Floppy Rulers and Light Pens—Reactor Mathematical Aids*

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ABSTRACT—Much of present day nuclear reactor mathematics is concerned with numerical methods which require the use of giant digital computers. Even as little as a quarter of a century ago sturdy rulers and ink pens were used as graphical aids to computation. These aids, or at least their computer analogues, still play a part in computation, although the present fashion is to use floppy rulers and light pens.

1. Preamble

If I were to tell you that I intend to talk about solution of the neutron reactor diffusion equation, even in the simplest form,

\[-\nabla \cdot D \nabla \phi + \sigma \phi = - \frac{1}{\nu} \frac{\partial \phi}{\partial t}\]

in the reactor material . . . (1)

\[n \cdot D \nabla \phi + a \phi = 0\]

on the reactor boundary . . . (2)

(where \(D, \sigma, \nu\) and \(a\) are known, \(n=\) outward normal and \(\phi\), the neutron flux, is required to be calculated as a function of space and time \((t)\)) then I am sure you would not want to stay. I will, therefore, talk about minor matters that are, nevertheless, part of the procedure used in solving the above equations (for a somewhat more general form see Pollard 1974) when using a giant digital computer. We will use such everyday objects as rulers and pens, and computer analogues of these, to bring some focus on ideas of mathematics pertinent to the use of digital computers for nuclear reactor computation.

2. Sturdy Rulers

You are all familiar with the old foot ruler. Perhaps you have used one to draw in a line of best fit to some experimental data. The idea here is that the ruler is moved around on a plot of the data until the eye is satisfied with the fit as in Figure 1.

Maybe you have even tried to fit a straight line, or several straight lines, to some other continuous function as in Figure 2.

Well, a digital computer does not usually include as part of its working parts, so-called "hardware", a ruler or an eye, so we must simulate these in order to use the machine for the type of approximation task briefly indicated so far.

![Figure 1](image1.png)  
*Figure 1.—A straight line fit to experimental data.*

![Figure 2](image2.png)  
*Figure 2.—A straight line fit to a given function.*

2.1 Sturdy ruler simulation

Simulation of a ruler is easy as any father of a high school student knows. We simply have

\[y(x) = mx + c \]

with \(m=\)slope and \(c=\)intercept for the line. A digital computer can add and multiply, hence the above is easy to set as a task or "program" for the machine to calculate.
2.2 Simulation of the eye

A mathematical analogue for an eye requires in its simplest form a measure of size of a function—that is, how big is a function? We extend the idea of size of a number $x$, which is simply the absolute value (magnitude) of the number, $|x|$. Mathematicians delight in the expressiveness of notation and since the size of a function $y(x)$ is a generalization of the absolute value we write for it $|y|$ and we call it a norm. Fortunately, when we make the generalization, many different norms exist. We might even say that our simulated eye sees things differently with different norms. One possible approach is to take the maximum value of the function $y(x)$ for a certain interval of $x$, say $[a, b]$, then

$$||y|| = \text{maximum } |y(x)| \quad \ldots \quad (4)$$

Figure 3 illustrates the idea involved here.

![Figure 3. A norm of a function.](image)

Fairly readily we see that our norm $||y||$ given by equation (4) satisfies the desirable size properties (Kantorovich and Akilov, 1964).

1. $||y|| > 0$, 1. $||y|| = 0$ implies $y = 0$,
2. $||cy|| = c||y||$ for $c$ a positive constant, and
3. $||y+z|| < ||y|| + ||z||$ (“triangle inequality”).

