ON WEBER TRANSFORMS.

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Summary.

There are two forms of the Weber Transform. The first, denoted by the operator $T_1$, is given by

$$A_1: \text{If } \nu \geq 0 \text{ and } x^\nu f(x) \text{ belongs to } L^1(a, \infty), a > 0,$$

$$F(s) = T_1[f(x)] = \int_a^\infty xC_\nu(xs, as)f(x)\,dx \quad \ldots \ldots \ldots \quad (S.1)$$

(the integral converges absolutely).

We have used the notation

$$C_\nu(\alpha, \beta) = J_\nu(\beta)Y_\nu(\alpha) - J_\nu(\alpha)Y_\nu(\beta), \quad \ldots \ldots \quad (S.2)$$

where $J_\nu(\alpha)$ and $Y_\nu(\alpha)$ are the Bessel functions of the first and second kind of order $\nu$.

The second form denoted by the operator $W_1$ is defined as follows:

$$B_1: \text{If } s^\nu G(s)/Q_\nu(as) \text{ belongs to } L^1(0, \infty),$$

$$g(x) = W_1[G(s)] = \int_0^\infty [Q_\nu(xa, sa)]^2 G(s)\,ds \quad \ldots \ldots \ldots \quad (S.3)$$

(the integral converges absolutely).

Here we have used the notation

$$[Q_\nu(\alpha)]^2 = [J_\nu(\alpha)]^2 + [Y_\nu(\alpha)]^2 \quad \ldots \ldots \ldots \quad (S.4)$$

The inversion theorem for the $T_1$-transform was given in [T.W.I.] (see references at end of paper) and states that, whenever $f(x)$ is of bounded variation near a point $x$,

$$\frac{1}{2}[f(x+0)+f(x-0)] = \lim_{\lambda \to \infty} \int_0^\lambda \frac{sC_\nu(xs, as)}{[Q_\nu(\alpha)]^2}F(s)\,ds \quad \ldots \ldots \ldots \quad (S.5)$$

$$\equiv T^{-1}[F(s)].$$

Similarly, the inversion of the $W_1$-transform is given by

$$\frac{1}{2}[G(s+0)+G(s-0)] = \lim_{\lambda \to \infty} \int_0^\lambda xC_\nu(xs, sa)g(x)\,dx$$

$$\equiv W^{-1}[g(x)] \quad \ldots \ldots \ldots \ldots \quad (S.6)$$

at any point at which $G(s)$ is of bounded variation.
The author of this paper has been unable to find formula (S.6) in the literature available to him and a proof has been given in the Appendix.

It will be seen that the transforms $T_1$ and $W_1^{-1}$ will be equivalent only if $x^f(x)$ belongs to $L^1 (a, \infty)$ and $s^q[T_1[f(x)]]/Q_\nu(a)gs$ belongs to $L^1 (0, \infty)$, since the integrals in equations (S.6) and (S.7) do not necessarily converge absolutely.

In §3 of the paper, we assume that $x^f(x)$ belongs to $L^2 (a, \infty)$ and $s^qG(s)/Q_\nu(a)gs$ belongs to $L^2 (0, \infty)$ and define a $T$ and a $W$-transform (denoted by $T_2$ and $W_2$ respectively) so that, if

(i) $x^f(x)$ belongs to both $L^1 (a, \infty)$ and $L^2 (a, \infty)$ then $T_1[f(x)]=T_2[f(x)]$,
(ii) $s^qG(s)/Q_\nu(a)gs$ belongs to both $L^1 (0, \infty)$, and $L^2 (0, \infty)$, then $W_1[G(s)]=W_2[G(s)]$.

These definitions will allow an identification between $T_2$ and $W_2^{-1}$ which can be expressed as:

To every $f(x)$ for which $x^f(x)$ belongs to $L^2 (a, \infty)$, there corresponds a unique function $F(s)$, for which $s^qF(s)/Q_\nu(a)gs$ belongs to $L^2 (0, \infty)$ (and conversely). The functions $f(x)$ and $F(s)$ are connected by

(a) $F(s)=T_2[f(x)]=W_2^{-1}[f(x)]$
(b) $=\lim_{p \to \infty} \int_a^p xC_\nu(xs, as)f(x)dx$,
(c) $f(x)=T_2^{-1}[F(s)]=W_2[F(s)]$
(d) $=\lim_{p \to \infty} \int_0^p \frac{sC_\nu(xs, as)}{[Q_\nu(a)gs]^2}F(s)ds$,

provided that we identify functions which are equal almost everywhere.

This result is contained as part of Theorem 3.1 in the text, which is clearly a theorem of the Plancherel type.

Incidently, in the proof of the theorem 3.1, the Parseval formula

$$\int_a^\infty xf(x)g(x)dx = \int_0^\infty \frac{sF(s)G(s)}{(Q_\nu(a)gs)^2}ds$$

is proved.

In §4, the $T_2$-transform of a function which is zero outside a finite interval is examined. The conclusion which connects these functions with a certain class of integral functions is expressed in the theorem:

In order that $x^g(x)$ belongs to $L^2(a, \infty)$ and

$$g(x)=0, \text{ for } x>b>a,$$

it is necessary and sufficient that

(i) $s^qG(s)/Q_\nu(a)gs$ belongs to $L^2 (0, \infty)$,
(ii) $G(z)$ is analytic in $z=s+i\beta$ for all $\beta$,
(iii) $G(ze^{i\pi})=G(z)$ and
(iv) $|zG(z)|=O(e^{b-a}|\beta|)$ as $|z| \to \infty$. 


