# RADIAL HEAT FLOW IN CIRCULAR CYLINDERS WITH A GENERAL BOUNDARY CONDITION. II.

By J. C. JAEGER, M.A., D.Sc.

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1. In these Proceedings[1] a number of results were given on conduction of heat in regions bounded internally or externally by circular cylinders with boundary condition

$$k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3v=k_4 \quad\quad (1)$$

at a surface. The solutions were obtained by a formal method using the Laplace transformation and it was remarked that it could be verified by a procedure previously developed elsewhere[2] that they did in fact satisfy the differential equations and boundary and initial conditions of their problems. The verification procedure described in II is applicable to a wide range of one-variable problems in conduction of heat, and, since only some special problems of those in III were verified, it seems worth while indicating that the complete set of results obtained in I may be verified in this way. These include most of the results of III as special cases.

In §§2, 3, 4 three results on the nature of the roots of certain equations involving Bessel functions, which were stated without proof in I and are of intrinsic interest, will be proved for a set of conditions including those of physical interest in I.

2. *The Roots of the Equation.*[3]

$$(lz^2-m)J_0(z)+nzJ_1(z)=0 \quad\quad (2)$$

*where* l, m, n *are real constants, are all real and simple* (*except possibly for* z=0) *provided*

$$l\geqslant 0,\ m\geqslant 0,\ n>0 \quad\quad (3)$$

In (2) we may without loss of generality take $l\geqslant 0$ and if $l=0$ we take $m>0$. This convention is implied, here and subsequently, in stating results such as (3), (6) and (8).

If some of $l$, $m$, $n$ vanish the equation (2) reduces to a simpler form. If $l=m=0$ the result is well known. If $n=0$ the equation becomes $(lz^2-m)J_0(z)=0$, which if $l>0$, $m>0$, may have double roots at $\pm(m/l)^{\frac{1}{2}}$, if $(m/l)^{\frac{1}{2}}$ is equal to a root of $J_0(z)=0$.

(i) A pure imaginary root $z=iy$ of (2) is a real zero of

$$(ly^2+m)I_0(y)+nyI_1(y) \quad\quad (4)$$

Now $I_0(y)$ and $I_1(y)$ are both positive for real positive $y$, so the expression (4) is certainly always positive if $y>0$ and conditions (3) are satisfied. Thus (4) has no real positive zero, and since it is an even function it has no real negative

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[1] *Journ. and Proc. Roy. Soc. N.S.W.*, 1940, *74*, 342. This paper will be referred to as I.

[2] *Proc. Cambridge Phil. Soc.*, 1939, *35*, 394. *Proc. London Math. Soc.*, 1940, *46*, 361. These papers will be referred to as II and III, respectively.

[3] This is I (13).

zero. If the conditions (3) are not satisfied there may be no, one, or two real positive zeros of (4).

(ii) The equation (2) has no complex roots if the conditions (3) are satisfied. For if $\xi$ and $\eta$ be conjugate complex roots of (2), we have

$$(l\xi^2-m)J_0(\xi)-n\xi J'_0(\xi)=0$$
$$(l\eta^2-m)J_0(\eta)-n\eta J'_0(\eta)=0.$$

Thus

$$l(\eta^2-\xi^2)J_0(\xi)J_0(\eta)+n\{\xi J'_0(\xi)J_0(\eta)-\eta J'_0(\eta)J_0(\xi)\}=0.$$

Therefore[4]

$$l(\eta^2-\xi^2)J_0(\xi)J_0(\eta)+n(\eta^2-\xi^2)\int_0^1 xJ_0(\xi x)J_0(\eta x)dx=0.$$

If $l\geqslant 0$, $n>0$ this is impossible, so there can be no complex root.

(iii) The equation (2) has no repeated roots, except possibly $z=0$, if the conditions (3) are satisfied. For writing[5]

$$y=(lz^2-m)J_0(z)+nzJ_1(z),$$

we find

$$yJ_1(z)+\frac{dy}{dz}J_0(z)=z\{(2l+n)J_0^2(z)+nJ_1^2(z)\}.$$

Thus if $z\neq 0$, $l\geqslant 0$, $n>0$, $y$ and $\frac{dy}{dz}$ cannot vanish simultaneously.