Using the above mathematical equipment we are able to measure the nearness of two functions $g(x)$ and $y(x)$ through the norm $||g-y||$, the size of the difference between the two functions. We may then simulate our eye fit by the mathematically precise, and computer feasible, process of calculating that $y_n(x)$ is closer to $g(x)$ than $y_m(x)$ when

$$||g-y_n|| < ||g-y_m|| \quad \ldots \quad (5)$$

The best fit of a class of functions $y_m(x)$ to a given function $g(x)$ is then the particular function $y_n(x)$ for which

$$||g-y_n|| = \text{minimum} \quad (6)$$

3. Ink Pens

Most of us are no doubt familiar with the idea of graphical display of data. At a glance, a graph is able to convey the general trend of behaviour of, say, a function $\varphi(t)$ plotted against $t$. Nowadays, it is fashionable and extremely useful to have graphs plotted automatically from results produced on a computer. Figures (4) to (6) illustrate the versatility of such automatic graph plotting. The results plotted are for the solution of a more general form of equation (1) from calculations on the AAEC reactor MOATA when a safety rod is suddenly dropped into the fuel in one half of the reactor in order to shut the reactor down, $\varphi(t) \rightarrow 0$ (Pollard, 1974). Figure 4 is a plot carried out by fifth form high school students at a recent AEAC Summer School (Barry et al., 1974).

4. Floppy Rulers

In Figure 2 we notice that, as an approximation to the given function, we obtained a better fit, as measured by our norms, when two straight lines were used. We could obviously join together many straight lines, but the resulting many cornered approximation would not be acceptable to our human eye as it would lack smoothness. Draftsmen have been able to achieve this smoothness property in their drawing of curves by using thin beams of flexible wood or plastic called splines—these are our floppy rulers. Figure 7 shows the idea of using a spline to achieve a smooth fit to a set of exact data points, $g(x_i)$ with $x_1(=a) < x_2 < \ldots < x_n(=b)$.

4.1 Floppy ruler simulation

Now when the spline is only bent slightly to pass through the given points, the potential energy is proportional to

$$\int_a^b \left( \frac{d^2y}{dx^2} \right)^2 dx \quad (\text{Handscomb, 1966}) \quad \ldots \quad (7)$$

and the shape taken up by the spline minimizes the above. Since mathematically $\frac{d^2y}{dx^2}$ (for small slopes $\frac{dy}{dx}$) is a measure of curvature, that is extent of bending, we could say that the spline enables a curve to be drawn through the points which minimizes curvature or is the smoothest curve through the points.
4.2 *Simulation of the eye with smoothness discernment*

In section 2.2 we sought a mathematical analogue of an eye based on the size of a function (equation (4)). Here we want to specify a smoothness measure—how about the candidate

\[ \|y\|_2 = \left( \int_a^b \left( \frac{d^2y}{dx^2} \right)^2 dx \right)^{1/2} \]  

which immediately gives us condition (9):

\[ \|y\|_2 < \|g\|_2. \]  

In fact, if the given points are obtained from a smooth-looking function \( g(x) \) then the spline fit is even smoother! We accept our candidate as an "eye" with suitable discrimination, although we must not take smoothness as an universally desirable property.

Our desirable size properties (1) to (3) of section 2.2 are met, provided we accept the lapse that

(1. (b)) \( \|y\|_2 = 0 \) implies \( y(x) = mx + c \) (an arbitrary straight line).

You could almost say that our new eye cannot see straight lines and, consequently, mathematicians term our candidate a pseudo-norm.

In addition the minimum potential energy condition (7) tells us that if \( g(x) \) is any curve through the given points and \( y(x) \) is the spline through the points then

\[ \|y\|_2 < \|g\|_2. \]  

An interesting property of our spline \( y(x) \) and pseudo-norm \( \|y\|_2 \) is that

\[ \|g - y\|_2 = \|g\|_2 - \|y\|_2 \]  

(Holladay, 1957), (10)

which immediately gives us condition (9):

\[ \|y\|_2 = \|g\|_2 - \|g - y\|_2 < \|g\|_2. \]

4.3 *Spline functions*

You might reasonably ask "what is this spline function \( y(x) \)?" It turns out (Ahlberg,
Nilson and Walsh, 1967) that \( y(x) \) is no at polynomial but rather consists of neighbouring smooth joining cubic polynomials with each cubic holding between neighbouring data points ("knots"), say corresponding to \( x_i \) and \( x_{i+1} \) and with a possible jump in \( \frac{d^3 y}{dx^3} \) at the knots.