When we are considering functions in $L^1$-space, there are essentially two different types of Weber Transforms. In order to state these in the form required for this paper, we will use the following notations mentioned in the summary:

If $J_v(x)$ and $Y_v(x)$ are the Bessel functions of the first and second kind, defined on p. 8 and p. 64 of [W.B.F.] respectively, we write

$$C_{v}(x, \beta) = J_v(x)Y_v(\beta) - J_v(\beta)Y_v(x) \quad \ldots \ldots \ldots \ldots (1.1)$$

and

$$[Q_v(x)]^2 = [J_v(x)]^2 + [Y_v(x)]^2. \quad \ldots \ldots \ldots \ldots (1.2)$$

The form of the Weber transform, which was treated in detail in [T.W.I.], is defined with its inverse in the theorem

$A_1$: If $v \geq 0$ and $x^t f(x)$ belongs to $L^1 (a, \infty), a > 0$, then we define

$$F_1[f(x)] = \int_{a}^{\infty} xC_v(xs, as)f(x)dx \quad \ldots \ldots \ldots \ldots (1.3)$$

and, if $f(t)$ is of bounded variation near $t = x$,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_{0}^{\infty} \frac{sC_v(xs, as)}{[Q_v(as)]^2}F(s)ds = T^{-1}[F(s)] \quad \ldots \ldots \ldots \ldots (1.4)$$

When we say that "a function $h(t)$ belongs to $L^p(a, b)$", we mean that $h(t)$ is measurable, and $\int_{a}^{b} |h(t)|^p dt$ exists in the Lebesgue sense. By the notation $\int_{a}^{b} h(t)dt$, we understand $\lim_{\lambda \to \infty} \int_{a}^{\lambda} h(t)dt$ where the integral may or may not converge absolutely. When we write $\int_{a}^{b} h(t)dt$ we will know that the integral does converge absolutely as in equation (1.3) above.

The second type of the Weber transform is that in which the $s$-function (say $G(s)$) is the initial function. The theorems of [W.B.F.] on p. 468 allow the derivation of a transform of this type. However, since these theorems demand that $G(s)$ should be zero in a neighbourhood of the origin, the transform derived from them is not sufficiently general for this paper. In the Appendix, we will prove a theorem, of which the following is an obvious corollary:

$B_1$: If $v \geq 0$, and $s^t G(s)/Q_v(as)$ belongs to $L^1 (0, \infty)$, then we define

$$W_1[G(s)] = g(x) = \int_{0}^{\infty} \frac{sC_v(xs, as)}{[Q_v(as)]^2}G(s)ds \quad \ldots \ldots \ldots \ldots (1.5)$$

and, if $G(t)$ is of bounded variation near $t = s > 0$,

$$\frac{1}{2}[G(s+0) + G(s-0)] = \int_{a}^{\infty} xC_v(xs, as)g(x)dx = W_1^{-1}[g(x)] \quad \ldots \ldots \ldots \ldots (1.6)$$
If we use the properties of $C_v(x, a)$ and $Q_v(a)$ mentioned in §2 of this paper, it will be clear that $F(s)$ defined by equation (1.3) and $g(x)$ defined by equation (1.5) will be continuous functions. However, it will be observed that since the inversion integrals do not necessarily converge absolutely, nothing is said concerning the $L$-class of $x^t g(x)$ or of $s^t G(s)/Q_v(a)$.

In §3, when we define the $T_2$-transform of a function $f(x)$ with $x^2 f(x)$ belonging to $L^2(a, \infty)$, it will follow that if $F(s) = T_2 \{f(x)\}$ then $s^t F(s)/Q_v(a)$ will belong to $L^2(a, \infty)$. A corresponding result will be found for the $W_2$-transform.

This will allow us to identify the $T_2$-transform with the $W_2^{-1}$-transform.

Now if we proceed formally, we may obtain the Parseval formula for the Weber Transforms. Thus

$$\int_a^\infty x f(x) g(x) dx = \int_a^\infty x f(x) dx \int_0^\infty s C_v(x, a) G(s) ds$$

$$= \int_0^\infty s G(s) ds \int_a^\infty x C_v(x, a) f(x) dx.$$  

This formula will be shown to hold for the $T_2$-transform (and of necessity for the $W_2$-transform) without any further restrictions being placed on the functions concerned.

In §4, we will examine the $T_2$-transforms of functions which are zero outside a finite interval. Two theorems (or more accurately two different statements of the one theorem) will be proved. These theorems are of the type found in [G], where the same problem was examined in the case of the Hankel $J$-transform.

This section contains some of the properties of certain of the functions, which will be used later in this paper.

(a) $C_v(x, \beta) = J_v(x) Y_v(\beta) - J_v(\beta) Y_v(x)$ (see equation (1.1)).

(i) $C_v(x, a)$ is an analytic function of $z$ if $x \geq a > 0$.

To prove this assertion, we observe that $J_v(t)$ and $Y_v(t)$ have no singularities except at $t=0$. This point is a branch point.

If we use

$$J_v(ze^{m \pi i}) = e^{mv \pi i} J_v(z)$$

and

$$Y_v(ze^{m \pi i}) = e^{-mv \pi i} Y_v(z) + 2i \sin mv \pi \cot v \pi J_v(z)$$

(from [W.B.F.], p. 75), it follows immediately that $C_v(x, a)$ is a single valued function of $z$.

In order to remove the singularity at the origin, it is necessary to define the value of $C_v(x, a)$ at $z=0$ by $\lim_{z \to 0} C_v(x, a)$. From the series expansions of the Bessel functions concerned, it is easy to derive

$$C_v(x, a) = -\frac{1}{v \pi} \left( \frac{x^v}{a^v} - \frac{a^v}{x^v} \right) + O(|z|^2), \quad v \neq 0 \quad \ldots \ldots (2.1)$$
as \(|z| \to 0\), and
\[
C_y(xz, az) = -\frac{2}{\pi} \ln \frac{x}{a} + O(|z|^2), \quad v=0 \quad \ldots \ldots (2.2)
\]
as \(|z| \to 0\).

Thus we define
\[
C_y(xz, az)_{z=0} = \begin{cases} 
-\frac{1}{\pi y} \left( \frac{a^y - x^y}{a^y} \right), & v > 0 \\
-\frac{2}{\pi} \ln \frac{x}{a}, & v = 0
\end{cases} \quad \ldots \ldots (2.3)
\]

(ii) From the equations quoted above, we may also derive
\[
C_y(xe^{i\pi}, \beta e^{i\pi}) = C_y(\alpha, \beta), \quad \ldots \ldots \ldots \ldots \ldots \ldots (2.4)
\]
showing that \(C_y(xz, az)\) is an even function of \(z\) for \(x \geq a > 0\).

