HANKEL TRANSFORMS OF FUNCTIONS ZERO OUTSIDE A FINITE INTERVAL.

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SUMMARY.

If we define the Hankel transform \( G(u) \) of a function \( g(x) \) by
\[
G(u) = \text{l.i.m.} \int_0^\infty x J_\nu(ux)g(x)dx, \quad \nu > -\frac{1}{2}
\]
it is proved that

(a) \( g(x) \) is zero for almost all \( x > A \), and
(b) \( x^\lambda g(x) \) belongs to \( L^2(0, A) \),

when and only when

(i) \( G(z) \) is analytic in \( z \) for \( 0 \leq \arg z \leq \pi \), \( |z| > \varepsilon > 0 \),
(ii) \( z^4 G(z) = O(e^{\delta \text{Im} z}) \) as \( |z| \to \infty \), \( \text{Im} z \geq 0 \),
(iii) \( G(ue^{i\tau}) = e^{i\tau \nu}G(u), \quad u > 0 \),
(iv) \( u^\lambda G(u) \) belongs to \( L^2(0, \infty) \), and
(v) \( |G(z)| = O(|z|^{\nu}) \) as \( z \to 0 \).

I. INTRODUCTION.

When using transform methods, it is most important to have at one's disposal theorems which allow the determination of non-trivial properties of the original functions from those of the image function.

In [P.W.] (theorem X, p. 13), Paley and Wiener proved the well-known theorem: If \( F(\xi) \) is the Fourier transform of \( f(x) \), then \( f(x) \) is zero almost everywhere for \( |x| > A \), and belongs to \( L^2( -A, A) \), when and only when \( F(z) \) is an integral function satisfying
\[
F(z) = o(e^{\delta |z|}) \quad \text{................. (1.1)}
\]
with \( F(\xi) \) belonging to \( L^2( -\infty, \infty) \).

When we say that "\( f(x) \) belongs to \( L^p(a, b) \)" we mean that \( f(x) \) is measurable in the Lebesgue sense over \( (a, b) \) and
\[
\int_a^b |f(x)|^p dx
\]
exists in the Lebesgue sense.

The aim of this paper is to prove the theorem quoted in the Summary, which is the analogue in the Hankel transform theory of Paley and Wiener's theorem.
Recalling that $J_\nu(t)$ is the Bessel function of the first kind of order $\nu$, defined by

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}t)^{\nu+2m}}{m! \Gamma(\nu+m+1)} \quad \ldots \ldots \ldots \ldots (1.2)$$

the form of the Hankel transform theorem which will be used will be the following:

Assuming that

$$\nu > -\frac{1}{2} \quad \ldots \ldots \ldots \ldots (1.3)$$

and that $x^\nu g(x)$ belongs to $L^2(0, \infty)$ then there exists a $G(u)$ such that $u^\nu G(u)$ belongs to $L^2(0, \infty)$ and that

$$T[g(x)] = G(u) = \int_0^\infty xJ_\nu(ux)g(x)dx \quad \ldots \ldots \ldots \ldots (1.4)$$

and

$$T^{-1}[G(u)] = g(x) = \int_0^\infty uJ_\nu(xu)G(u)du \quad \ldots \ldots \ldots \ldots (1.5)$$

where both the integrals in equations (1.4) and (1.5) are to be taken in the sense of the limit in the mean square.

By saying that the equation (1.4) exists as a limit in the mean square, we understand that

$$\lim_{A\to \infty} \int_0^\infty |G(u) - \int_0^A xJ_\nu(ux)g(x)dx|^2du = 0$$

This form of the transform theorem is that used in [T.F.I.], chapter VIII, and [B.C.], chapter V.

In order to save repetition, we will consider functions as identical if they are equal almost everywhere. In particular, if we say that a function is zero for $x > A$, we mean that it is equal almost everywhere to a function which is zero for $x > A$.

In sections 3 and 4, the following Parseval theorems will be needed:

**O1:** If $x^\nu f(x)$ and $x^\nu g(x)$ both belong to $L^2(0, \infty)$ and

$$T[f(x)] = F(u)$$

and

$$T[g(x)] = G(u)$$

then

$$\int_0^\infty x^\nu f(x)g(x)dx = \int_0^\infty uF(u)G(u)du, \quad \ldots \ldots (1.6)$$

the integrals holding in the $L^1$-sense. This is Theorem 1 of [P.M.O.] with a notation change.

