[
C_2 = \frac{\nu^5}{(1 + \frac{\nu^2}{4})^3} \left( \frac{1}{2^3.3 + 25.3^2.5^7} \nu^2 + \frac{127}{2^4.5.7} \nu^4 - \frac{1}{2^{9.5^2.7} \nu^6} \right)
\]

The disadvantage of this method of solution is that even in the range of appropriateness of the nearly parabolic solution \( \nu \) may become large and the tabulation for the coefficients correspondingly extended.

The form of the equations determining the coefficients in \((22)\) suggests another solution. The equation determining \( C_1 \) is of form

\[
C_1(1 + \frac{\nu^2}{4}) = -\nu^3 \quad \text{(polynomial of degree 2 in} \nu).\]

The polynomial has no term of the first power in \( \nu \), so that if we divide throughout by \( 1 + \frac{\nu^2}{4} \) we can obtain a remainder of the form \( \gamma_1 \nu^3 \) where \( \gamma_1 \) is a numerical constant, so that if we add a term \( \gamma_1 \nu^3 \epsilon \) to the \( 12\nu + \nu^3 \) side of the original equation the coefficient of \( \epsilon \) would become a polynomial of degree 2 and \( \gamma_1 \) a constant to be determined in the process of equating coefficients. Similarly it is possible to obtain a form of coefficient avoiding fractions for the higher powers of \( \epsilon \) by equating coefficients in a solution of the form

\[
D = 12\sigma + \sigma^2 \epsilon^2
\]

\[
= 12\mu + \mu^3(1 + \epsilon)6 \left( \frac{1}{2.3} + \frac{1.3}{2.4} \frac{\epsilon \mu^2}{5} + \ldots \right),
\]

\[
e^2 = 1 + \gamma_1 \epsilon + \gamma_2 \epsilon^2 + \ldots, \quad \mu = \sigma(1 + G_1 \epsilon + G_2 \epsilon^2 + \ldots),
\]

with

\[
G_1 = g_{12} \sigma^2, \quad G_2 = g_{22} \sigma^2 + g_{24} \sigma^4,
\]

\[
(23)
\]
where $G_n$ is a polynomial of degree $2n$ in $\sigma$ an auxiliary quantity determined by the first equation. Here the first coefficients are

\[
G_1 = -\frac{3}{2^{2.5}} \sigma^2,
\]

\[
G_2 = -\frac{1}{2^{2.5} 7} \sigma^2 + \frac{1}{2^{3.5} 7} \sigma^4,
\]

\[
G_3 = -\frac{13}{3^{2.5} 7} \sigma^2 + \frac{71}{2^{3.5} 7} \sigma^4 + \frac{1}{2^{4.3} 5^2 7} \sigma^6,
\]

\[
G_4 = -\frac{107}{2^{4.3} 5^2 7.11} \sigma^2 + \frac{6679}{2^{4.3} 5^2 7.11} \sigma^4 + \frac{221}{2^{6.5} 7.11} \sigma^6 + \frac{43}{2^{8.5} 7.11} \sigma^8,
\]

\[
G_5 = -\frac{2^{2.103}}{3^{5.7} 11.11.13} \sigma^2 + \frac{214601}{2^{4.3} 5^2 7.11.11.13} \sigma^4 + \frac{178849}{2^{6.3} 5^2 7.11.11.13} \sigma^6
\]

\[
+ \frac{123791}{2^{8.3} 5^2 7.11.11.13} \sigma^8 + \frac{1213}{2^{10.5} 7.11.13} \sigma^{10},
\]

\[
\epsilon^2 = 1 - \frac{4}{5} \epsilon - \frac{6}{5^2} \epsilon^2 - \frac{52}{3.5^3} \epsilon^3 - \frac{107}{3.5^7 2^{11.1.13}} \epsilon^4 - \frac{2^{10.5} 7.11.13}{5^3} \epsilon^6 - \ldots,
\]

where the exponent is placed on $\epsilon$ so that the first equation can be conveniently written in the same form as that for the parabola and solved by the use of the same table. The extreme value of $\epsilon$ for which the nearly parabolic solution is necessary is $0.07$, for which the first neglected term of $\epsilon^2$ is $0.8 \times 10^{-8}$.