3. *The expression*[6]

$$(lz^2+m)K_0(z)-nzK_1(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

*has no zeros for* R(z)$\geqslant 0$, *provided*

$$l\geqslant 0,\ m\geqslant 0,\ n<0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

As in §2 we take $l\geqslant 0$, and if $l=0$, $m>0$. If $l=m=0$ the result is well known. If $n=0$, $l>0$, $m>0$ there are zeros at $\pm i(m/l)^{\frac{1}{2}}$.

(i) The expression (5) has no zeros for real positive $z$ if the conditions (6) are satisfied, since $K_0(z)>0$, $K_1(z)>0$, for real positive $z$.

(ii) The expression (5) has no complex zero $\xi$. For if $\eta$ is the conjugate of $\xi$, using the argument of §2 (ii) with G. and M., p. 70 (30), we have

$$(\xi^2-\eta^2)lK_0(\xi)K_0(\eta)-n(\xi^2-\eta^2)\int_1^\infty xK_0(\xi x)K_0(\eta x)dx=0,$$

and if $l\geqslant 0$, $n<0$ we have a contradiction.

(iii) The expression (5) has no pure imaginary zero $z=iy$, for this implies

$$(ly^2-m)[J_0(y)-iY_0(y)]-ny[J'_0(y)-iY'_0(y)]=0.$$

It follows that

$$J_0(y)Y'_0(y)-Y_0(y)J'_0(y)=0,$$

but this is equal to $(2/\pi y)$ and so we have a contradiction.

4. *The Zeros of*[7]

$$F(z)=[(lz^2-m)J_0(az)+nzJ_1(az)][(l'z^2-m')Y_0(bz)+n'zY_1(bz)]$$
$$-[(l'z^2-m')J_0(bz)+n'zJ_1(bz)][(lz^2-m)Y_0(az)+nzY_1(az)] \quad \ldots\ldots \quad (7)$$

*are all real and simple* (*except possibly for* z$=0$), *provided*

$$l\geqslant 0,\ l'\geqslant 0,\ m\geqslant 0,\ m'\geqslant 0,\ n<0,\ n'>0 \quad \ldots\ldots\ldots\ldots \quad (8)$$

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[4] Using Gray and Mathews, *Treatise on Bessel Functions*, p. 69 (23). This work will be referred to as G. and M.

[5] I am indebted to a referee for this argument.

[6] This result is needed in I, §§5 and 6.

[7] This is I (30).

We suppose $b>a$ in the discussion. The cases in which $n$ or $n'$ vanish are discussed in (iv) below.

(i) A pure imaginary zero $z=\pm iy$ of (7) is a real positive zero of

$$[(ly^2+m)I_0(ay)+nyI_1(ay)][(l'y^2+m')K_0(by)-n'yK_1(by)]$$
$$-[(l'y^2+m')I_0(by)+n'yI_1(by)][(ly^2+m)K_0(ay)-nyK_1(ay)]=0 \quad \ldots (9)$$

which may be written

$$\begin{aligned}&(ly^2+m)(l'y^2+m')[I_0(ay)K_0(by)-K_0(ay)I_0(by)]\\&-nn'y^2[I_1(ay)K_1(by)-K_1(ay)I_1(by)]\\&+ny(l'y^2+m')[I_1(ay)K_0(by)+K_1(ay)I_0(by)]\\&-n'y(ly^2+m)[I_0(ay)K_1(by)+I_1(by)K_0(ay)] \quad \ldots\ldots\ldots\ldots (10)\end{aligned}$$

It is known that $I_n(ay)K_n(by)-I_n(by)K_n(ay)$, $n=0$ and 1, have no real positive zeros. Taking $b>a$, it follows from the asymptotic expansions that they are negative for real positive $y$. Also $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$ are all positive for real positive $x$. Thus if the conditions (8) are satisfied, all four terms of (10) are $\leqslant 0$ for real positive $y$ and thus there is no real positive zero of (9).

(ii) Suppose $\alpha$ is a zero of (7), then

$$U=[(l\alpha^2-m)Y_0(a\alpha)+n\alpha Y_1(a\alpha)]J_0(\alpha r)-[(l\alpha^2-m)J_0(a\alpha)+n\alpha J_1(a\alpha)]Y_0(\alpha r)$$

is a non-zero solution of the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)+\alpha^2U=0, \quad a<r<b \quad \ldots\ldots\ldots\ldots (11)$$

with boundary conditions

$$\left.\begin{aligned}(l\alpha^2-m)U-n\frac{dU}{dr}=0, \quad r=a\\(l'\alpha^2-m')U-n'\frac{dU}{dr}=0, \quad r=b\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12)$$