(iii) The asymptotic expansions of the Bessel functions give
\[
C_y(xz, az) = \frac{2}{\pi z(ax)^{1/4}} \left\{ -\sin z(x-a) + \frac{4y^2 - 1}{8z} \left( \frac{1}{a} - \frac{1}{x} \right) \cos z(x-a) + O(|z|^{-2} e^{(x-a)\text{Im}z}) \right\} \quad \ldots \ldots (2.5)
\]
as \(|z| \to \infty\).

\(b\) \(E_\nu(\alpha, \beta) = J_{\nu+1}(\alpha)Y_\nu(\beta) - J_\nu(\beta) J_{\nu+1}(\alpha)\).

(i) By direct use of the formula \(\frac{d}{dt} [t^{\nu+1} \psi(t)] = t^{\nu+1} \psi(t)\), where \(\psi(t)\)
is any cylinder function, it follows that
\[
\frac{d}{dx} \{x^{\nu+1} E_\nu(xs, as)\} = sx^{\nu+1}C_\nu(xs, as). \quad \ldots \ldots (2.6)
\]

(ii) The indefinite integral
\[
\int zC_\nu(xz, az)dz = \frac{zxE_\nu(xz, az) + zaE_\nu(az, xz)}{a^2 - a^2} \quad \ldots \ldots (2.7)
\]
is obtained from \([W.B.F.], p. 134 (8)\).

(iii) Formula (12) of \([W.B.F.], p. 77\), gives
\[
E_\nu(x, \alpha) = j_{\nu+1}(x)Y_\nu(\alpha) - j_{\nu}(\alpha) Y_{\nu+1}(\alpha) = \frac{2}{\pi x} \quad \ldots \ldots (2.8)
\]

(iv) The series definitions of the Bessel functions lead to
\[
E_\nu(xz, az) = \frac{a_\nu}{\pi^{\nu+1}} z^{-1} + O(|z|) \quad \ldots \ldots (2.9)
\]
as \(|z| \to 0\).
(c) \([Q_\nu(x)]^2 = [J_\nu(x)]^2 + [Y_\nu(x)]^2\) (see equation (1.2)).

(i) \([Q_\nu(x)]^2 = \frac{2}{\pi x} [1 + O(x^{-2})]\) \(\quad\) ........................................ (2.10)

\(\quad\) as \(x \to \infty\).

(ii) \((\alpha^2 x Q_\nu(x)]^2 = \pi^{-2} 2^{2\nu} [\Gamma(\nu)]^2 + o(1), \quad \nu > 0\) \(\quad\) ............... (2.11)

\(\quad\) \((\ln x)^{-2} [Q_\nu(x)]^2 = 4 \pi^{-2} + o(1), \quad \nu = 0\) \(\quad\) as \(x \to 0^+\).

(d) The Bessel function of the third kind defined by \(H^{(3)}_\nu(z) = J_\nu(z) + i Y_\nu(z)\) has no singularities for \(\text{Im}z \geq 0\), and

(i) \(\frac{H^{(1)}_{\nu}(az)}{H^{(1)}_{\nu}(az)} = \frac{a^\nu}{x^\nu} + o(1)\) \(\quad\) ........................................ (2.12)

\(\quad\) as \(|z| \to 0\); 

(ii) \(\frac{H^{(1)}_{\nu}(az)}{H^{(1)}_{\nu}(az)} = a^{-1} x^{-1} e^{iz(x-o)} \{1 + o(|x|^{-1})\} \) \(\quad\) ........................................ (2.13)

\(\quad\) as \(|z| \to \infty\); 

(iii) \(\frac{H^{(1)}_{\nu+1}(az)}{H^{(1)}_{\nu}(az)} = \frac{1}{z} \left( \frac{2\nu a^\nu}{z^\nu + o(1)} \right) \) \(\quad\) ........................................ (2.14)

\(\quad\) as \(|z| \to 0\); 

(iv) \(\frac{H^{(1)}_{\nu+1}(az)}{H^{(1)}_{\nu}(az)} = -i a^{-1} x^{-1} e^{iz(x-o)} \{1 + o(|x|^{-1})\} \) \(\quad\) ........................................ (2.15)

\(\quad\) as \(|z| \to \infty\).

(e) As the unit function, we define

\[ U(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \] \(\quad\) ........................................ (2.16)

3

In this section, we define a \(T_f\)-transform, for functions which belong to an \(L^2\)-space. This transform will coincide with the \(T_1\)-transform when \(f(x)\) is a function which satisfies Theorem \(A_1\). The final results are stated in a theorem of the Plancherel type, Theorem 3.1.

Suppose that \(x^f(x)\) and \(x^g(x)\) belong to \(L^2(a, \infty)\). Then, there exist approximating sequences \(\{x^f_n(x)\}\) and \(\{x^g_n(x)\}\) such that \(x^f_n(x)\) and \(x^g_n(x)\) are functions which are continuous over a finite interval and zero outside that interval, and

\[ \lim_{n \to \infty} \int_a^\infty x [f(x) - f_n(x)]^2 dx = 0, \] \(\quad\) ........................................ (3.1)

and

\[ \lim_{n \to \infty} \int_a^\infty x [g(x) - g_n(x)]^2 dx = 0. \] \(\quad\) ........................................ (3.2)
Writing $F_n(s)=T_1[f_n(x)]$ and $G_n(s)=T_1[g_n(x)]$, we find that
\[
\int_0^\lambda \frac{F_m(s)G_n(s)}{[Q_\nu(as)]^2} ds = \int_0^\lambda \frac{sF_m(s)}{[Q_\nu(as)]^2} ds \int_a^\infty xg_n(x)C_\nu(xs, as)dx
\]
\[
= \int_a^\infty xg_n(x)dx \int_0^\lambda \frac{sF_m(s)C_\nu(xs, as)}{[Q_\nu(as)]^2} ds.
\]

The inversion of the order of integration is justified since the range of integration is actually finite and the integrand is continuous.