In the process leading to the previous solution it would have been possible to carry on the division by $(1 + \nu^2/4)$ a step further to give a remainder of the form $\beta_1 \nu$ and destroyed the fractional form of the $C_1$ term in the solution in Taylor’s series by adding a term $\beta_1 \nu \epsilon$ to the left-hand member of the equation determining the coefficients. This suggests a solution of the form

\[
12\tau + \tau^3 b = 12 \mu + \mu^3 (1 + \epsilon) \left( \frac{1}{2.3} + \frac{1.3 \epsilon \mu^2}{2.4} + \ldots \right),
\]

with

\[
b = 1 + \beta_1 \epsilon + \beta_2 \epsilon^2 + \ldots,
\]

\[
\mu = \nu (1 + H_1 \epsilon + H_2 \epsilon^2 + \ldots),
\]

where $H_n$ is a polynomial of degree $2n$ in $\tau$. The first terms of this solution are given by

\[
H_1 = \frac{4}{3.5} - \frac{3}{2^{2.5}} \tau^2,
\]

\[
H_2 = \frac{2^{42}}{3^{2.5} 7} \frac{43}{2^{5.7}} \tau^2 + \frac{1}{2^{3.5} 7} \tau^4,
\]

\[
\ldots
\]

\[
b = 1 + \frac{4}{3.5} \epsilon + \frac{2^{42}}{3^{2.5} 7} \epsilon^2 + \ldots
\]

It is a pleasure to acknowledge the helpful conversations I had with Mr. W. B. Smith-White, who kindly read a draft of this section before it was offered as a contribution to him in his capacity of editor.
Numerical Considerations.

Kepler's equation may for purposes of calculation be written in a number of ways according to the tastes of the computer or the tables he has available; among these are

\[ [M] + \{e\} \{\sin E\} = E \]

\[ \left[ \frac{M}{e} \right] + \left( \frac{1}{e} \right) (E) = \sin E \]

\[ [M] + \{e\} (Y) = \sin^{-1} Y \]

\[ [M] - \{1 - e\} (Y) = \sin^{-1} Y - Y \]

where \( \sin E = Y \). The quantity in the square bracket is put into the product register of the calculating machine, that in the curly bracket in the setting register and the quantity in the plain bracket built up in the multiplier register till the right-hand member of the equation appearing in the product register has the value corresponding to that in the plain bracket. The first two forms, due to Comrie and Strömgren respectively (see Möller, 1933), require a table of sines with argument in decimals of a degree or in radians. The recently published Chambers Six-Figure Mathematical Tables (Comrie, 1949) are most suitable. Reasonably near to perihelion the third and fourth forms are useful. The tables of Möller (1940) and Strömgren (1945) are particularly suitable for the last form, which is valuable in the range of values, to be discussed shortly, when computation with an extra figure is necessary. The last three forms have the advantage that the quantity to appear in the multiplier register, being the argument of the mathematical table, can be more conveniently built up to its full number of tabulated figures. It should be added that the second method can be used even if \( e \) is small, since although its reciprocal is large and the significant figures of \( E/e \) and \( M/e \) are moved relative to those \( \sin E \) the same thing occurs with the equation in its original form and indeed (if only \( E \) were required) for a five-figure solution of \( M = E - e \sin E \) with \( e < 0.1 \) the value of \( \sin E \) would be needed to only four decimals.

If the tables for \( \sin E \) extend only to \( \pi/2 \) the equation may be used past this point in the form

\[ (\pi - M) - e \sin (\pi - E) = \pi - E \]

or

\[ \frac{\pi - M}{e} - \frac{\pi - E}{e} = \sin (\pi - E) \]

It is well known that the accuracy of solution of Kepler's equation falls off when \( e \) tends towards 1. Suppose equation (7) has been solved by using a table of sines. Let the solution obtained be \( E_1, \sin_1 E_1 \) the value of its sine simultaneously obtained from the table and \( E \) the accurate solution.