Also, for any $\beta$,

$$V=[(l\beta^2-m)Y_0(a\beta)+n\beta Y_1(a\beta)]J_0(\beta r)-[(l\beta^2-m)J_0(a\beta)+n\beta J_1(a\beta)]Y_0(\beta r)$$

satisfies

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)+\beta^2V=0, \quad a<r<b \quad \ldots\ldots\ldots\ldots (13)$$

with $$(l\beta^2-m)V-n\frac{dV}{dr}=0, \quad r=a \quad \ldots\ldots\ldots\ldots (14)$$

From (11) and (13) it follows that

$$(\alpha^2-\beta^2)\int_a^b rUV\,dr+\left[rV\frac{dU}{dr}-rU\frac{dV}{dr}\right]_a^b=0,$$

and hence, using (12), (14) and the notation (7), we have

$$(\alpha^2-\beta^2)\left\{\int_a^b rUVdr+\frac{bl'}{n'}[UV]_{r=b}-\frac{al}{n}[UV]_{r=a}\right\}=\frac{b}{n'}F(\beta)[U]_{r=b}$$
$$\ldots\ldots\ldots\ldots (15)$$

Suppose $\alpha$ is a complex zero of (7) and $\beta$ its conjugate. Then $F(\beta)=0$, and (15) becomes

$$(\alpha^2-\beta^2)\left\{\int_a^b r\,|\,U\,|^2dr+\frac{bl'}{n'}|\,U\,|^2_{r=b}-\frac{al}{n}|\,U\,|^2_{r=a}\right\}=0.$$

Thus if $l\geqslant 0$, $l'\geqslant 0$, $n<0$, $n'>0$ we have a contradiction, and no complex root is possible.

(iii) To show that (7) has no repeated zeros, let $\alpha$ be a zero (real) of (7) and let $\beta$ be real and tend to $\alpha$. Then as $\beta\to\alpha$, $V\to U$ and $F(\beta)/(\beta-\alpha)\to F'(\alpha)$.

Thus (15) gives

$$2\alpha\left\{\int_a^b rU^2dr+\frac{bl'}{n'}[U^2]_{r=b}-\frac{al}{n}[U^2]_{r=a}\right\}=-\frac{b}{n'}F'(\alpha)[U]_{r=b}$$

If $\alpha$ is a repeated zero of (7), $F'(\alpha)=0$. Thus if $\alpha\neq0$, and the conditions (8) are satisfied, we have a contradiction.

(iv) If $n=0$, $l'\geqslant0$, $m'\geqslant0$, $n'>0$ we have

$$F(z)=(lz^2-m)G(z),$$

where

$$G(z)=J_0(az)[(l'z^2-m')Y_0(bz)+n'zY_1(bz)] \\ -Y_0(az)[(l'z^2-m')J_0(bz)+n'zJ_1(bz)].$$

The method of (ii) and (iii) may be used to show that the zeros of $G(z)$ are all real and simple. If $(m/l)^{\frac{1}{2}}$ is equal to a zero of $G(z)$, $F(z)$ will have double zeros at $\pm(m/l)^{\frac{1}{2}}$. A similar result holds for the case $n'=0$, $l\geqslant0$, $m\geqslant0$, $n<0$. If $n=n'=0$, we have

$$F(z)=(lz^2-m)(l'z^2-m')C_0(az,\ bz)$$

where $$C_0(az,\ bz)=J_0(az)Y_0(bz)-Y_0(az)J_0(bz).$$

The zeros of $C_0(az,\ bz)$ are known to be all real and simple. $F(z)$ has a repeated zero if $(m/l)^{\frac{1}{2}}$ or $(m'/l')^{\frac{1}{2}}$ coincides with one of them.

5. The method of solution used in I consisted of forming from the original differential equation and boundary conditions a subsidiary equation and boundary conditions, from the solution $\bar{v}(p)$ of which the solution $v(t)$ of the original problem was derived formally by the use of the inversion theorem, namely

$$v(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\bar{v}(\lambda)d\lambda, \qquad (16)$$

and the solution was obtained in its final form from the line integral in (16) by using the contour of Fig. 1 or Fig. 2. To make the solutions rigorous we