If we take the limit $\lambda \to \infty$, which is easily justified, see [T.T.F.], p. 390 (iv), and then use the inversion theorem in the form A.1, we obtain
\[
\int_0^\infty \frac{sF_m(s)G_n(s)}{[Q_\nu(as)]^2} ds = \int_0^\infty xf_m(x)g_n(x)dx. \quad \ldots \quad (3.3)
\]

Thus we have
\[
\int_0^\infty \frac{s[F_n(s)]^2}{[Q_\nu(as)]^2} ds = \int_0^\infty x[f_n(x)]^2 dx.
\]

and
\[
\int_0^\infty \frac{s[F_m(s) - F_n(s)]^2}{[Q_\nu(as)]^2} ds = \int_0^\infty x[f_m(x) - f_n(x)]^2 dx. \quad \ldots \quad (3.4)
\]

Since the right side of equation (3.4) tends to zero when both $m$ and $n$ tend to infinity together, then so does the left side. Thus $\{s^2F_n(s)\}$ converges in the mean square to a function $\frac{s^2F(s)}{Q_\nu(as)}$ (say) which belongs to $L^2(0, \infty)$.

As a consequence, we make the definition
\[
F(s)=T_1[f(x)]. \quad \ldots \quad (3.5)
\]

We have also obviously proved that
\[
\int_0^\infty \frac{s[F(s)]^2}{[Q_\nu(as)]^2} ds = \lim_{n \to \infty} \int_0^\infty \frac{s[F_n(s)]^2}{[Q_\nu(as)]^2} ds
\]
\[
= \lim_{n \to \infty} \int_a^\infty x[f_n(x)]^2 dx
\]
\[
= \int_a^\infty x[f(x)]^2 dx. \quad \ldots \quad (3.6)
\]

Now, in equation (3.6), we replace $f(x)$ by $f(x) + g(x)$ where both $x^2f(x)$ and $x^3g(x)$ belong to $L^2(a, \infty)$. Since the transform is obviously linear, we obtain
\[
\int_0^\infty \frac{s[F(s) + G(s)]^2}{[Q_\nu(as)]^2} ds = \int_a^\infty x[f(x) + g(x)]^2 dx
\]

and ultimately, after again using equation (3.6), arrive at the Parseval equation
\[
\int_0^\infty \frac{s^2F(s)G(s)}{[Q_\nu(as)]^2} ds = \int_a^\infty xf(x)g(x) dx. \quad \ldots \quad (3.7)
\]

This result will be recorded later in Theorem 3b.
Now the definition of $F_n(s)$ written in full is

$$F_n(s) = \int_a^\infty x f_n(x) C_\nu(xs, as) \, dx$$

and then

$$\int_1^s \xi F_n(\xi) \, d\xi = \int_1^s d\xi \int_a^\infty x f_n(x) \xi C_\nu(x \xi, a \xi) \, dx \quad \ldots \ldots \ (3.8)$$

where $s > 0$.

Since the range of integration is finite and all functions concerned are continuous, we may invert the order of integration and use equation (2.7) to obtain

$$\int_1^s \xi F_n(\xi) \, d\xi = \int_a^\infty x f_n(x) D_\nu(x, s) \, dx \quad \ldots \ldots \ (3.9)$$

where

$$D_\nu(x, s) = \left[ \frac{\xi x E_\nu(x \xi, a \xi) + \xi a E_\nu(a \xi, x \xi)}{x^2 - a^2} \right]_{\xi=1}^{\xi=s} \quad \ldots \ldots \ (3.10)$$

All numerator terms of $D_\nu(x, s)$ are analytic functions of $x$ for $s > 0$, and equation (2.8) shows that $D_\nu(x, s)$ is bounded as $x \rightarrow a$. Then, the asymptotic expansions of the Bessel Functions shows immediately that $x^4 D_\nu(x, s)$ belongs to $L^2(a, \infty)$. We may thus take the limit $n \rightarrow \infty$ in equation (3.9).

So

$$\int_1^s \xi F(\xi) \, d\xi = \int_a^\infty x f(x) D_\nu(x, s) \, dx \quad \ldots \ldots \ (3.11)$$

whence

$$F(s) = \frac{1}{s} \frac{d}{ds} \int_a^\infty x f(x) D_\nu(x, s) \, dx, \quad s > 0 \quad \ldots \ldots \ (3.12)$$

almost everywhere.

The importance of equation (3.12) is that it shows that $F(s)$ is independent of the choice of the $\{f_n(x)\}$, and depends only on $f(x)$.

Now, it is clear that $x^4 s C_\nu(xs, as)$, considered as a function of $x$, is uniformly bounded for $s \geq 0$. Thus, if $x^4 f(x)$ belongs to $L^4(a, \infty)$, then the differentiation indicated in equation (3.12) can be carried out to show that, if $x^4 f(x)$ belongs to both $L^4(a, \infty)$ and $L^2(a, \infty)$, then $T_1[f(x)] = T_2[f(x)]$.

If we now assume that $x^4 f(x)$ belongs to $L^2(a, \infty)$, then $x^4 f(x) U(p-x)$ belongs to both $L^2(a, \infty)$ and $L^1(a, \infty)$.

Then

$$F(s) - F(s, p) = T_2[f(x) U(x-p)],$$

where

$$F(s, p) = \frac{1}{s} \frac{d}{ds} \int_a^p x f(x) D_\nu(x, s) \, dx$$

$$= \int_a^p x f(x) C_\nu(xs, as) \, dx.$$
So equation (3.6) or (3.7) gives
\[ \int_0^\infty \frac{s[F(s) - F(s, p)]^2}{[Q_\nu(as)]^2} \, ds = \int_p^\infty x[f(x)]^2 \, dx \rightarrow 0, \]
as \( p \to \infty \).

