BOUNDARY STRESSES IN AN INFINITE HUB OF SPECIAL SHAPE.

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With four Text-figures.

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INTRODUCTION.

This paper makes use of the notation and results of Muskhelishvili (1953) to find the stresses on the boundary of a hub, considered as an infinite plane with a hole in the shape of an equiangular curvilinear triangle. The stresses are caused by a moment, applied to a rigid kernel fitted into the hole. Muskhelishvili solves the fundamental problem of plane elasticity when the stresses are given on the boundary for the case of a finite region mapped on to unit circle by a polynomial. The general method used here is a modification of the above method for the case of an infinite region when the displacements are given on the boundary, and when the infinite region is mapped onto unit circle by a function of the form $z = \frac{c_0}{\zeta} + c_1\zeta + c_2\zeta^2 + \ldots$.

THE CONFORMAL TRANSFORMATION.

Each of the circular arcs forming the sides of the curvilinear triangle has the opposite corner as centre. A function of the desired form which maps the infinite region in the $z$-plane onto unit circle in the $\zeta$-plane in an approximate manner can be obtained as follows: Choose $z = 0$ as geometric centre, and with the orientation shown (Text-fig. 1) join points $z = R e^{i\phi}$ on the boundary, separated by an argument of $\frac{\pi}{12}$. The outside of the 24-sided polygon which results can be
mapped on to the inside of the unit circle by the Schwarz-Christoffel formula. The sizes of the exterior angles shown are $p\pi, q\pi, r\pi, s\pi, t\pi$ where $p = 0.38388, q = 0.04660, r = 0.04030, s = 0.03673, t = 0.03546$ and

$$3p + 6q + 6r + 6s + 3t = 2.$$ 

The Schwarz-Christoffel formula then gives

$$z = -A \int_{\zeta_0}^{\zeta} \frac{(1-x^3)(1+x^6)^{\gamma}(1-\sqrt{2x^3+x^6})^\delta(1+\sqrt{2x^3+x^6})^\mu}{x^2} \, dx$$

where $\zeta = \rho e^{i\theta}$ and $\zeta_0$ is so chosen that the integral vanishes at the lower limit. $A$ is a constant which can be used to alter the size and orientation of the hole.

By expanding and integrating term by term, a rapidly convergent series results. By truncating the series after five terms and choosing $A$ so that the radius of the circumscribed circle of the hole will be $a$, we obtain

$$z = \omega(\zeta) = a \left\{ \frac{0.825}{\zeta} + 0.149\zeta^2 + 0.017\zeta^5 + 0.006\zeta^8 + 0.003\zeta^{11} \right\}.$$ 

By plotting the curve whose parametric equations are

$$x = a\{ 0.825 \cos \theta + 0.149 \cos 2\theta + 0.017 \cos 5\theta + 0.006 \cos 8\theta + 0.003 \cos 11\theta \}$$
$$y = a\{-0.825 \sin \theta + 0.149 \sin 2\theta + 0.017 \sin 5\theta + 0.006 \sin 8\theta + 0.003 \sin 11\theta \}$$

we see (Text-fig. 2) that the area mapped onto unit circle approximates the equiangular curvilinear triangle, the greatest error in $R$ being about 3%. A rounding off of sharp corners has occurred, the radius of curvature at $\Phi = 0$ (corresponding to $\theta = 0$) being 0.05.
As $\zeta$ moves around unit circle anti-clockwise, $z$ moves around the boundary of the region clockwise. The relation between angles $\Phi$ in the $z$-plane and angles $\theta$ in the $\zeta$-plane are shown in Text-figure 3. Remembering the sign difference, we show positive $\Phi$ corresponding to positive $\theta$.