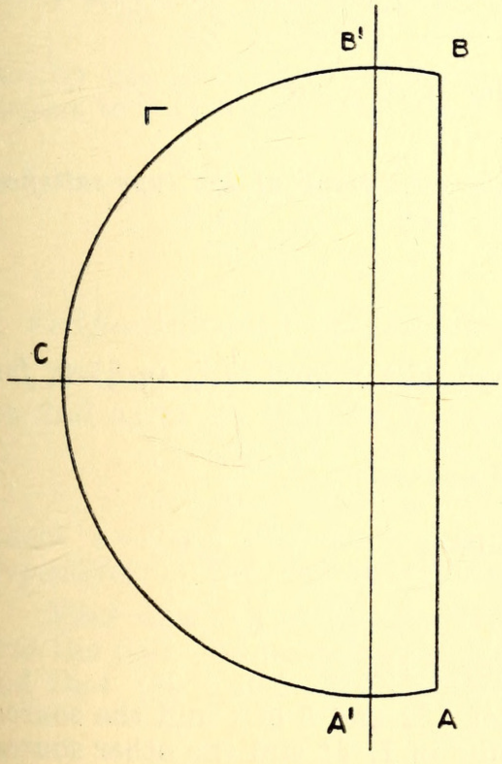


Fig. 1.

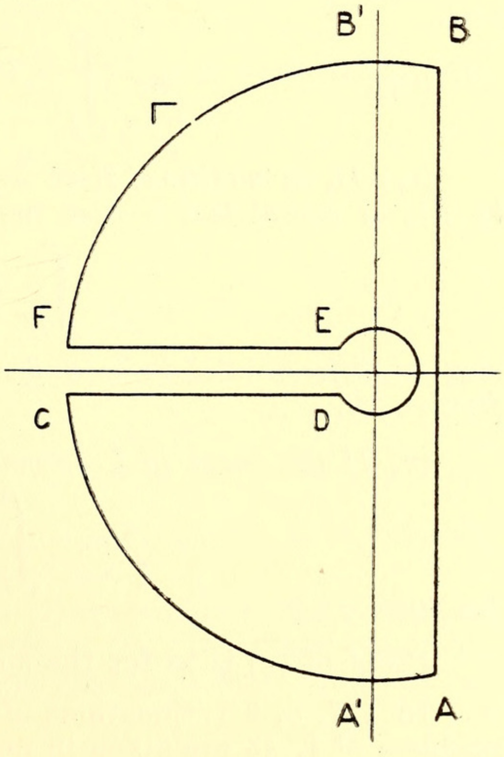


Fig. 2.

verify (*a*) that $v(t)$ given by (16) satisfies the conditions of the problem, and (*b*) that the integrals over the large circles of Figs. 1 and 2 tend to zero as the radius tends to infinity.

6. The method of verifying that solutions obtained in the form (16) satisfy their differential equations and initial and boundary conditions consists of transforming the path $L$, $(\gamma - i\infty, \gamma + i\infty)$, of (16) into a path $L'$ which begins at infinity in the direction $\arg\lambda = -\beta$, $\pi > \beta > \frac{1}{2}\pi$, keeps all singularities of the integrand to the left and ends in the direction $\arg\lambda = \beta$. The verification is then performed on the integrals over $L'$. Most of the verification is performed by the use of Theorem 2 of II, which is restated here for convenience and to include two small extensions proved as in II.

THEOREM 2. *If* $f(\lambda, \xi)$ *is an analytic function of* $\lambda$ *on and to the right of the path* $L'$, *and if*

$$|f(\lambda, \xi)| < CR^k \exp[-\xi R^{\frac{1}{2}} \cos \tfrac{1}{2}\theta],$$

*when* $\lambda = Re^{\pm i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$, $R > R_0$, *where* $C$, $k < 1$, $R_0$, *and* $\theta_0 > \frac{1}{2}\pi$ *are constants, then*

$$\text{(i)} \qquad \int_L e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda} = \int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda},$$

*provided that either* $t \geqslant 0$, $\xi > 0$, *or* $t > 0$, $\xi \geqslant 0$.

$$\text{(ii)} \qquad \int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda}$$

*is uniformly convergent with respect to* $t$ *in* $t \geqslant 0$ *for fixed* $\xi > 0$, *and with respect to* $\xi$ *in* $\xi \geqslant 0$ *for fixed* $t > 0$. *Also the integral may be differentiated under the integral sign with respect to* $t$ *in* $t \geqslant 0$ *for fixed* $\xi > 0$, *or in* $t \geqslant t_0 > 0$ *for fixed* $\xi \geqslant 0$, *and the resulting integral is uniformly convergent with respect to* $\xi$ *in* $\xi \geqslant 0$, *for fixed* $t > 0$.

$$\text{(iii)} \qquad \lim_{t \to 0} \int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda} = 0, \text{ for fixed } \xi > 0.$$

(iv) *If, in addition,* $\partial f/\partial \xi$ *and* $\partial^2 f/\partial \xi^2$ *satisfy conditions of the type satisfied by* $f(\lambda, \xi)$ *except that* $k$ *need not be less than* 1, *then*

$$\int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda}$$

*may be differentiated twice under the integral sign with respect to* $\xi$, *in* $\xi \geqslant 0$, *for fixed* $t > 0$.