Now \( \frac{s}{[Q_\nu(as)]^2} \geq 0 \) for all \( s > 0 \). Thus
\[ \lim_{p \to \infty} \int_0^\infty \frac{[F(s) - F(s, p)]^2}{[Q_\nu(as)]^2} \, ds = 0, \]
that is
\[ F(s) = \text{l.i.m.}_{p \to \infty} \int_a^p x f(x) C_\nu(xs, as) \, dx. \tag{3.13} \]

In order to obtain an inversion theorem, we write
\[ \nu(x) = \nu(x, \xi) = \begin{cases} \nu, & a < x < \xi, \\ 0, & x > \xi, \end{cases} \tag{3.14} \]
for which \( x^\nu g(x) \) obviously belongs to both \( L^1(a, \infty) \) and \( L^2(a, \infty) \). The transform of \( \nu(x, \xi) \) is
\[ G_\nu(s, \xi) = T_\nu[g_\nu(x, \xi)] = \int_a^\xi x^{\nu+1} C_\nu(xs, as) \, dx \]
\[ = s^{-1} \left[ \nu^{\nu+1} E_\nu(\xi s, as) - \frac{2a^\nu}{\pi s} \right]. \tag{3.15} \]
(from equations (2.6) and (2.8)).

Therefore equation (3.7) gives
\[ \int_a^\xi x^{\nu+1} f(x) \, dx = \int_0^\infty sF(s) G_\nu(s, \xi) \, ds, \]
that is
\[ f(x) = \frac{1}{x^{\nu+1}} \frac{d}{dx} \int_0^\infty sF(s) G_\nu(s, x) \, ds. \tag{3.16} \]
almost everywhere.

We may now summarize the results obtained up to this point by
\[ \text{A}_2 : \text{If} \, x^\nu f(x) \text{ belongs to} \, L^2(a, \infty) \text{ and} \, \{x^\nu f_n(x)\} \text{ is a sequence of functions which are continuous over a finite range and which are zero outside that range} \]
and which approximate to $x^f(x)$ in the mean, then there exists a function $F(s)$, such that

\[(a) \quad \frac{s^f F(s)}{Q^\nu(as)} \text{ belongs to } L^2(0, \infty)\]

\[(b) \quad F(s) = \lim_{n \to \infty} \int_a^\infty x f_n(x) C^{\nu}(x, as) dx\]

\[(c) \quad = \lim_{p \to \infty} \int_a^p x f(x) C^{\nu}(x, as) dx\]

\[(d) \quad = \frac{1}{s} \int_a^\infty x f(x) D^{\nu}(x, s) dx\]

\[(e) \quad f(x) = \frac{1}{x^{\nu+1}} \frac{d}{dx} \int_0^\infty \frac{s F(s) G^{\nu}(s, x)}{[Q^\nu(as)]^2} ds, \text{ almost everywhere.}\]

We have thus defined a mapping of the whole set of functions $f(x)$ for which $x^f(x)$ belongs $L^2(a, \infty)$ on to a subset of the functions $F(s)$ for which $\frac{s^f F(s)}{Q^\nu(as)}$ belongs to $L^2(0, \infty)$.

In order to show that this subset is actually the whole set of $F(s)$ for which $\frac{s^f F(s)}{Q^\nu(as)}$ belongs to $L^2(0, \infty)$, we use the Theorem B1. The procedure corresponds very closely to that just given. The conclusion will be

**B2**: If $\frac{s^f F(s)}{Q^\nu(as)}$ belongs to $L^2(0, \infty)$ and $\{\frac{s^f F_n(s)}{Q^\nu(as)}\}$ is a sequence of functions which are continuous over a finite range and which are zero outside that range and which converges to $\frac{s^f F(s)}{Q^\nu(as)}$ in the mean, then there exists a function $f(x)$ such that

\[(a) \quad x^f(x) \text{ belongs to } L^2(a, \infty),\]

\[(b) \quad f(x) = \lim_{n \to \infty} \int_0^\infty \frac{s F_n(s) C^{\nu}(x, as)}{[Q^\nu(as)]^2} ds,\]

\[(c) \quad = \lim_{p \to \infty} \int_0^p \frac{s F(s) C^{\nu}(x, as)}{[Q^\nu(as)]^2} ds,\]

\[(d) \quad = \frac{1}{x^{\nu+1}} \frac{d}{dx} \int_0^\infty \frac{s F(s) G^{\nu}(s, x)}{[Q^\nu(as)]^2} ds,\]

\[(e) \quad F(s) = \frac{1}{s} \int_a^\infty x f(x) D^{\nu}(x, s) dx, \text{ almost everywhere.}\]

The only point worthy of note in the derivation of B2 is that the function which replaces $g^{\nu}(x, \xi)$ in equation (3.14) would be

$$H(s, \xi) = \begin{cases} [Q^\nu(as)], & 1 < s < \xi \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} -[Q^\nu(as)]^2, & \xi < s < 1 \\ 0 & \text{otherwise} \end{cases}$$

Comparing A2 and B2, we are able to enunciate.
Theorem 3.1.

To every \( f(x) \) for which \( x^{t}f(x) \) belongs to \( L^{2}(a, \infty) \), there corresponds a unique function \( F(s) \), for which \( s^{2}F(s)/Q_{v}(as) \) belongs to \( L^{2}(0, \infty) \) (and conversely). The functions \( f(x) \) and \( F(s) \) are connected by the formulae

\[ (a) \quad F(s) = T_{2}[f(x)] = W_{2}^{-1}[f(x)] \]

\[ (b) \quad = \lim_{p \to \infty} \frac{1}{s} \int_{a}^{p} xf(x)C_{v}(xs, as) \, dx \]

\[ (c) \quad = \lim_{s \to 0} \frac{d}{ds} \int_{a}^{\infty} xf(x)D_{v}(x, s) \, dx. \]

\[ (d) \quad f(x) = T_{2}^{-1}[F(s)] = W_{2}[F(s)], \]

\[ (e) \quad = \lim_{p \to \infty} \int_{0}^{p} \frac{s^{2}F(s)C_{v}(xs, as)}{[Q_{v}(as)]^{2}} \, ds \]

\[ (f) \quad = \lim_{x \to 0} \frac{d}{dx} \int_{0}^{\infty} \frac{s^{2}F(s)G_{v}(s, x)}{[Q_{v}(as)]^{2}} \, ds. \]

Also corresponding to equation (3.7), we have

Theorem 3.2.