![Text-figure 3.](image)

**The Boundary Condition.**

The boundary condition for the fundamental problem of plane elasticity when the displacements on the boundary of the region are given may be represented in complex form by

$$\lambda \varphi_1(z) - z \varphi_1'(z) - \psi_1(z) = 2\mu(g_1 + ig_2) \text{ on } L \quad \ldots \ldots \quad (1)$$

where $L$ is the contour enclosing the region considered in the complex $z$-plane, $\varphi_1(z)$ and $\psi_1(z)$ are arbitrary functions arising from the solution of the bi-harmonic equation, $\mu$ is the shear modulus, and $\lambda = 3 - 4\sigma$ ($\sigma$ being Poisson’s Ratio); $g_1 = g_1(s)$ and $g_2 = g_2(s)$ are the given displacements of points on the contour $L$.

If the region in the $z$-plane is mapped onto unit circle in the $\zeta$-plane by a transformation $z = \omega(\zeta)$, the boundary condition may be rewritten as

$$\lambda \varphi(\sigma) - \frac{\omega'(\sigma)}{\omega(\sigma)} \varphi'(\sigma) - \psi(\sigma) = 2\mu(g_1 + ig_2) = 2\mu g \text{ on } \gamma \quad \ldots \ldots \quad (2)$$

where $\gamma$ is unit circle in the $\zeta$-plane and $\sigma = e^{i\theta}$; $g_1$ and $g_2$ have the same value as in (1) but are referred to the angle $\theta$.

**General Method of Solution.**

The mapping function $z = \omega(\zeta)$ is such that $\omega'(\zeta) \neq 0$ inside or on $\gamma$. We have to find functions $\varphi(\zeta)$, $\psi(\zeta)$ satisfying (2). Since the region to be considered is an infinite region we assume that the stresses at infinity vanish. We assume also that the resultant vector of the external forces applied to the boundary of
the region vanish, so that \( \varphi(\zeta) \) and \( \psi(\zeta) \) are holomorphic inside and on \( \gamma \). The state of stress is unaltered if we assume \( \varphi(0) = 0 \).

Taking the conjugate of (2), we have

\[
\overline{\varphi}(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)} \overline{\varphi}'(\sigma) - \overline{\psi}(\sigma) = 2\mu(g_1 - ig_2) = 2\mu \tilde{g} \quad \ldots \ldots \quad (3)
\]

Rewriting (2) in the form

\[
\overline{\varphi}(\sigma) = x\varphi(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)} \overline{\varphi}'(\sigma) - 2\mu g \quad \ldots \ldots \ldots \ldots \quad (4)
\]

and denoting the right-hand side of (4) by \( G(\sigma) \) we note that \( G(\sigma) \) represents the boundary value of a function \( \psi(\zeta) \) holomorphic inside \( \gamma \). The necessary and sufficient condition for this to be true is

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{G(\sigma)}{\sigma - \zeta} d\sigma = \tilde{a} \quad \text{for all } \zeta \text{ inside } \gamma \quad \ldots \ldots \quad (5)
\]

where \( \tilde{a} = G(0) = \overline{\psi}(0) \) (i.e. \( a = \psi(0) \)).

Thus, substituting in (5) from the right-hand side of (4) we have

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(\sigma)}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\sigma)}{\omega'(\sigma)} \frac{\varphi'(\sigma)}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_{\gamma} \frac{2\mu g}{\sigma - \zeta} d\sigma = \tilde{a}
\]

Thus we obtain the functional equation

\[
x\varphi(\zeta) - \frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\sigma)}{\omega'(\sigma)} \frac{\varphi'(\sigma)}{\sigma - \zeta} d\sigma - \frac{2\mu}{2\pi i} \int_{\gamma} \frac{g d\sigma}{\sigma - \zeta} = \tilde{a} \quad \ldots \ldots \quad (6)
\]

It will be seen that (6) completely determines \( \varphi(\zeta) \) as well as \( \tilde{a} \). Having found \( \varphi(\zeta) \) we can obtain \( \psi(\zeta) \) directly from (3). \( \psi(\sigma) \) is the boundary value of the function \( \psi(\zeta) \) holomorphic inside \( \gamma \), so that

\[
\psi(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(\sigma) d\sigma}{\sigma - \zeta}
\]