(v) *If the range of* $\xi$ *extends to infinity,*

$$\lim_{\xi \to \infty} \int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda} = 0,$$

*for fixed* $t \geqslant 0$.

Proof of (v) is as for the special case in Paper II.

In §§7, 8, 9 verifications of the solutions of §§2 and 5 of I and the source problem of I, §3 are given in detail. The results of I, §4 and the other source problems of I may be treated in the same way.

7. *Verification that I* (11) *satisfies the conditions of I*, §2.

We write $f(\lambda)=(k_1\lambda+k_3)I_0(\mu a)+k_2\mu I_1(\mu a)$ .................... (17)
where $\mu=\sqrt{(\lambda/\varkappa)}$.

From the asymptotic expansions of the Bessel functions it follows that when $\lambda=\varkappa\rho e^{i\theta}$, $\pi>\theta_0\geqslant\theta\geqslant 0$ .................... (18)

$$|f(\lambda)|>C\rho^{\alpha}\exp[a\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],\text{ if }\rho>\rho_0 \quad (19)$$

where[8] $\alpha$ is 3/4, 1/4, or $-1/4$ according as $k_1\neq 0$; $k_1=0$, $k_2\neq 0$; or $k_1=k_2=0$; respectively.

Also since $|I_0(z)|\leqslant\exp|R(z)|$ .................... (20)
we have, when $\lambda$ has the value (18),

$$|I_0(\mu r)|<C\exp[r\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta].$$

Thus $$\left|\frac{I_0(\mu r)}{f(\lambda)}\right|<C\rho^{-\alpha}\exp[-(a-r)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],\ \rho>\rho_0,\ 0\leqslant r\leqslant a \quad (21)$$
where $\alpha$ has the values 3/4, 1/4 or $-1/4$.

The derivatives with respect to $r$ of the left-hand side of (21) satisfy similar inequalities. Thus for all values of the $k$ the integrand of I (11) satisfies the conditions of Theorem 2 (these are taken, here and subsequently, to include those of Theorem 2 (iv)). It follows immediately from the Theorem that

$$v=\frac{k_4}{2\pi i}\int_{L'}\frac{e^{\lambda t}I_0(\mu r)d\lambda}{\lambda f(\lambda)},\ \text{when } t\geqslant 0,\ 0\leqslant r<a \quad (22)$$
$$\text{or } t>0,\ 0\leqslant r\leqslant a,$$

that $\lim\limits_{t\to o} v=0$, for fixed $r$ in $0\leqslant r<a$, and that $v$ satisfies its differential equation.

To verify the boundary condition I (4) we take $v$ in the form (22) and observe that by Theorem 2 (ii) we may differentiate under the integral sign with respect to $r$ in $0\leqslant r\leqslant a$ for fixed $t>0$, and with respect to $t$ in $t\geqslant t_0>0$ for fixed $r$ in $0\leqslant r\leqslant a$. Thus

$$k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3 v=\frac{k_4}{2\pi i}\int_{L'}\frac{e^{\lambda t}\{(k_1\lambda+k_3)I_0(\mu r)+k_2\mu I_1(\mu r)\}d\lambda}{\lambda f(\lambda)}$$

and by (ii) and (iv) of Theorem 2 this integral is uniformly convergent with respect to $r$ in $0\leqslant r\leqslant a$ for fixed $t>0$. Thus

$$\lim_{r\to a}\left(k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3 v\right)=\frac{k_4}{2\pi i}\int_{L'}\frac{e^{\lambda t}d\lambda}{\lambda}=k_4.$$

8. *Verification that I* (36) *satisfies the conditions of I*, §5.