If \( F(s) = T_{2}[f(x)] \), and \( G(s) = T_{2}[g(x)] \), then

\[ \int_{a}^{\infty} xf(x)g(x) \, dx = \int_{0}^{\infty} \frac{s^{2}F(s)G(s)}{[Q_{v}(as)]^{2}} \, ds. \]

4

In this section, we examine the transform of a function which is zero outside a finite interval. If we suppose that \( g(x) = 0 \) for \( x > b \), then

\[ G(s) = T_{2}[g(x)] \]

\[ = T_{1}[g(x)] \]

\[ = \int_{a}^{b} xC_{v}(xs, as)g(x) \, dx \quad \cdots \cdots \cdots (4.1) \]

and we may define

\[ G(z) = \int_{a}^{b} xC_{v}(xz, az)g(x) \, dx \quad \cdots \cdots \cdots (4.2) \]

where \( z = s + it \).

Then, assuming that \( C_{v}(xz, az)_{z=0} \) is defined by equation (2.3), we see that \( G(z) \) will be defined for \( z = 0 \).

There are two forms of the theorem. If \( G(z) \) is considered to be defined in whole \( z \)-plane we have

Theorem 4.1.

In order that \( x^{4}g(x) \) belongs to \( L^{2}(a, \infty) \) and

\[ g(x) = 0, \text{ for } x > b > a, \quad \cdots \cdots \cdots (4.3) \]
it is necessary and sufficient that

(i) \( s^1 G(s)/Q_\nu(as) \) belongs to \( L^2(0, \infty) \);
(ii) \( G(z) \) is analytic in \( z = s + i\beta \) for all \( z \);
(iii) \( G(z e^{i\pi}) = G(z) \);
(iv) \[ |z G(z)| = O(e^{(b-a)\beta}) \text{, as } |z| \to \infty. \]

If it is preferred to consider \( G(z) \) defined in the half-plane \( \beta \geq 0 \), then the enunciation is

**Theorem 4.2.**

In order that \( x^1 g(x) \) belongs to \( L^2(a, \infty) \) and

\[ g(x) = 0 \text{ for } x > b > a, \]

it is necessary and sufficient that

(ia) \( s^1 G(s)/Q_\nu(as) \) belongs to \( L^2(0, \infty) \);
(iiia) \( G(z) \) is analytic in \( z = s + i\beta \) for \( \beta \geq 0 \);
(iiiia) \( G(se^{i\pi}) = G(s) \), for real \( s \);
(iva) \[ |z G(z)| = O(e^{(b-a)\beta}) \text{, as } |z| \to \infty, \, \beta \geq 0. \]

It is easy to see that the set (i)-(iv) implies (ia)-(iva) and conversely. We prove the theorem in the form 4.1.

**Proof.**

**Necessary Conditions.**

Theorem 3.1 gives (ia) immediately.

Since \( x^1 g(x) \) belongs to \( L^2(a, b) \), it belongs to \( L^1(a, b) \), and since as was shown in §2 (a) (i), \( x^1 C_\nu(xz, az) \) is an analytic function of \( z \); (iiia) is obvious.

Now

\[ G(se^{i\pi}) = \int_a^b x C_\nu(xse^{i\pi}, ase^{i\pi}) g(x) \, dx \]
\[ = \int_a^b x C_\nu(xs, as) g(x) \, dx \]
(by equation (2.4))
\[ = G(z), \]
which proves (iiiia).

Now to prove (iva), we use §2 (a) (iii), which shows that, provided \( |z| \) is taken sufficiently large,

\[ |G(z)| < A \int_a^b x^1 |g(x)| \frac{e^{(x-a)\beta}}{|z|} \, dx, \text{ for some constant } A; \]
\[ < A |z|^{-1} e^{(b-a)\beta} \int_a^b x^1 |g(x)| \, dx, \]
that is
\[ |z G(z)| = O(e^{(b-a)\beta}) \]
as \( |z| \to \infty. \)
Sufficient Conditions.

Now in order to prove the converse, we assume that (ia)-(iva) hold. This method follows the lines of \([G]\). We will prove that

\[ T_2[g(x)] = T_2[g(x) U(p-x)] \quad \text{for all } p > b, \quad \ldots \ldots \quad (4.4) \]

then, using the inversion theorem \(3.1\), we will obtain

\[ g(x) = g(x) U(p-x), \quad \text{for all } p > b, \quad \ldots \ldots \quad (4.5) \]

that is

\[ g(x) = 0, \quad \text{for } x > b \quad \ldots \ldots \quad (4.6) \]

(we, of course, identify functions which are equal almost everywhere).

We write

\[ P(s, t, p) = T_2[C_{\nu}(xt, at) U(p-x)] \]

\[ = \int_a^p xC_{\nu}(xs, as)C_{\nu}(xt, at)dx \]

\[ = \frac{p}{s^2-t^2} \{ s E_{\nu}(ps, as) C_{\nu}(pt, at) - t E_{\nu}(pt, at) C_{\nu}(ps, as) \}, \]

where \( E_{\nu}(x, \beta) \) has been defined in \( \S 2 \) \((b)\).

Then the Parseval Theorem \(3.2\) gives

\[ \int_0^\infty \frac{\int P(t, s, p) G(t) dt}{[Q_{\nu}(at)]^2} \int_a^\infty xC_{\nu}(xs, as) U(p-x) g(x)dx, \]

which can be interpreted as

\[ \int_0^\infty \frac{\int P(t, s, p) G(t) dt}{[Q_{\nu}(at)]^2} \int_a^\infty \int \]

\[ = T_2[g(x) U(p-x)], \quad \ldots \ldots \quad (4.7) \]

We will show that the left side of equation (4.7) equals \( G(s) \), if (ia)-(iva) hold.

It will be convenient to write \( z = t + i \beta \), and consider the integral

\[ \int \frac{pzG(z)}{z^2-s^2} \left\{ \frac{z H^{(1)}_{\nu}(pz)}{H^{(1)}_{\nu}(az)} C_{\nu}(ps, as) - \frac{s H^{(1)}_{\nu}(pz)}{H^{(1)}_{\nu}(az)} E_{\nu}(ps, as) \right\} \int \]

taken over the contour shown in the figure.

The semi-circles above \( z = +s \), \( z = -s \) and \( z = 0 \) are assumed to have the same radius \( \varepsilon \), and the large semi-circle is assumed to have radius \( R \). In the final step the limits \( \varepsilon \to 0 \) and \( R \to \infty \) are taken.
Since $H^{(1)}_\nu(az)$ has no zeros above the real axis, the integral vanishes by Cauchy’s Theorem.