Substituting from (3) we have

\[
\psi(\zeta) = -\frac{2\mu}{2\pi i} \int_{\gamma} \frac{g d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\sigma)}{\omega'(\sigma)} \frac{\varphi'(\sigma)}{\sigma - \zeta} d\sigma \quad \ldots \ldots \quad (7)
\]

remembering that

\[
\frac{x}{2\pi i} \int_{\gamma} \frac{\varphi(\zeta)}{\sigma - \zeta} d\sigma = x\varphi(0) = 0.
\]

**Solution in the Special Case**

The problem will be solved for the special case when the mapping function has the form

\[
z = \omega(\zeta) = a \left\{ \frac{0.825}{\zeta} + 0.149\zeta^2 + 0.017\zeta^3 + 0.006\zeta^4 + 0.003\zeta^5 \right\}.
\]
Towards solving (6) we note that \( \frac{\omega(\sigma)}{\omega'(\sigma)} \) is the boundary value of the rational function

\[
\frac{\omega(\zeta)}{\omega'(\zeta)} = \gamma^9 \left( \frac{0.825 + 0.149\zeta^2 + 0.107\zeta^3 + 0.006\zeta^4 + 0.003\zeta^5}{-0.825\zeta^2 + 0.298\zeta^4 + 0.085\zeta^5 + 0.048\zeta^7 + 0.033} \right)
\]

\[= -\{0.004\zeta^4 + 0.008\zeta^6 + 0.024\zeta^9 + 0.190\} + \text{terms in } \frac{1}{\zeta^3}, \frac{1}{\zeta^5} \ldots \text{etc.} \]

which is holomorphic outside \( \gamma \) except at \( \zeta = \infty \), where it has a pole of order 9.

Now \( \varphi(\zeta) \) is holomorphic inside \( \gamma \), so that \( \varphi'(\sigma) \) is the boundary value of \( \varphi \left( \frac{1}{\zeta} \right) \), holomorphic outside \( \gamma \). Hence

\[
\frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) \text{ is the boundary value of } \varphi \left( \frac{1}{\zeta} \right), \text{ holomorphic outside } \gamma \text{ except at } \zeta = \infty, \text{ where it has a pole of order 9.}
\]

Assuming \( \varphi(\zeta) = a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n + \ldots \) in \( |\zeta| < 1 \), then

\[
\varphi \left( \frac{1}{\zeta} \right) = \bar{a}_1 + \frac{2\bar{a}_2}{\zeta} + \ldots + \frac{n\bar{a}_n}{\zeta^{n-1}} + \ldots \text{ for } |\zeta| > 1.
\]

Therefore, in the present case, for \( |\zeta| > 1 \)

\[
\frac{\omega(\zeta)}{\omega'(\zeta)} \varphi \left( \frac{1}{\zeta} \right) = -K_0 - K_1\zeta - K_2\zeta^2 - \ldots - K_9\zeta^9 + 0 \left( \frac{1}{\zeta} \right) \quad \ldots (9)
\]

where \( K_0 = 0.072\bar{a}_2 + 0.048\bar{a}_3 + 0.036\bar{a}_9 \)
\( K_1 = 0.048\bar{a}_2 + 0.040\bar{a}_3 + 0.032\bar{a}_8 \)
\( K_2 = 0.024\bar{a}_1 + 0.032\bar{a}_4 + 0.028\bar{a}_7 \)
\( K_3 = 0.024\bar{a}_3 + 0.024\bar{a}_6 \)
\( K_4 = 0.016\bar{a}_2 + 0.020\bar{a}_5 \)
\( K_5 = 0.008\bar{a}_1 + 0.016\bar{a}_4 \)
\( K_6 = 0.012\bar{a}_3 \)
\( K_7 = 0.008\bar{a}_2 \)
\( K_8 = 0.004\bar{a}_1 \)