**O2:** If $u^\nu G(u)$ belongs to $L^1(0, \infty)$ and $u^\nu F(u)$ belongs to $L^1(0, c)$ and $u^{-1}F(u)$ belongs to $L^1(c, \infty)$ where $c > 0$, then equation (1.6) holds. This is Theorem 4 of [P.M.O.].
2. The Necessary Conditions.

We assume now that \((a)\) and \((\beta)\) hold.

It is immediately obvious from \((\beta)\) that \(x^\delta g(x)\) belongs to \(L^1(0, A)\).

In this section we write \(z = u + i\beta\) and restrict our discussion to the half plane \(\beta \geq 0\).

Referring to equation (1.4), we have now

\[
z^\delta G(z) = \int_0^A (xz)^\delta J_\nu(xz)[x^\delta g(x)]\,dx \quad \cdots \cdots \cdots \cdots \quad (2.1)
\]

The only singularities of \(J_\nu(t)\) and \(J'_\nu(t)\) are at the origin and from [W.B.F.], p. 49, we derive

\[
\left\{ (xz)^\delta J_\nu(xz) \right\} \left| \frac{z |\nu+i\beta x|}{2\nu\Gamma(\nu+1)} \right| \cdots \cdots \cdots \cdots \quad (2.2)
\]

and

\[
\frac{\partial}{\partial z} \left\{ (xz)^\delta J_\nu(xz) \right\} \left| \leq \frac{1+2|\nu| + x^2 |z|^2}{\Gamma(\nu+1) + \Gamma(\nu+2)} \right| z^{-\nu-1} x^{\nu+i} \left| \nu - i\beta x \right| \cdots \cdots \cdots \cdots \quad (2.3)
\]

Thus (i) follows immediately.

Now we have

\[
G(z) = \int_0^\infty xJ_\nu(xz)g(x)\,dx
\]

and using again the inequality (2.2), obtain

\[
|G(z)| \leq \frac{z |\nu e^{i\beta}|}{2\nu\Gamma(\nu+1)} \int_0^A x^{\nu+1} |g(x)| \, dx
\]

Since \(\nu > -\frac{1}{2}\) and \(x^\delta g(x)\) belongs to \(L^1(0, A)\), the integral on the right exists and

\[
G(z) = O(|z|^{\nu})
\]

as \(|z| \to 0\). This is (v).

To obtain further information concerning the behaviour of \(G(z)\) when \(|z|\) is large, we write

\[
G(z) = \left[ \int_0^\eta + \int_\eta^A \right] xJ_\nu(xz)g(x)\,dx \quad \cdots \cdots \cdots \cdots \quad (2.4)
\]

with \(0 < \eta < A\).

Inequality (2.2) gives

\[
|I_1| \leq \frac{2^{-\nu}}{\Gamma(\nu+1)} z |\nu e^{i\beta}| \int_0^\eta x^{\nu+1} |g(x)| \, dx
\]

\[
\leq \frac{2^{-\nu}}{\Gamma(\nu+1)} z |\nu e^{i\beta}| \int_0^A x^{\nu+1} |g(x)| \, dx
\]

\[
= O(|z|^{\nu e^{i\beta}}), \quad \eta < A \quad \cdots \cdots \cdots \cdots \quad (2.5)
\]

as \(|z| \to \infty\).
In $I_2$, we observe that $x \neq 0$, then the asymptotic formula for the Bessel function [W.B.F.], p. 199, gives

$$ |(zx)^{1/2}J_\nu(zx)| < Ce^{\beta x} $$

for sufficiently large $|zx|$, where $C$ is a constant independent of $z$ and $x$. Thus

$$ |I_2| \leq E \ |z|^{-1} e^{1/2} \int_0^A x^{1/2} \ g(x) \ dx $$

for some finite constant $E$, provided that $|z|$ is sufficiently large. That is

$$ |I_2| \leq E \ |z|^{-1} e^{1/2} \int_0^A x^{1/2} \ g(x) \ dx = O(|z|^{-1} e^{1/2}) \ \ \ \ \ \ \ \ \ \ \ \ \ \ (2.6) $$

as $|z| \to \infty$.