Then

\[ M = E_1 - e \sin_1 E_1, \]

\[ \sin_1 E_1 = \sin E_1 + \Delta, \]

\[ E_1 = E + \delta E, \]

and

\[ \sin_1 E_1 = \sin E + \delta \sin E \]

where the last three equations are definitions of \( \Delta \), the error of the table and of \( \delta E \) and \( \delta \sin E \), the errors in the solutions for \( E \) and \( \sin E \) respectively. From these equations with (7)

\[ \delta E = \frac{e \Delta}{1 - e \cos E} \quad \text{and} \quad \delta \sin E = \frac{\Delta}{1 - e \cos E} \]

If we are working with a table to \( n \) figures the maximum error (now called \( \Delta \)) of the table will be \( 0.5 \times 10^{-n} \) and its average value would be half of this.
The rectangular coordinates in the plane of the orbit are

\[ x_0 = a (\cos E - e) \]
\[ y_0 = a(1 - e^2)^{1/2} \sin E \]

and the maximum errors in these arising from errors in solving Kepler's equation and (another \( \Delta \)) in extracting \( \cos E \) from the tables are

\[ \delta x_0 = a \Delta (e \sin E + 1 - e \cos E)/(1 - e \cos E) \]
\[ \delta y_0 = a(1 - e^2)^{1/2} \Delta/(1 - e \cos E) \]

The effect that these errors have on the position of a body on the celestial sphere depends on their relation in direction and magnitude to the geocentric distance vector, but as this relation differs for every body and for the same one at different times it is best for the purpose of establishing a measure of the influence of the errors on the position of an object to compare their magnitude with that of the heliocentric distance, \( r \). Now

\[ r = a(1 - e \cos E) \]

so that using the relations above

\[ \frac{(\delta x_0)^2 + (\delta y_0)^2}{r} = \frac{(2 + 2e \sin E)^{1/2}}{(1 - e \cos E)^{3/2}} \Delta \]  

(24)

The value of the coefficient of \( \Delta \) in (24) which may be tolerated is arbitrary, but 3 seems a reasonable figure and one which will not introduce errors from this cause more serious than must be admitted in almost any extended calculation. If the values of \( M \) and \( e \) are plotted as abscissa and ordinate on a plane, the curve given by (7) and

\[ \frac{(2 + 2e \sin E)^{1/2}}{(1 - e \cos E)^{3/2}} = 3 \]

defines the boundary of the region in which Kepler's equation will give a satisfactory result. Beyond this region is an area in which it is profitable if we want a result accurate to \( n \) figures to work with a table to \( n + 1 \) decimals. The boundary of this region is given by

\[ \frac{(2 + 2e \sin E)^{1/2}}{(1 - e \cos E)^{3/2}} = 30 \]

and beyond this methods adapted for nearly parabolic solutions should be used. Also consider the equation

\[ x_0 = a(\cos E - e) \]

and suppose the possible error in the tabulation of \( \cos E \) to be \( \Delta_1 \), then

\[ \frac{\delta x_0}{r} = \Delta_1/(1 - e \cos E) \]  

(25)

along the extra figure boundary \((1 - e \cos E)^{-1}\) has values from 1.8 to 2.5 and along the nearly parabolic boundary from 7.8 to 8.5, which shows that it is necessary to work with the extra figure, when it is appropriate, even after the determination of \( E \) (or \( \sin E \)). For small geocentric distances, say less than 0.3 astronomical unit, it would be necessary to use the extra figure apart from the considerations arising above.

If we work through the case of the hyperbolic orbit, equation (8), in the same way as has been done for the ellipse we find the boundary beyond which extra figure computation is necessary is given by

\[ \frac{(2e \cosh F + 2e \sinh F)^{1/2}}{(e \cosh F - 1)^{3/2}} = 3 \]
and that beyond which nearly parabolic solution must be used by
\[
\frac{(2e \cosh F + 2e \sinh F)^{1/2}}{(e \cosh F - 1)^{1/2}} = 30
\]
Both equations are taken with equation (8) if \( M \) is to be plotted against \( e \).

Figure 2 shows these curves which define the regions in which the various
types of computation are appropriate. In Figure 3 the area in which nearly
parabolic solution is desirable is represented on the plane of \( e \) and \( \sin E (= \varepsilon^{1/2} \mu) \)
\( M \) and \( \sin E \) are always available when calculating in the ordinary way or with

![Fig. 2.—Method of solution according to values of \( e \) and \( M \).](image)

the extra figure and a quantity approximating to \( \varepsilon^{1/2} \mu \) is available at a stage in
the nearly parabolic solution which I hope to publish soon so that these diagrams
facilitate a decision as to which type of computation is to be used, or in some
cases when a change of process is necessary. In doubtful cases either of the
possible alternatives should be satisfactory. It may be pointed out that the
nearly parabolic solution is not necessary except when \( \sin E < 0.53 \) or
\( \sinh F < 0.58 \) and by the previous discussion the development in series of the
solution for the co-ordinate \( \mu \) is then permissible.