Since $G(z)$ is analytic near $z=0$, it is bounded there. Then §2 (d) (i) and (iii) show that the contribution due to the semi-circle at the origin vanishes in the limit.

If we adopt the usual notation $H^{(2)}_\nu(z) = J_\nu(z) - iY_\nu(z)$, we find from [W.B.F.] p. 75, that when $t$ is real,

$$H^{(1)}_\nu(te^{i\pi}) = -e^{-\nu\pi i}H^{(2)}_\nu(t). \quad \text{(4.9)}$$

This result and assumption (iiia) allows us to show that the contribution from the real axis is

$$\int_{-\infty}^{\infty} \frac{pt^{2}G(t)}{t^2 - s^2} \left[ \frac{H^{(1)}_{\nu+1}(pt)}{H^{(1)}_{\nu}(pt)} - \frac{H^{(2)}_{\nu+1}(pt)}{H^{(2)}_{\nu}(pt)} \right] C_\nu(ps, as) dt$$

$$\quad - \int_{-\infty}^{\infty} \frac{pst}{t^2 - s^2} \left[ \frac{H^{(1)}_\nu(pt)}{H^{(1)}_\nu(at)} - \frac{H^{(2)}_\nu(pt)}{H^{(2)}_\nu(at)} \right] E_\nu(ps, as) dt$$

$$\quad = -2i \int_{0}^{\infty} \frac{t P(t, s, p) G(t)}{[Q_\nu(at)]^2} dt.$$

The points $z=\pm s$ are poles of the integrand and the contribution due to the semi-circles above $z=\pm s$ is

$$-\pi ip \left\{ \frac{1}{2} G(s) \left( \frac{sH^{(1)}_{\nu+1}(ps)}{H^{(1)}_\nu(as)} C_\nu(ps, as) - \frac{sH^{(1)}_\nu(ps)}{H^{(2)}_\nu(as)} E_\nu(ps, as) \right) \right. \right.$$}

$$\left. + \frac{1}{2} G(s) \left( \frac{sH^{(2)}_{\nu+1}(ps)}{H^{(2)}_\nu(as)} C_\nu(ps, as) - \frac{sH^{(2)}_\nu(ps)}{H^{(2)}_\nu(as)} E_\nu(ps, as) \right) \right\}$$

$$- \pi ipsG(s) \left[ \frac{C_\nu(ps, as)\{J_{\nu+1}(ps)J_\nu(as) + Y_{\nu+1}(ps)Y_\nu(as)\}}{[Q_\nu(as)]^2} - E_\nu(ps, as)\{J_\nu(ps)J_\nu(as) + Y_\nu(ps)Y_{\nu+1}(as)\} \right]. \quad \text{(5.10)}$$

If we expand the expression in square brackets and regroup the terms, we find that this contribution is equal to

$$\left\{ \frac{\pi ipsG(s)}{[Q_\nu(as)]^2} \right\} \left\{ [Q_\nu(as)]^2 E_\nu(ps, ps) \right\}$$

$$\quad = \pi ipsG(s) \left( \frac{2}{\pi ps} \right)$$

$$\quad = 2iG(s).$$

In the simplification we have used equation (2.8).

If we now use §2 (d) (ii) and (iv) together with assumption (iva) we find that the integrand is $O(1/z|z|^{-\nu} e^{-\beta(x-b)})$ on the large semi-circle. Then applying the Lemma of §3 of [G] we see that the integral taken along this large semi-circle vanishes when $R \to \infty$ for $p > b$.
Thus, combining our results, we have
\[-2i \int_0^\infty \frac{tP(t, s, p)G(t)}{(Q(\nu(at))^2} dt + 2iG(s) = 0,\]
that is
\[\int_0^\infty \frac{tP(t, s, p)G(t)}{(Q(\nu(at))^2} dt = G(s), \quad p > b,\]
which is the required result and the theorem is proved.

APPENDIX.

In this appendix, we prove a theorem from which the transform \( W_1 \) and its inverse may be derived. A theorem of this type is found in [W.B.F.], p. 468 et seq. This theorem of Watson is most unsuitable for our purpose since it demands that \( F(s) \) must be zero in a neighbourhood of the origin and the restrictions on \( F(s) \), for \( s \to \infty \), are somewhat too heavy.

The theorem we will prove will be:

**Theorem W.**

Assuming that \( \nu \geq 0 \), \( s^1F(s)/Q(\nu(\nu s)) \) belongs to \( L^1(0, \infty) \) and that \( F(s) \) is of bounded variation near the point \( s = u \), then
\[\frac{1}{2} \left\{ F(u+0) + F(u-0) \right\} \]

By equations (2.1) and (2.2), \( s^1C(\nu s, \nu s)/Q(\nu(\nu s)) \) is uniformly bounded for \( a \leq x \leq \lambda \). Thus \( \int_0^\infty sC(\nu s, \nu s)F(s)/[Q(\nu(\nu s))^2] ds \) converges uniformly for \( a \leq x \lambda \).

So we multiply by \( xC(\nu x, \nu x) \) and integrate under the integral sign to obtain
\[\int_a^\lambda xC(\nu x, \nu x)dx \int_0^\infty \frac{sF(s)C(\nu x, \nu x)}{[Q(\nu(\nu s))^2} ds \]
\[= \int_0^\infty \frac{sF(s)ds}{[Q(\nu(\nu s))^2} \int_a^\lambda xC(\nu x, \nu x)C(\nu x, \nu x)dx \quad \ldots \ldots \ldots \ldots \quad (A.2) \]
\[= \int_0^\infty \frac{sF(s)}{[Q(\nu(\nu s))^2} \left\{ \frac{\lambda}{s^2-u^2} \left[ sE(\nu s, \nu s)C(\nu \nu s, \nu x) - uE(\nu s, \nu x)C(\nu s, \nu x) \right] \right\} ds. \]
\[\ldots \ldots \ldots \ldots \quad (A.3) \]