Writing $g(\lambda)=(k_1\lambda+k_3)K_0(\mu a)-k_2\mu K_1(\mu a)$ .................... (23)
we find as in §7 that for $\lambda=\varkappa\rho e^{i\theta}$, $\pi\geqslant\theta\geqslant 0$,

$$\left|\frac{K_0(\mu r)}{g(\lambda)}\right|<C\rho^{\alpha}\exp[-(r-a)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],\ \rho>\rho_0 \quad (24)$$

where $\alpha=-1$, $-\frac{1}{2}$, or 0 according as $k_1\neq 0$; $k_1=0$, $k_2\neq 0$; or $k_1=k_2=0$, respectively. The derivatives satisfy similar conditions.

Thus in all cases the conditions of Theorem 2 are satisfied and it follows that the path can be deformed into $L'$, that $v$ satisfies the differential equation, and that $\lim\limits_{t\to o} v=0$. It is verified as in §7 that the boundary condition at

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[8] $C$ is used for any positive constant, $\rho_0$, $\rho_1$, . . . for fixed values of $\rho$, etc.

$r=a$ is satisfied. The remaining condition

$$\lim_{r\to\infty} v=0$$

follows from Theorem 2 (v).

9. *Verification of the solution for an instantaneous cylindrical surface source over* r=r′ *in the solid cylinder* $0\leqslant r<a$.

From the results of I, §3, with the notation (17) and (23) we have

$$v=-\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_0(\mu r')\{I_0(\mu r)g(\lambda)-K_0(\mu r)f(\lambda)\}e^{\lambda t}d\lambda}{f(\lambda)},\ r'\leqslant r<a \qquad \text{.................. (27)}$$

and

$$w=-\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_0(\mu r')I_0(\mu r)g(\lambda)e^{\lambda t}d\lambda}{f(\lambda)},\ 0\leqslant r<a \qquad \text{.......... (28)}$$

We have to verify that $v$ satisfies I (16) and that $w$ satisfies I (19) and I (20).

When $\lambda=\kappa\rho e^{i\theta}$, $\pi>\theta_0\geqslant\theta\geqslant 0$,

$$\left|\frac{I_0(\mu r')I_0(\mu r)g(\lambda)}{f(\lambda)}\right|<C\rho^{-\frac{1}{2}}\exp[(r+r'-2a)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],\ \rho>\rho_0 \ \dots\ (29)$$

$$\left|\frac{I_0(\mu r')\{I_0(\mu r)g(\lambda)-K_0(\mu r)f(\lambda)\}}{f(\lambda)}\right|<C\rho^{-\frac{1}{2}}\exp[(r'-r)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],$$

$$r'\leqslant r\leqslant a,\ \rho>\rho_1 \ \text{............ (30)}$$

with similar results for the derivatives.

It follows from (29) and Theorem 2 that $w$ satisfies I (19) and I (20). Also, it follows from (30) that the path of integration in (27) may be deformed into $L'$, and that the integral over $L'$ may be differentiated under the integral sign with respect to $r$ in $r'<r\leqslant a$ for fixed $t>0$, and with respect to $t$ in $t\geqslant t_0>0$ for fixed $r$ in $r'<r\leqslant a$. Thus

$$k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3 v=-\frac{Q}{4\pi^2 i\kappa}\int_{L'}\frac{\varphi(\lambda,\ r)e^{\lambda t}}{f(\lambda)}d\lambda,$$

where $\varphi(\lambda,\ r)=I_0(\mu r')\{[(k_1\lambda+k_3)I_0(\mu r)+k_2\mu I_1(\mu r)]g(\lambda)$
$-[(k_1\lambda+k_3)K_0(\mu r)-k_2\mu K_1(\mu r)]f(\lambda)\}$

and the integral is uniformly convergent with respect to $r$ in $r'<r\leqslant a$ for fixed $t>0$. Therefore

$$\lim_{r\to a}\left(k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3 v\right)=0.$$

Since we have used the inversion theorem purely formally, and not established conditions for its validity, to complete the proof it is necessary to show that the application of the inversion theorem to I (21) gives I (18).

We consider the region $0\leqslant r\leqslant r'$. Applying the inversion theorem to I (21) gives

$$\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}I_0(\mu r)K_0(\mu r')e^{\lambda t}d\lambda$$

Now on $\lambda=\kappa\rho e^{i\theta}$

$|I_0(\mu r)K_0(\mu r')|<C\rho^{-\frac{1}{2}}\exp[-(r'-r)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta],\ 0\leqslant r\leqslant r',\ \rho>\rho_0.$

Thus by II, Theorem 1 (footnote), the integrals over the arcs $BB'F$ and $AA'C$ of Fig. 2, tend to zero as $\rho\to\infty$ for $t>0$, $0\leqslant r\leqslant r'$.