Many transformations of Kepler's equation have been used to provide
solutions, ones of importance being by Tietjen (see Bauschinger, 1934), Howe
(see Plummer, 1919) and Oppolzer and Marth (see Marth, 1890b). Oppolzer
and Marth, independently, wrote the equation in the form
\[
\tan (E - M) = \frac{e \sin M}{\lambda - e \cos M},
\]
where
\[
\lambda = \frac{E - M}{\sin (E - M)}.
\]
Tables for this solution were given by the two authors mentioned and more
recently Subbotin (1929) has used it tabulating \( \log \lambda \) to seven decimal places
with argument \(\tan (E-M)\). It is of interest to examine the range of applicability of this transformation of Kepler's equation as was done for the equation itself in the previous paragraphs.

We obtain

\[
\delta E = \tan (E-M) \frac{\tan (E-M)}{\sec^2 (E-M) \{\cos (E-M) + (E-M) \sin (E-M) - e \cos M\}}
\]

\[
= \frac{\frac{1}{2} \sin 2 (E-M)}{\cos (E-M) + (E-M) \sin (E-M) - e \cos M}
\]

Where \(\triangle\) is the error of the tabulated value of \(\lambda\). The ratio of the error arising from \(\delta E\) in the position of the body on the plane of its orbit to the radius vector

![Graph showing method of solution according to values of \(e\) and \(\sin E\) (or \(\sinh F\)).](image)

\[\text{Fig. 3. — Method of solution according to values of } e \text{ and } \sin E \text{ (or } \sinh F).\]

\[
\frac{\{(\delta x_0)^2 + (\delta y_0)^2\}^{1/2}}{r} = \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} \delta E
\]

\[
= \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} \frac{\frac{1}{2} \sin 2 (E-M)}{\cos (E-M) + (E-M) \sin (E-M) - e \cos M}\]

If we are prepared to allow the coefficient of \(\triangle\) in this equation to attain the value 3 and as before map the boundary of the area for which the transformation is applicable, we are applying a rather less severe test since the error in the equation includes the effect arising only from the error in calculating \(E\) and neglects errors in the tables subsequently used for computing \(x_0\) and \(y_0\). Nevertheless, the boundary which is shown by the dotted curve on Figure 2 indicates that while the method of solution does represent an improvement on that using a table of sines to the same number of figures, it does not improve on the accuracy obtained by using the sine table to an extra figure nor encroach appreciably on the area in which a nearly parabolic solution is needed.

It is thus not possible to dispense with extra figure calculation unless we are willing to extend unduly the tabulation for nearly parabolic solutions. Equations (25) and (26) show that however accurately \(E\) may be determined the solution in terms of eccentric anomaly leads to difficulties when \(e\) is nearly 1 and \(E\) is small.
Bibliography.

The following bibliography is supplementary to the references given by Radau (1900), Bauschinger (1901 and 1934), Herglotz (1906) and Wintner (1941). Material listed by these authors has not been repeated unless referred to in my article. The titles of the references do not always show satisfactorily the aspect of Kepler’s Problem with which they deal, so they are omitted and the contents indicated by a code, which has the further advantage of economy.

G, indicates a general account such as might be given by a text book, a review article or an article chiefly didactic in purpose,

T, that the main interest is theoretical,

S, an article whose main purpose is to describe or discuss a method of obtaining a solution of Kepler’s Problem, and

I, a solution depending on numerical integration or the application of finite differences.

e, indicates that the article refers to elliptic motion,

n, to nearly parabolic motion, and

p, to parabolic motion.

s, indicates a solution in power series,

f, a solution in trigonometric series,

g, that the solution is graphical or mechanical and

c, that a numerical solution is given (c is only used if two methods are described otherwise solutions may be assumed to involve numerical methods),

t, indicates that tables are given to aid the purpose of the article and

z, that I have not seen the article myself and depend on a review or abstract for its description.

This bibliography is meant to be fairly comprehensive for categories S and T. Developments whose main application lies in the study of perturbed motion have usually been omitted except for some borderline cases under the heading I. The historical aspect has been neglected entirely but those who are interested will find valuable guidance in Radau (1900), Herglotz (1906) and Wintner (1941); and the bibliography of Houzeau and Lancaster (1887, 1889) gives many references, nearly all of which were inaccessible to me. The collected works of Kepler have been edited by Frisch (1858-71) and a recent account of “de motibus stellae Martis” is given by Pannekoek (1948).

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