Hence

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) d\sigma = -K_0 - K_1\zeta - \ldots - K_9\zeta^9, \text{ so that, substituting in (6), we have}
\]

\[
x \varphi(\zeta) + K_0 + K_1\zeta + \ldots + K_9\zeta^9 - \bar{a} = \frac{2\mu}{2\pi i} \int_{\gamma} \frac{g d\sigma}{\sigma - \zeta} \quad \ldots \ldots \ldots (11)
\]

It is assumed that the shaft (i.e. the kernel) in the region of the hub is rigid in comparison with the hub, so that if a moment acting on the shaft causes the
shaft in the region of the hub to rotate about its centre through a small angle $\varepsilon$, the components of displacement on the edge of the shaft will be

$$dx = -\varepsilon y, \quad dy = \varepsilon x.$$  

It is further assumed that points on the hub remain in contact with the same points on the shaft after deformation as before. This will certainly be true in the case of a welded join. The displacements $g_1, g_2$ on the edge of the hole are then $g_1 = -\varepsilon y$, $g_2 = \varepsilon x$. Therefore $g = g_1 + ig_2 = i\varepsilon(x + iy) = i\varepsilon z = i\varepsilon(\sigma)$. Thus

$$g = ia\varepsilon\left(\frac{0.825}{\sigma} + 0.149\sigma^2 + 0.017\sigma^5 + 0.006\sigma^8 + 0.003\sigma^{11}\right) \ldots (12)$$

Now $g(\sigma)$ is the boundary value of the function $g(\zeta)$ holomorphic in $|\zeta|<1$ except at $\zeta=0$, where it has a simple pole. Thus the Cauchy Integral

$$\frac{2\mu}{2\pi i} \int \frac{g\,d\sigma}{\sigma - \zeta} = 2\mu i a (0.149)$$

Thus, equation (11) becomes

$$\chi(a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n) + (K_0 - \bar{a}) + K_1\zeta + K_2\zeta^2 + \ldots + K_9\zeta^9 = 2\mu i a (0.149) \ldots \ldots (13)$$

Comparing coefficients on both sides and substituting for $K_1, K_2, \ldots, K_9$, we have

$$\chi a_1 + 0.038a_3 + 0.048a_8 = 0$$

$$\chi a_2 + 0.014a_3 + 0.040a_5 + 0.032a_8 = 2\mu i a (0.149)$$

$$\chi a_3 + 0.024a_4 + 0.032a_6 + 0.028a_7 = 0$$

$$\chi a_4 + 0.024a_6 + 0.024a_8 = 0$$

$$\chi a_5 + 0.016a_2 + 0.020a_5 = 2\mu i a (0.017) \ldots \ldots (15)$$

$$\chi a_6 + 0.008a_4 + 0.016a_7 = 0$$

$$\chi a_7 + 0.012a_3 = 0$$

$$\chi a_8 + 0.004a_6 = 2\mu i a (0.006)$$

$$\chi a_9 + 0.004a_7 = 0$$

$$\chi a_{10} = 0$$

$$\chi a_{11} = 2\mu i a (0.003).$$

Putting $a_1 = a_1 + i\beta_1, \ldots, a_n = a_n + i\beta_n$ and equating real and imaginary parts we find that all $x$'s are zero and all $\beta$'s are zero except $\beta_2, \beta_5, \beta_8$, and $\beta_{11}$. Thus the only equations which concern us here are

$$(x - 0.048)\beta_2 - 0.040\beta_5 - 0.032\beta_8 = 2\mu a (0.149)$$

$$(x - 0.020)\beta_5 - 0.016\beta_8 = 2\mu a (0.017) \ldots \ldots (16)$$

$$\chi^2 \beta_8 - 0.008\beta_8 = 2\mu a (0.006)$$

$$\chi^2 \beta_{11} = 2\mu a (0.003).$$

Solving (16), neglecting terms of order $10^{-5}$, we have

$$\beta_2 = 2\mu a \left[\frac{0.149x - 0.002}{x^2 - 0.068x}\right]$$

$$\beta_5 = 2\mu a \left[\frac{0.017x + 0.001}{x^2 - 0.088x + 0.001}\right] \ldots \ldots \ldots \ldots (17)$$