Therefore $$\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty} I_0(\mu r)K_0(\mu r')e^{\lambda t}d\lambda$$

$$=-\frac{2\kappa Q}{4\pi^2 i\kappa}\int_0^\infty e^{-\kappa u^2 t}uJ_0(ur)[K_0(iur')-K_0(-iur')]du$$

$$=\frac{Q}{2\pi}\int_0^\infty e^{-\kappa u^2 t}J_0(ur)J_0(ur')udu$$

$$=\frac{Q}{4\pi\kappa t}\exp\left(-\frac{r'+r'^2}{4\kappa t}\right)I_0\left(\frac{rr'}{2\kappa t}\right),\quad t>0,\ 0<r<r'.$$

The proof for the other range is similar.

10. It remains to show that for the problems of I the integrals round the arcs $BB'C$ and $AA'C$ of the circle $\Gamma$ of I, Fig. 1, or $BB'F$ and $AA'C$ of I, Fig. 2, tend to zero as the radii tend to infinity. When Fig. 1 is used the radius is to tend to infinity through a sequence of values avoiding the poles of the integrand; these poles have been discussed in §§2, 3, 4. In all cases we show that the integrands of the line integrals for $v$ satisfy the conditions of II, Theorem 1, and the result follows. The problems of I, §§2, 3, 5, are discussed in §§12, 13, 14; the remaining problems are treated in the same way.

11. *Lemma. For* $\lambda=\kappa(n+\frac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta}$, $\mu=\sqrt{(\lambda/\kappa)}$, $\pi\geqslant\theta\geqslant0$.

$$\left|\cosh\left(\mu a-\frac{1}{4}\pi i\right)\right|>C\exp[(n+\tfrac{1}{2})\pi\cos\tfrac{1}{2}\theta]\quad\ldots\ldots\ldots\ldots\ (31)$$

where $C$ is a constant independent of $n$.

$$\left|\cosh\left(\mu a-\frac{1}{4}\pi i\right)\right|^2=\left|\cosh\left[(n+\tfrac{1}{2})\pi e^{i\theta/2}-\frac{1}{4}\pi i\right]\right|^2$$

$$=\tfrac{1}{2}\left\{\cosh[(2n+1)\pi\cos\tfrac{1}{2}\theta]+\cos\left[(2n+1)\pi\sin\tfrac{1}{2}\theta-\frac{\pi}{2}\right]\right\}$$

$$=\tfrac{1}{2}\cosh[(2n+1)\pi\cos\tfrac{1}{2}\theta]\{1+\sin[(2n+1)\pi\sin\tfrac{1}{2}\theta]\,\mathrm{sech}[(2n+1)\pi\cos\tfrac{1}{2}\theta]\}$$

Now let $\beta=2\sin^{-1}\dfrac{2n+3/4}{2n+1}$.

Then $0\leqslant\sin[(2n+1)\pi\sin\frac{1}{2}\theta]\,\mathrm{sech}\,[(2n+1)\pi\cos\frac{1}{2}\theta]<2^{-\frac{1}{2}}$, when $\pi\geqslant\theta\geqslant\beta$.

Also, when $\beta\geqslant\theta\geqslant0$,

$|\sin[(2n+1)\pi\sin\frac{1}{2}\theta]\,\mathrm{sech}\,[(2n+1)\pi\cos\frac{1}{2}\theta]|<\mathrm{sech}\,[(2n+1)\pi\cos\frac{1}{2}\beta]<C<1$

Thus, when $\pi\geqslant\theta\geqslant0$,

$$\left|\cosh\left(\mu a-\frac{1}{4}\pi i\right)\right|>C\exp[(n+\tfrac{1}{2})\pi\cos\tfrac{1}{2}\theta]$$

The same argument gives, when $\lambda=\kappa(n+\frac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta}$,

$$\left|\cosh\left(\mu a-\frac{3}{4}\pi i\right)\right|>C\exp[(n+\tfrac{1}{2})\pi\cos\tfrac{1}{2}\theta]\quad\ldots\ldots\ (32)$$

12. *The problem of I,* §2.