If now $\beta > 0$, we combine equations (2.5) and (2.6) and obtain

$$ |z^{1/2}G(z)| = O(e^{1/2}) \ \ \ \ \ \ \ \ \ \ \ \ \ \ (2.7) $$

as $|z| \to \infty$.

When $\beta = 0$, $(zx)^{1/2}J_\nu(zx)$ is bounded ($< M$, say). Then

$$ |z^{1/2}G(z)| < M \int_0^A x^{1/2} \ g(x) \ dx = O(1) \ \ \ \ \ \ \ \ \ \ \ \ \ \ (2.8) $$

as $|z| \to \infty$.

Equations (2.7) and (2.8) give (ii).

If, in equation (1.2), we replace $t$ by $zx$, we easily find

$$ G(z) = z^{1/2} \int_0^A x^{1/2} + \nu P(x, z) g(x) dx $$

where $P(x, z)$ is analytic and even in both $z$ and $x$. Conclusion (iii) of the summary follows immediately.

We finally observe that (iv) follows from the fundamental theorem quoted in the introduction.

3. **THE SUFFICIENT CONDITIONS.**

In this section we will show that the necessary conditions are also sufficient. We shall show that if $G(z)$ satisfied (i)-(v), then

$$ T[g(x)E(a-x)] = T[g(x)], \ a > A, \ \ \ \ \ \ \ \ \ \ \ \ \ \ (3.1) $$

where

$$ E(s) = \begin{cases} 1, & s > 0. \\ 0, & s < 0. \end{cases} $$

The inversion theorem will give the result.

Using equation (8) of [W.B.F.], p. 134, and recalling that $\nu > -1/2$, we obtain

$$ \int_0^a uJ_\nu(ux)J_\nu(tx) dx = \frac{a}{u^2 - t^2} \left\{ uJ_{\nu+1}(ua)J_\nu(ta) - tJ_\nu(ua)J_{\nu+1}(ta) \right\} $$

$$ = u(a, t, u), \ \ \ \text{(say)}. \ \ \ \ \ \ \ \ \ \ \ \ \ \ (3.2) $$
That is
\[ T[J_\nu(tx)E(a-x)] = w(a, t, u) \quad \ldots \ldots \ldots \ldots \ldots (3.3) \]

It is clear that \( J_\nu(tx)E(a-x) \) belongs to both \( L^2(0, \infty) \) and \( L^1(0, \infty) \).

We now suppose that \( u^k G(u) \) and (\emph{a fortiori}) that \( x^k y(x) \) belong to \( L^2(0, \infty) \).

The Parseval Theorem O₁ gives
\[
\int_0^\infty \int_0^\infty v w(a, u, v) G(v) dv = \int_0^\infty x [J_\nu(ux)E(a-x)] g(x) dx \\
= \int_0^\infty x J_\nu(ux) [g(x) E(a-x)] dx ;
\]

that is
\[ T[g(x)E(a-x)] = \int_0^\infty t w(a, u, t) G(t) dt \quad \ldots \ldots \ldots \ldots \ldots (3.4) \]

We now examine
\[
\int_0^\infty t w(a, u, t) G(t) dt \quad \ldots \ldots \ldots \ldots \ldots (3.5)
\]

where as mentioned above \( G(z) \) possesses the properties (i)-(v). It will be convenient in this section to write \( z = t + i\beta \).

The method we will use will follow closely that of [W.B.F.], p. 423 \emph{et seq.}

Consider
\[
\int_{c+} \frac{z G(z) }{z^2 - u^2} [zH^{(1)}_{\nu+1}(za)J_\nu(ua) - u H^{(1)}_{\nu}(za)J_{\nu+1}(ua)] dz \ldots \ldots \ldots \ldots \ldots (3.6)
\]

where
\[ H^{(1)}_{\nu}(z) = J_{\nu}(z) + iY_\nu(z) \]

is the Bessel Function of the third kind [W.B.F.], p. 73, and the contour consists of a semi-circle (radius \( R \)) above the real axis, and the real axis between \(-R \) and \( +R \), with small indentations (radius \( \varepsilon \)) at \(-u, +u \) and \( O \).

Since the integrand is analytic inside the contour, the integral vanishes.