We have
\[
y(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad x_i < x < x_{i+1} \quad \ldots \quad (11)
\]
and the constants may be determined from the conditions

1. Continuity of \( y, \frac{dy}{dx}, \) and \( \frac{d^2 y}{dx^2} \) at the internal knots
2. Interpolation through the given data points
\[
y(x_i) = g(x_i), \quad i = 1, 2, \ldots, n, \quad \text{and}
\]
3. Specification of two arbitrary end conditions, one at each end \( x_1 \) and \( x_n \), say
\[
\frac{d^3 y}{dx^3} = 0.
\]

Calculation of the coefficients is not a difficult task as the process may be reduced to solution of a tridiagonal set of equations, readily solvable using a forward elimination and backward substitution scheme (Ahlberg, Nilson and Walsh, 1967).

The advantage of spline approximation over polynomial approximation employing neighbouring straight lines, quadrics, etc., up to a polynomial of order \( n-1 \) through \( n \) data points, is its greater insensitivity to the dropping of data points and its ability to uniformly approximate not only the function but also the low order derivatives of the function.

A generalization of the spline approximation briefly outlined here, which attempts only to pass near experimentally inexact data points rather than to pass through each one, is used in reactor studies for producing smooth fits to basic cross section data \( \sigma \) as a function of reacting neutron energy (Parker, 1970). A further generalization of the idea to surface smoothing is used in vehicle body design and,
indeed, one of the earliest applications for splines was in the “fairing” of ship lines (Theilheimer and Starkweather, 1961).

Figure 8 shows a spline fit to an idealized neutron resonance cross section

$$\sigma(x) = \frac{1}{1+x^2}$$

with knots at

$$x = -8, -7, \ldots, 0, \ldots, 7, 8.$$  (12)

In the plot the actual cross section $$\sigma(x)$$ and the spline fit essentially coincide. A comparison plot is also given of the (Lagrangian) polynomial of degree 16 that passes through the same knots—note that the off scale peaks are $$\sigma = 4.8$$ and $$\sigma = -50$$!

Full marks to floppy rulers—the advantage of the spline approximation over polynomial approximation is indeed marked for the example given. In general, the advantage is not as pronounced as indicated here.

5. Light Pens

In section 3 on Ink Pins we were introduced briefly to computer driven graph plotting. The graphs may also be produced on a cathode ray (TV) screen for viewing and may be photographed for later perusal. Here we meet a device, a light pen, which in essence may communicate to the computer by drawing actions carried out with the pen in near contact to the cathode ray screen. Some pens emit a light beam for user orientation, but communication to the computer is by the pen detecting the dynamic light changes of the phosphor glow on the screen with subsequent pulse signal transmission to the computer. Use of a light pen opens up a new mode of communication to a computer—we can communicate with each other through pictures!

Use of a TV screen and light pen enables new approaches to be adopted when using a computer. Draughtsmen can recall a drawing to be modified and quickly make the change using the light pen. Copies of the new drawing can then be obtained for distribution. Civil engineers can simulate driving along a road as yet still only in the planning stages. The blending of the road with the countryside can be assessed. Mechanical engineers can simulate the flight of aircraft. Body surface shapes can be quickly changed and new computer assessments tackled. Nuclear physicists can judge...
the smoothness of their fits to neutron cross sections, $\sigma$. New "knots" may be chosen and new fits obtained. This rapidly developing field of "computer graphics" offers not only picture presentation of already understood phenomena but, because of its interactive nature (men and computer working together), new insights can be gained into the phenomena being studied. The extensive book by Parslow and Green (1971) shows further applications covering a wide range of disciplines. Figure 9 is a photograph of the type of equipment being used for computer graphics work.

References


A.A.E.C. Research Establishment,
Lucas Heights,
New South Wales.

6. What Next?

It is to be expected that sometime during their school life present kindergarten children will use digital computers. The teaching aid could consist of a TV screen, light pen and typewriter, all connected to a computer. Not only mathematics, but perhaps unexpected subjects, geography, economics and geology, will use the teaching aid.

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