Here, using the notation (17)

$$v=\frac{k_4}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}I_0(\mu r)d\lambda}{\lambda f(\lambda)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

Now it follows from the asymptotic expansions of the Bessel functions that

$$f(\lambda)=\frac{2(k_1\lambda+k_3)e^{\frac{1}{4}\pi i}}{(2\pi\mu a)^{\frac{1}{2}}}\cosh\left(\mu a-\frac{1}{4}\pi i\right)+\frac{2k_2\mu e^{\frac{3}{4}\pi i}}{(2\pi\mu a)^{\frac{1}{2}}}\cosh\left(\mu a-\frac{3}{4}\pi i\right)$$

$$+\text{similar terms } O\left(\frac{1}{\mu}\right) \text{ compared with the above.}$$

Thus,[9] if $\lambda=\varkappa(n+\frac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta},\ \pi\geqslant\theta>0$

$$\left|f(\lambda)\right|>Cn^{\alpha}\exp[(n+\tfrac{1}{2})\pi\cos\tfrac{1}{2}\theta),\ n>n_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

where the results (31) and (32) have been used and $\alpha$ is 3/2, $\frac{1}{2}$ or $-\frac{1}{2}$ according as $k_1\neq0$; $k_1=0$, $k_2\neq0$; or $k_1=k_2=0$.

Also $|I_0(z)|\leqslant\exp|R(z)|$.

Thus on $\lambda=\varkappa(n+\frac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta}$

$$\left|\frac{I_0(\mu r)}{f(\lambda)}\right|<Cn^{\alpha}\exp\left\{(n+\tfrac{1}{2})\pi\,.\,\frac{(r-a)}{a}\cos\tfrac{1}{2}\theta\right\},\ \pi\geqslant\theta\geqslant0,\ 0\leqslant r\leqslant a,\ n>n_3$$

where $\alpha$ is $-3/2$, $-\frac{1}{2}$ or $\frac{1}{2}$ according as $k_1\neq0$; $k_1=0$, $k_2\neq0$; or $k_1=k_2=0$.

In all cases the conditions of II, Theorem 1, are satisfied and thus the integral over $\Gamma$ tends to zero as its radius tends to infinity if

either $0\leqslant r\leqslant a,\ t>0$

or $0\leqslant r<a,\ t\geqslant0$.

13. *The source problem of I,* §3.

Here, in the notation (16),

$$v=-\frac{Q}{4\pi^2 i\varkappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_0(\mu r')\{I_0(\mu r)g(\lambda)-K_0(\mu r)f(\lambda)\}e^{\lambda t}d\lambda}{f(\lambda)},\ r'\leqslant r<a.$$

From the asymptotic expansions it follows that, for

$$\lambda=\varkappa(n+\tfrac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta},\ \pi\geqslant\theta\geqslant0$$

$$|I_0(\mu r')\{I_0(\mu r)g(\lambda)-K_0(\mu r)f(\lambda)\}|<Cn^{\alpha}\exp\left((n+\tfrac{1}{2})\pi\frac{(r'-r+a)}{a}\cos\tfrac{1}{2}\theta\right),$$

$$r'\leqslant r<a,\ n>n_1$$

where $\alpha=\frac{1}{2}$, $-\frac{1}{2}$, $-3/2$ according as $k_1\neq0$; $k_1=0$, $k_2\neq0$; $k_1=k_2=0$.

Thus, using (34), we have when

$$\lambda=\varkappa(n+\tfrac{1}{2})^2\frac{\pi^2}{a^2}e^{i\theta},\ \pi\geqslant\theta\geqslant0$$

$$\left|\frac{I_0(\mu r')\{I_0(\mu r)g(\lambda)-K_0(\mu r)f(\lambda)\}}{f(\lambda)}\right|<\frac{C}{n}\exp\left\{(n+\tfrac{1}{2})\pi\frac{(r-r')}{a}\cos\tfrac{1}{2}\theta\right\}$$

$$r'\leqslant r<a,\ n>n_2.$$

Thus the conditions of II, Theorem 1, are satisfied for $t>0$ if $r'\leqslant r\leqslant a$, and similarly they are satisfied if $0\leqslant r\leqslant r'$.

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[9] These circles do not pass through any pole of the integrand of (33).

14. *The problem of I,* §5.

Here in the notation (23)

$$v = \frac{k_4}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} K_0(\mu r) d\lambda}{\lambda g(\lambda)}$$

and since the order property (24) holds in $\pi \geqslant \theta \geqslant 0$ the conditions of II, Theorem 1, are satisfied.

The University of Tasmania.