Using (iii) and (v), and
\[
H^{(1)}_{\nu}(te^{i\pi}) = e^{-\nu\pi i} H^{(1)}_{\nu}(t) - 2e^{-\nu\pi i} J_{\nu}(t)
\]

([W.B.F.], p. 75, equation 5) we obtain for the contribution from the real axis (after the limits \( \varepsilon \rightarrow 0 \) and \( R \rightarrow \infty \) have been taken)
\[
\int_0^\infty \frac{t G(t) }{t^2 - u^2} [t2J_\nu(ua)J_{\nu+1}(ta) - 2uJ_{\nu+1}(ua)J_\nu(ta)] dt \ldots \ldots (3.7)
\]

The indentation at \(-u \) gives
\[
(-\frac{1}{2}\pi i)G(u)[u J_\nu(ua)[H^{(1)}_{\nu+1}(ua) - 2J_{\nu+1}(ua)] \\
- u J_{\nu+1}(ua)[H^{(1)}_{\nu}(ua) - 2J_{\nu}(ua)]] \ldots \ldots (3.8)
\]
The indentation at \( +u \) gives
\[
(-\frac{1}{2}\pi i)G(u)\{uJ_{\nu}(ua)H_{\nu+1}^{(1)}(ua) - uJ_{\nu+1}(ua)H_{\nu}^{(1)}(ua)\} \quad \ldots \ldots \quad (3.9)
\]

There is no contribution from the indentation at the origin.

To determine the contribution from the large semi-circle, we need a rather obvious lemma.

**Lemma.**

If \( S' \) is the semi-circle with centre the origin, and radius \( R \) and
\[
|G(z)| = O(|z|^{-\eta} e^{-b|z|}) \quad \text{where} \quad \eta > 0, \quad b > 0
\]
then
\[
\lim_{R \to \infty} \int_{S} G(z)dz = 0
\]

**Proof.**—When \( R \) is sufficiently large,
\[
|\int_{S} G(z)dz| < K \int_{0}^{\pi} e^{-bR \sin \theta} R^{1-\eta} d\theta \quad \text{for some} \quad K > 0
\]
\[
< 2K \int_{0}^{\pi} e^{-2R \sin \theta} d\theta
\]
\[
= K \pi R^{-\eta} [1 - e^{-R}]
\]
\[
\to 0, \quad \text{as} \quad R \to \infty.
\]

From [W.B.F.], p. 201, we have
\[
H_{\nu}^{(1)}(z) \sim M z^{-\frac{1}{2}} e^{iz} \quad \text{as} \quad |z| \to \infty \quad \ldots \ldots \quad (3.10)
\]
(for constant \( M \)).

Assumption (ii), equation (3.10) and the Lemma show that the contribution from the large semi-circle vanishes (in the limit) when \( a > A \), which we now assume.

Combining our results, we obtain
\[
\int_{0}^{\infty} tw(a, u, t)b(t)dt
\]
\[
= ( -\frac{1}{2}\pi i) auG(u)\{J_{\nu+1}(ua)H_{\nu}^{(1)}(ua) - J_{\nu}(ua)H_{\nu+1}^{(1)}(ua)\}
\]
\[
= \frac{1}{2}\pi auG(u)\{J_{\nu+1}(ua)Y_{\nu}(ua) - J_{\nu}(ua)Y_{\nu+1}(ua)\}
\]
\[
= G(u)
\]
by Lommel's formula ([W.B.F.], p. 77).

Referring back to equation (3.4), we see that we have proved equation (3.1).

To complete this section we have only to remark that the inversion theorem shows that \( x_{t}g(x) \) belongs to \( L^{2}(0, A) \).

4. **THE \( L^{1} \)-CASE.**

It is obvious that if \( x_{t}g(x) \) belongs to \( L^{1}(0, A) \), the whole of section 2 is valid except the last paragraph. We cannot say that \( u_{t}G(u) \) belongs to any particular \( L \)-class.
In section 3, if we replace (iv) by \( u^4 G(u) \) belongs to \( L^1(0, \infty) \) \(^{115}\) we may derive equation (3.4) by using theorem O\(_2\) of the Introduction. The whole of section 3 now follows up to the point at which we proved \( T[g(x)] = T[g(x)E(a - x)] \). To obtain \( \alpha \) we must use the inversion theorem in the form of [T.F.I.], theorem 135. An examination of the proof of [T.F.I.], theorem 135, indicates that since \( g(x) = 0 \) for \( x > A \), \( x^\frac{1}{2}g(x) \) will belong to \( L^1(0, A) \).

**REFERENCES.**


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