$$\beta_8 = 2\mu a \left[\frac{0.006x + 0.001}{x^2 - 0.068x}\right]$$

$$\beta_{11} = 2\mu a \left[\frac{0.003}{x}\right]$$

and

$$\varphi(\zeta) = i[\beta_2\zeta^2 + \beta_5\zeta^5 + \beta_8\zeta^8 + \beta_{11}\zeta^{11}] \ldots \ldots (18)$$
and since \( a_2 = i \beta_2 \), \( a_5 = i \beta_5 \), etc., we have from (10):

\[
\begin{align*}
K_2 &= -0.048i \beta_2 - 0.010i \beta_5 - 0.032 \beta_8 \\
K_5 &= -0.016i \beta_2 - 0.020i \beta_5 \\
K_8 &= -0.008i \beta_2,
\end{align*}
\]

all other \( K \)'s being zero, \( K_0 \) not being required. Hence (9) can be rewritten as follows:

In \( | \zeta | > 1 \)

\[
\begin{align*}
\frac{\omega(\zeta)}{\omega'(\zeta)} \varphi' \left( \frac{1}{\zeta} \right) &= -K_2 \zeta^2 - K_5 \zeta^5 - K_8 \zeta^8 + 0 \left( \frac{1}{\zeta} \right) \quad \ldots \quad (9')
\end{align*}
\]

Taking the conjugate of this equation we see that in \( | \zeta | < 1 \),

\[
\begin{align*}
\omega \left( \frac{1}{\zeta} \right) = -K_2 \zeta^2 - \zeta^5 - \zeta^8 + \text{a Holomorphic Function} \quad .. \quad (19)
\end{align*}
\]

Considering the second integral in (7), the function \( \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) \) is the boundary value of \( \frac{\omega(\zeta)}{\omega'(\zeta)} \varphi'(\zeta) \), holomorphic inside \( \gamma \) except \( \zeta = 0 \), where it has a pole of order 8. Thus the Cauchy Integral

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) d\sigma = \text{the holomorphic function in (19)}.
\]

\[
\begin{align*}
\omega \left( \frac{1}{\zeta} \right) = \frac{\omega(\zeta)}{\omega'(\zeta)} \varphi'(\zeta) + \frac{\overline{K}_2}{\zeta^2} + \frac{\overline{K}_5}{\zeta^5} + \frac{\overline{K}_8}{\zeta^8} \quad \ldots \quad (20)
\end{align*}
\]

Now, on the boundary of the hub, using (12),

\[
\hat{g} = -ia\varepsilon \left\{ 0.825\sigma + 0.149 \frac{1}{\sigma^2} + 0.017 \frac{1}{\sigma^5} + 0.006 \frac{1}{\sigma^8} + 0.003 \frac{1}{\sigma^{11}} \right\} \quad \ldots \quad (21)
\]

so that

\[
\begin{align*}
\frac{-2\mu}{2\pi i} \int_{\gamma} \frac{\hat{g} d\sigma}{\sigma - \zeta} &= +2\mu ia\varepsilon (0.825\zeta).
\end{align*}
\]

Thus equation (7) yields

\[
\begin{align*}
\psi(\zeta) &= 2\mu a\varepsilon (0.825\zeta) - \frac{\omega \left( \frac{1}{\zeta} \right)}{\omega'(\zeta)} \varphi'(\zeta) - \frac{\overline{K}_2}{\zeta^2} - \frac{\overline{K}_5}{\zeta^5} - \frac{\overline{K}_8}{\zeta^8} \quad \ldots \quad (22)
\end{align*}
\]