Thus our theorem is proved, if we can prove that
\[\frac{1}{2} \left\{ F(u+0) + F(u-0) \right\} \]

\[= \lim_{\lambda \to \infty} \int_0^\infty \frac{sF(s)}{[Q(\nu(\nu s))^2} \left\{ \frac{\lambda}{s^2-u^2} \left[ sE(\nu s, \nu s)C(\nu \nu s, \nu x) - uE(\nu s, \nu x)C(\nu s, \nu x) \right] \right\} ds \]
\[\ldots \ldots \ldots \ldots \quad (A.4) \]
For this purpose we split the range of integration into

$$\int_0^\infty = \int_0^\eta + \int_\eta^{u-\delta} + \int_{u-\delta}^{u+\delta} + \int_{u+\delta}^\infty \quad \ldots \ldots \quad (A.5)$$

Now the integrand in equation (A.4) can be expanded in the form

$$\frac{s^4 F(s)}{Q_\nu(as)} \cdot \frac{\lambda s^3}{s^2 - u^2} \left\{ s \left[ J_{\nu+1}(\lambda s)Y_\nu(as) - J_\nu(as)Y_{\nu+1}(\lambda s) \right] \left[ J_\nu(\lambda u)Y_\nu(au) - J_\nu(au)Y_\nu(\lambda u) \right] \right.$$ 

$$- u \left[ J_{\nu+1}(\lambda u)Y_\nu(au) - Y_{\nu+1}(\lambda u)J_\nu(au) \right] \left[ J_\nu(\lambda s)Y_\nu(as) - J_\nu(as)Y_\nu(\lambda s) \right] \right\},$$

from which we observe that the factor following the point is bounded for all $s$ and $\lambda$. Thus we can make the contribution from $\int_0^\eta$ arbitrarily small by choosing $\eta$ sufficiently small. We now consider this done.

The contribution from the integral $\int_{\eta}^{u-\delta}$ may be written as the sum of eight integrals of the form

$$\int_{\eta}^{u-\delta} \frac{s^4 F(s)}{Q_\nu(as)} H_i(s) P_i(s, \lambda) ds, \quad \ldots \ldots \quad (A.6)$$

where the $H_i(s)$ are bounded functions of $s$ (independent of $\lambda$), and $P_i(s, \lambda)$ is one of the functions $\lambda J_{\nu+1}(\lambda s)J_\nu(\lambda u)$, $\lambda J_{\nu+1}(\lambda s)Y_\nu(\lambda u)$, $\lambda Y_{\nu+1}(\lambda s)J_\nu(\lambda u)$, $\lambda Y_{\nu+1}(\lambda s)Y_\nu(\lambda u)$, $\lambda J_\nu(\lambda s)J_{\nu+1}(\lambda u)$, $\lambda J_\nu(\lambda s)Y_{\nu+1}(\lambda u)$, $\lambda Y_\nu(\lambda s)J_{\nu+1}(\lambda u)$, $\lambda Y_\nu(\lambda s)Y_{\nu+1}(\lambda u)$.

Each of these integrals will vanish as $\lambda \to \infty$. Since the method of proof is the same for each, we will prove this for the first.

If we use the usual notation $\omega_\nu = \frac{1}{2} \nu \pi + \frac{1}{4} \pi$, the asymptotic expansions of the Bessel Functions show that this integral may be written as

$$\int_{\eta}^{u-\delta} A(s) \left\{ \sin (\lambda s - \omega_\nu) \cos (\lambda u - \omega_\nu) + O\left(\frac{1}{\lambda}\right) \right\} ds, \quad \ldots \ldots \quad (A.7)$$

where obviously $A(s)$ belongs to $L^1(\eta, u-\delta)$. The Riemann-Lebesgue theorem for Fourier integrals ([T.F.I.], p. 11, Th. 1) shows immediately that this integral vanishes as $\lambda \to \infty$.

By a similar argument, the contributions, due to integrals of the form

$$\int_{u+\delta}^\infty \frac{s^4 F(s)}{Q_\nu(as)} H_i(s) P_i(s, \lambda) ds,$$

also vanish for $\lambda \to \infty$. 

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Now considering the integral $\int_{u-\delta}^{u+\delta} sF(s)$ and supposing that $\lambda$ is sufficiently large, we may write the contribution as

$$\frac{2}{\pi} \int_{u-\delta}^{u+\delta} \frac{sF(s)}{[Q\psi(as)]^2} \frac{1}{s^2 - u^2} \left\{ J_\psi(as)J_\psi(au) \left[ -\frac{s^2}{u^2} \cos(\lambda s - \omega v) \sin(\lambda u - \omega v) + \frac{u^2}{s^2} \sin(\lambda s - \omega v) \cos(\lambda u - \omega v) \right] ight. $$

$$+ Y_\psi(as)Y_\psi(au) \left[ \frac{u^2}{s^2} \sin(\lambda s - \omega v) \cos(\lambda u - \omega v) - \frac{s^2}{u^2} \cos(\lambda s - \omega v) \sin(\lambda u - \omega v) \right] $$

$$+ Y_\psi(as)J_\psi(au) \left[ -\frac{s^2}{u^2} \sin(\lambda s - \omega v) \sin(\lambda u - \omega v) - \frac{u^2}{s^2} \cos(\lambda s - \omega v) \cos(\lambda u - \omega v) \right]$$

$$+ J_\psi(as)Y_\psi(au) \left[ \frac{s^2}{u^2} \cos(\lambda s - \omega v) \cos(\lambda u - \omega v) + \frac{u^2}{s^2} \sin(\lambda s - \omega v) \sin(\lambda u - \omega v) \right]$$

$$+ O\left(\frac{1}{\lambda}\right)ds.$$

Now if we apply the Riemann-Lebesgue Theorem we see that we may neglect all terms which do not contain $\sin(\lambda(x-u))/x-u$ as a factor.

Then considering the remaining terms, we use Fourier's single integral formula ([T.F.I.], p. 25, Th. 12) and obtain as the contribution from $\int_{u-\delta}^{u+\delta}$

$$\frac{1}{2}\{F(u+0) + F(u-0)\}$$

and the theorem is proved.

REFERENCES.