**Relation between Moment of Applied Couple and \( \varepsilon \).**

The moment \( M \) of forces acting on the boundary of the hub about the centre will be equal and opposite to the moment of forces acting on the shaft in the vicinity of the hub about the centre. This moment is given by the real part
of the change in $\chi_1(z) - z\psi_1(z) - z\overline{\psi_1}(z)$ as $z$ moves around the contour $L$ clockwise (cf. Muskhelishvili's formula 33.3, in which there is a misprint), where

$$\chi_1(z) = \int \psi_1(z)dz + \text{constant}.$$ 

Since $\varphi(\zeta)$, $\varphi'(\zeta)$ and $\psi(\zeta)$ are holomorphic inside $\gamma$, $\varphi_1(z)$ and $\psi_1(z)$ are holomorphic outside $L$, so that we need only calculate the real part of the increase in

$$\int \psi_1(z)dz = \int \psi(\zeta)\omega'(\zeta)d\zeta$$

as $\zeta$ moves around $\gamma$ anti-clockwise. Thus we need the multivalued term in

$$\text{Re} \int \psi(\zeta)\omega'(\zeta)d\zeta,$$ i.e. in

$$\text{Re} \left\{ 2\mu \varepsilon (0.825)\zeta \omega'(\zeta) - \omega'(\zeta) \psi'(\zeta) - \left( \frac{K_2}{\zeta^2} + \frac{K_5}{\zeta^5} + \frac{K_8}{\zeta^8} \right) \omega'(\zeta) \right\} d\zeta. \quad (23)$$

This multivalued term is

$$\text{Re} \left\{ -2\mu \varepsilon a^2 (0.825)^2 - ia (0.298\beta_2 + 0.085\beta_5 + 0.048\beta_8 + 0.033\beta_{11}) - a (0.298K_2 + 0.085K_5 + 0.048K_8) \ln \zeta \right\} \quad (24)$$

From (10),

$$K_2 = 0.048i\beta_2 + 0.040i\beta_5 + 0.032i\beta_8,$$

$$K_5 = 0.016i\beta_2 + 0.020i\beta_5,$$

$$K_8 = 0.008i\beta_2.$$

Substituting in (24) and simplifying, we find

$$M = 2\pi \{ 2\mu \varepsilon a^2 (0.825)^2 + a\beta_2 (0.313) + a\beta_5 (0.099) + a\beta_8 (0.058) + a\beta_{11} (0.033) \} \quad (25)$$

### CALCULATION OF STRESS COMPONENTS.

The components of stress in orthogonal curvilinear co-ordinates $(\varphi)$, $(\theta)$ are given by the formulae (50.9), (50.10) in Muskhelishvili. A rearrangement gives

$$\tilde{\varphi}_\varphi + \tilde{\theta}_\theta = 4\text{Re} \left\{ \frac{\varphi'(\zeta)}{\omega'(\zeta)} \right\} \quad (26)$$

and

$$\tilde{\theta}_\varphi - \tilde{\varphi}_\theta + 2i\tilde{\varphi}_\varphi = \frac{2\zeta^2}{\varphi \omega'(\zeta)} \left[ \omega'(\zeta) \frac{d}{d\zeta} \left( \frac{\varphi'(\zeta)}{\omega'(\zeta)} \right) + \psi'(\zeta) \right]. \quad (27)$$

When $\varphi = 1$, $\tilde{\varphi}_\varphi$, $\tilde{\theta}_\theta$, $\tilde{\varphi}_\varphi$ give the normal, tangential and shear stresses on the boundary of the hub corresponding to values of $\theta$ in the $\zeta$-plane.

Equation (26) presents no difficulty. We find, putting $\varphi = 1$,

$$\tilde{\varphi}_\varphi + \tilde{\theta}_\theta = \frac{4a (B \sin 3\theta + C \sin 6\theta + D \sin 9\theta + E \sin 12\theta)}{a^2 (0.780 - 0.430 \cos 3\theta - 0.106 \cos 6\theta - 0.060 \cos 9\theta - 0.054 \cos 12\theta)} \quad (28)$$
where \( B=1.820\beta_2 - 1.250\beta_5 - 0.416\beta_8 - 0.528\beta_{11} \)
\( C=0.096\beta_2 + 4.290\beta_5 - 2.384\beta_8 - 0.935\beta_{11} \)
\( D=0.066\beta_2 + 6.600\beta_8 - 3.278\beta_{11} \)
\( E=9.075\beta_{11} \).

Towards evaluating (27), we note that

\[
\frac{\omega(\zeta)}{\omega'(\zeta)} \frac{d}{d\zeta} \left( \varphi'(\zeta) \right) = \frac{\omega(\zeta)\omega'(\zeta)\varphi''(\zeta) - \omega(\zeta)\varphi'(\zeta)\omega''(\zeta)}{[\omega'(\zeta)]^2} \quad \ldots \quad (29)
\]

and from the expression (22) for \( \varphi(\zeta) \) we find

\[
\varphi'(\zeta) = 2\mu a e \left( 0.825 \right) + \frac{2K_2}{\zeta^3} + \frac{5K_5}{\zeta^6} + \frac{8K_8}{\zeta^9} - \frac{\omega'(\zeta)\varphi'(\zeta)}{[\omega'(\zeta)]^2} \\
\omega'(\zeta)\varphi'(\zeta) \omega''(\zeta) - \frac{\omega'(\zeta)}{[\omega'(\zeta)]^2} \omega(\zeta)\varphi'(\zeta)\omega''(\zeta) \quad \ldots \quad (30)
\]

Substituting from (29) and (30) in (27) we find

\[
\omega(\zeta) - \varphi(\zeta) = 2e^{2i\theta} \left[ \frac{2\mu a e \left( 0.825 \right)}{\omega'(\zeta)\omega''(\zeta)} - \frac{\omega(\zeta)\varphi'(\zeta)}{\omega'(\zeta)\omega''(\zeta)} \right] \omega'(\zeta) - \frac{d}{d\sigma} \left\{ \omega(\sigma) \varphi'(\sigma) \right\} \omega''(\zeta) \quad \ldots \quad (31)
\]

Since on the boundary of \( \gamma, \omega(\sigma) = \omega(1) \) we have, putting \( \rho = 1 \) and \( \sigma = e^{i\theta} \),

\[
0 \omega(\zeta) - \varphi(\zeta) + 2ie^\theta = \left[ \frac{2e^{2i\theta}}{\omega'(\sigma)\omega''(\sigma)} \right] \left\{ \frac{2\mu a e \left( 0.825 \right)}{\omega'(\sigma)\omega''(\sigma)} \right\} \omega'(\sigma) - \frac{d}{d\sigma} \left\{ \omega(\sigma) \varphi'(\sigma) \right\} \omega''(\sigma) \quad \ldots \quad (31)
\]

Substituting in (31) for \( \omega'(\sigma), \varphi'(\sigma), K_2, K_5, K_8, \) etc., and separating the real and imaginary parts of each term in turn we find firstly

\[
\varphi(\zeta) = \frac{a}{\omega'(\zeta)} \left[ A' + B' \cos 30 + C' \cos 60 + D' \cos 90 + E' \cos 120 \right] \quad \ldots \quad (32)
\]

where

\( A' = 0.635\beta_2 + 0.458\beta_5 + 0.403\beta_8 + 0.363\beta_{11} - 2\mu e a \left( 0.681 \right) \)
\( B' = -1.516\beta_2 + 1.706\beta_5 + 0.896\beta_8 + 0.528\beta_{11} + 2\mu e a \left( 0.246 \right) \)
\( C' = 0.057\beta_2 - 4.026\beta_5 + 2.387\beta_8 + 0.935\beta_{11} + 2\mu e a \left( 0.070 \right) \)
\( D' = 0.016\beta_2 + 0.003\beta_5 - 6.598\beta_8 + 3.278\beta_{11} + 2\mu e a \left( 0.040 \right) \)
\( E' = -9.075\beta_{11} + 2\mu e a \left( 0.027 \right) \).
Combining the expression for $\theta - \rho \varphi$ from (31) with $\theta + \rho \varphi$ from (28), we find

$$\rho \varphi = a \left| \frac{1}{\omega'(\sigma)} \right|^2 [B'' \sin 30 + C'' \sin 60 + D'' \sin 90 + E'' \sin 120] \quad \ldots \quad (33)$$

where

$$B'' = 1.884 \beta_2 - 1.202 \beta_5 - 0.358 \beta_8 - 0.528 \beta_{11} + 2\mu \varepsilon a (0.246)$$
$$C'' = 0.151 \beta_2 + 4.380 \beta_5 - 2.381 \beta_8 - 0.935 \beta_{11} + 2\mu \varepsilon a (0.070)$$
$$D'' = 0.122 \beta_2 + 0.003 \beta_5 + 6.602 \beta_8 - 3.278 \beta_{11} + 2\mu \varepsilon a (0.040)$$
$$E'' = 9.075 \beta_{11} + 2\mu \varepsilon a (0.027).$$

Also

$$\theta = a \left| \frac{1}{\omega'(\sigma)} \right|^2 [B'' \sin 30 + C'' \sin 60 + D'' \sin 90 + E'' \sin 120] \quad \ldots \quad (34)$$

where

$$B'' = 5.396 \beta_2 - 3.798 \beta_5 - 1.306 \beta_8 - 1.584 \beta_{11} - 2\mu \varepsilon a (0.246)$$
$$C'' = 0.233 \beta_2 + 12.780 \beta_5 - 7.155 \beta_8 - 2.805 \beta_{11} - 2\mu \varepsilon a (0.070)$$
$$D'' = 0.142 \beta_2 - 0.003 \beta_5 + 19.798 \beta_8 - 9.834 \beta_{11} - 2\mu \varepsilon a (0.040)$$
$$E'' = 27.225 \beta_{11} - 2\mu \varepsilon a (0.027).$$

Text-fig. 4.

RESULTS FOR SPECIAL MATERIAL. GRAPHS.

Choosing steel as the special material, with Poisson’s Ratio 0.30, we have $\chi = 1.80$. Then, from (17)

$$\beta_2 = 2\mu \varepsilon a (0.085)$$
$$\beta_5 = 2\mu \varepsilon a (0.010)$$
$$\beta_8 = 2\mu \varepsilon a (0.004)$$
$$\beta_{11} = 2\mu \varepsilon a (0.002).$$

The value of $\varepsilon$ for a given couple with moment $M$ is given from (25)

$$\varepsilon = \frac{M}{4\pi \varepsilon a^2 (0.709)} \quad \ldots \quad (36)$$
From (32), (33), (34)

\[
\bar{\varphi} - \bar{\varphi} = \frac{-\mu \varepsilon (1.240 - 0.277 \cos 30 - 0.087 \cos 60 - 0.043 \cos 90 - 0.025 \cos 120)}{(0.780 - 0.430 \cos 30 - 0.106 \cos 60 - 0.060 \cos 90 - 0.054 \cos 120)} \\
\bar{\varphi} = \frac{\mu \varepsilon (0.784 \sin 30 + 0.246 \sin 60 + 0.141 \sin 90 + 0.083 \sin 120)}{(0.780 - 0.430 \cos 30 - 0.106 \cos 60 - 0.060 \cos 90 - 0.054 \cos 120)} \\
\bar{\theta} = \frac{\mu \varepsilon (0.335 \sin 30 + 0.102 \sin 60 + 0.063 \sin 90 + 0.033 \sin 120)}{(0.780 - 0.430 \cos 30 - 0.106 \cos 60 - 0.060 \cos 90 - 0.054 \cos 120)}
\]

(37) \hspace{2cm} (38) \hspace{2cm} (39)

The graphs of \(\bar{\varphi}, \bar{\theta}, \bar{\phi}\) against \(\theta\) are shown in Text-figure 4. By choosing the positive directions of \(\bar{\varphi}, \bar{\theta}, \bar{\phi}\) as shown we can show the stresses along the edge of the hole for values of angle \(\Phi\) from 0 to 120°. The stresses are repeated on each of the other two edges. The values of \(\Phi\) corresponding to the values of \(\theta\) from the graphs can be obtained from Text-figure 3.

**References.**


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